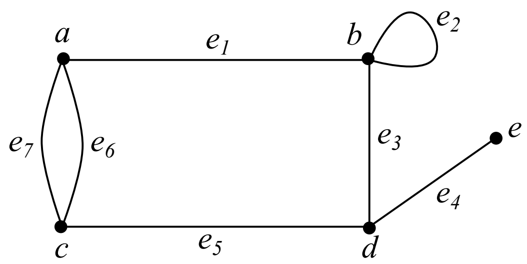
Unit 5

Graphs

5.1 Graphs

5.1.1 Graph Basics

An **undirected graph** G is an ordered pair (V, E) where V is a set of vertices and E is a set of undirected edges each of which joins a pair of vertices.
For example, $G = (V, E)$ where $V = \{a, b, c, d, e\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ is an undirected graph as given below:



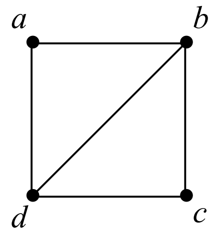
The pair of vertices $\{a, b\}$ that an edge joins are called its **endpoints** and the vertices a and b are said to be **adjacent**.

Loop: An edge that has the same vertex as both its endpoints, is called a loop. For example, e_2 is a loop in the above graph.

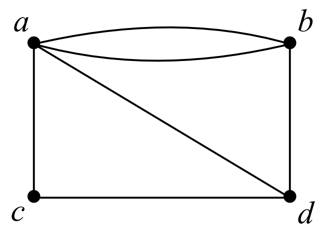
Multiple/Parallel Edges: Two or more edges are said to be multiple or parallel edges if they join the same pair of vertices. For example, e_6 and e_7 are multiple edges.

Types of Undirected Graphs:

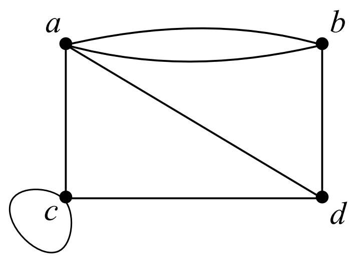
1. **Simple Graph:** An undirected graph that has neither loops not multiple edges is called a simple graph. For example, the following graph is a simple graph.



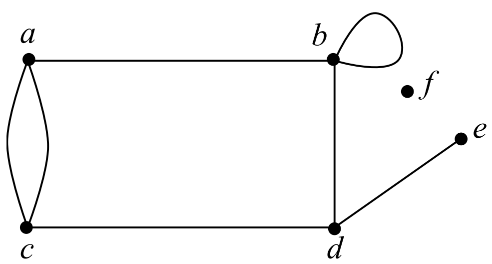
2. **Multigraph:** An undirected graph that can have multiple edges but not loops, is called a multigraph. For example, the following graph is a multigraph.



3. **Pseudograph:** An undirected graph that can have both loops as well as multiple edges is called a pseudograph. For example, the following graph is a pseudograph.



Degree of a vertex: The degree of a vertex x in an undirected graph is the number of edges in the graph which has x as one of its endpoints with loops counted twice. It is denoted by $\deg(x)$. For example in the following graph, we have $\deg(a) = \deg(c) = \deg(d) = 3$, $\deg(b) = 4$, $\deg(e) = 1$ and $\deg(f) = 0$.



Regular Graph: A simple graph in which every vertex has the same degree is called a regular graph. If that degree is n, then it is called an n-regular graph.

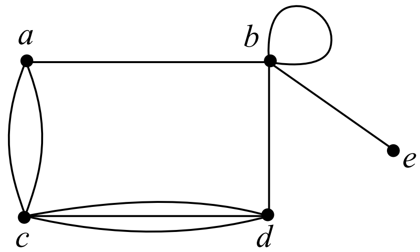
Isolated Vertex and Pendant Vertex: A vertex of degree zero is called an isolated vertex. A vertex of degree one is called a pendant vertex.

Theorem 1 (Handshaking Theorem): Let $G = (V, E)$ be an undirected graph with m edges. Then $2m = \sum_{v \in V} \deg(v)$.

Proof: Since one edge contributes exactly two to the sum of the degrees of vertices, so the sum of the degrees of all the vertices is twice the total number of edges i.e., $2m = \sum_{v \in V} \deg(v)$. □

Example:
In the graph below, $m = 9$, $\deg(a) = 3$, $\deg(b) = \deg(e) = 5$, $\deg(c) = 1$, $\deg(d) = 4$. Therefore,

$$\sum_{v \in V} \deg(v) = 3 + 5 + 5 + 1 + 4 = 18 = 2m.$$



Theorem 2: An undirected graph has an even number of vertices of odd degree.

Proof: Let $G = (V, E)$ be an undirected graph. Let V_e be the set of vertices of even degree and V_o be the set of vertices of odd degree. Then by handshaking theorem,

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_e} \deg(v) + \sum_{v \in V_o} \deg(v).$$

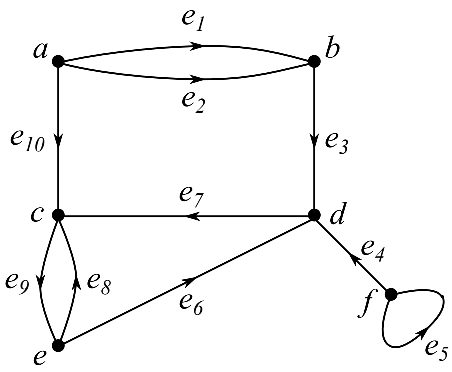
But $\deg(v)$ is even for all vertices $v \in V_e$, so $\sum_{v \in V_e} \deg(v)$ is an even number. Therefore, $2m - \sum_{v \in V_e} \deg(v) = \sum_{v \in V_o} \deg(v)$ is an even number, being the difference of two even numbers. Since $\deg(v)$ is an odd number for all $v \in V_o$, so $\sum_{v \in V_o} \deg(v)$ is an even number which is a sum of odd numbers. Hence there must be an even number of terms in $\sum_{v \in V_o} \deg(v)$ i.e., the number of vertices in V_o must be even. Therefore the number of vertices of odd degree is even. □

Example:
Here in the above graph, d is the only vertex of even degree and all the other four vertices are of odd degree. So the number of vertices of odd degree is even.

Directed Graph: A **directed graph** (or **digraph**) G is an ordered pair (V, E) where V is a set of vertices and E is a set of directed edges each of which is associated with an ordered pair of vertices.
For example, $G = (V, E)$ where $V = \{a, b, c, d, e, f\}$ and

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$$

is a directed graph as below:



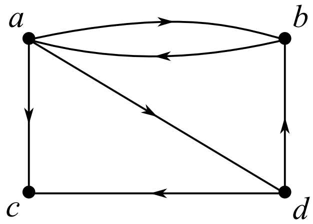
If a directed edge is associated with an ordered pair (a, b), then a is called the **initial vertex** and b is called the **final/terminal vertex** of that edge. Also, we say that a **is adjacent to** b or b **is adjacent from** a.

Loop: A directed edge that has the same initial and terminal vertices is called a loop. For example, e_5 in the above graph is a loop.

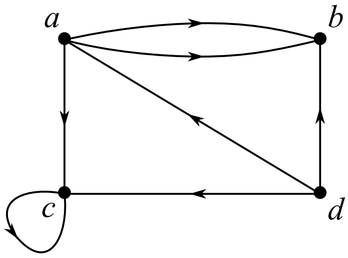
Multiple directed edges: Two or more directed edges are called multiple directed edges if they have the same pair of initial and terminal vertices. For example, e_1 and e_2 are multiple directed edges.

Types of Directed Graphs:

1. **Simple Directed Graph:** A directed graph that has neither loops nor multiple directed edges is called a simple directed graph. For example, the following is a simple directed graph.

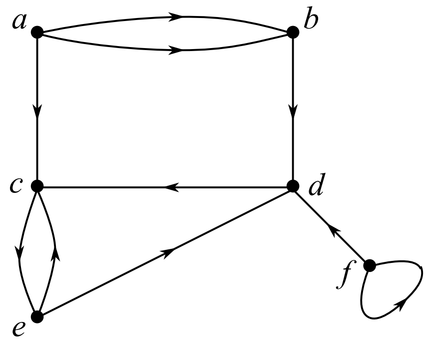


2. **Directed Multigraphs:** A directed graph that can have loops as well as multiple directed edges is called a directed multigraph. For example, the following graph is a directed multigraph.



Indegree and Outdegree of a vertex: Let x be a vertex of a directed graph G. Then the indegree of x, denoted by $\deg^-(x)$, is the number of edges with x as their terminal vertex. The

outdegree of x, denoted by $\deg^+(x)$, is the number of edges with x as their initial vertex.
For example, in the following directed graph, the indegree and outdegree of each vertex is listed below:



$$\begin{array}{ll} \deg^-(a) = 0 & \deg^+(a) = 3 \\ \deg^-(b) = 2 & \deg^+(b) = 1 \\ \deg^-(c) = 3 & \deg^+(c) = 1 \\ \deg^-(d) = 3 & \deg^+(d) = 1 \\ \deg^-(e) = 1 & \deg^+(e) = 2 \\ \deg^-(f) = 1 & \deg^+(f) = 2 \end{array}$$

Theorem 3: Let $G = (V, E)$ be a directed graph with m edges. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = m.$$

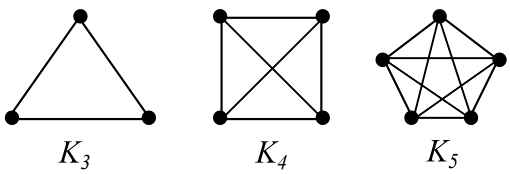
Proof: Since each directed edge has exactly one initial vertex, so each directed edge contributes exactly one to the sum of the indegrees of the vertices. Therefore, $m = \sum_{v \in V} \deg^-(v)$. Similarly, each directed edge has exactly one terminal vertex, so each directed edge contributes exactly one to the sum of the outdegrees of the vertices. Therefore $m = \sum_{v \in V} \deg^+(v)$. □

For example, in the previous digraph,

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = m = 10.$$

5.1.2 Graph Types

1. Complete Graph: The complete graph on n vertices, denoted by K_n, is the simple graph that has exactly one edge between each pair of distinct vertices. For example, the following graphs are K_3, K_4 and K_5.

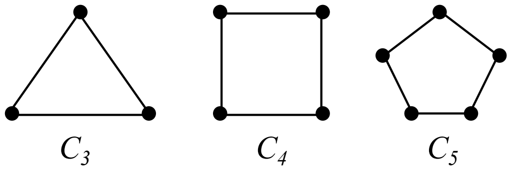


A complete graph K_n has n vertices. Each of these vertices has degree $n-1$ and so

$$\sum_{v\in V} \deg(v) = n(n-1).$$

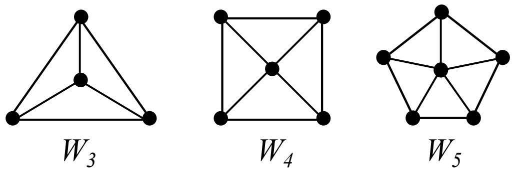
Therefore by the Handshaking theorem, the number of edges in K_n is $\dfrac{n(n-1)}{2}$.

2. Cycle: For $n \geq 3$, the cycle C_n is a simple graph consisting of n vertices $v_1, v_2, \cdots, v_n$ and n edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\cdots$, $\{v_{n-1}, v_n\}$ and $\{v_n, v_1\}$. For example,



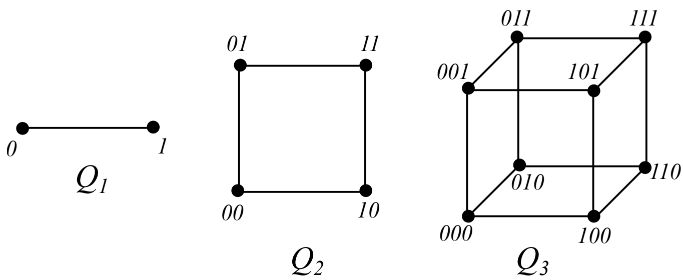
The cycle C_n has n vertices and n edges.

3. Wheel: For $n \geq 3$, the wheel W_n is the simple graph obtained from the cycle C_n by adding one new vertex and n new edges connecting this new vertex to all the other n vertices in C_n. For example,



Note that in a wheel W_n, there are $n+1$ vertices and $2n$ edges.

4. n-Cube The n-cube, denoted by Q_n, is the graph whose vertices are labeled using length n bit strings and the two vertices joined by an edge if and only if their labels differ in exactly one bit position. For example,



Note that an n-cube Q_n has 2^n vertices. Each of these vertices are connected with n other vertices corresponding to a change in one bit position among n bits. So the degree of each vertex is n and therefore $\sum_{v\in V} \deg(v) = n2^n$. By the Handshaking theorem, we therefore have

$$\frac{n2^n}{2} = n2^{n-1}$$

edges.

5. Bipartite Graphs: Let G be a simple graph. If the vertex set V of G can be divided into two subsets U and W such that

(i) U and W are nonempty,

(ii) $U \cup W = V$,

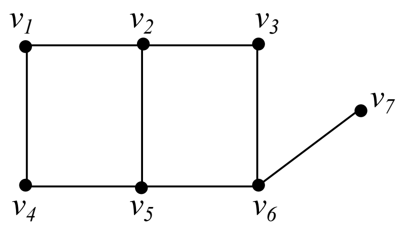
(iii) $U \cap W = \emptyset$ and

(iv) every edge in G connects a vertex in U and a vertex in W

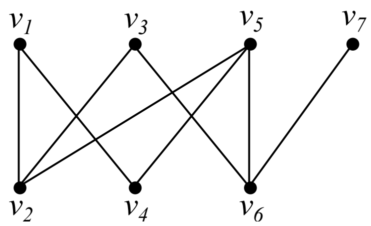
then G is called a bipartite graph.

Examples:

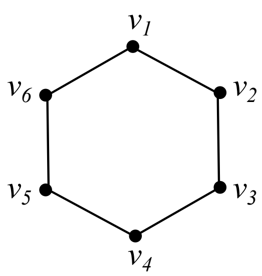
1. The graph below



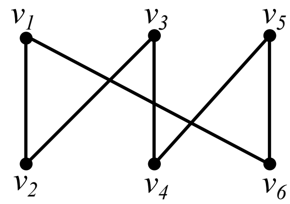
is a bipartite graph because the set of vertices $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ can be partitioned into two subsets $U = \{v_1, v_3, v_5, v_7\}$ and $W = \{v_2, v_4, v_6\}$ such that each edge connects a vertex in U with a vertex in W as below:



2. The cycle C_6 as given below is bipartite.



The set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ can be partitioned into two subsets $U = \{v_1, v_3, v_5\}$ and $W = \{v_2, v_4, v_5\}$ such that each edge connects a vertex in U with a vertex in W as below:



In fact, any cycle C_n with n even, is bipartite.

3. The cycle C_5 is not a bipartite graph. In fact, any cycle C_n with n odd, is not bipartite.

To determine whether a graph $G = (V, E)$ with $V = \{v_1, v_2, \cdots, v_n\}$ is bipartite or not, we follow these steps:

STEP I. Assign a vertex v_1 to the subset U.

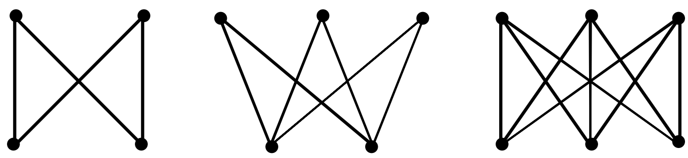
STEP II. Assign all the vertices adjacent to v_1 to the subset W.

STEP III. For a vertex in W, assign all the vertices adjacent to that vertex to the subset V.

STEP IV. Continue this process until all the vertices have been assigned to either U or W. If the subsets U and W so formed satisfy the required conditions, then G is a bipartite graph.

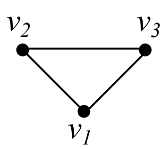
Theorem 4: A simple graph is bipartite if and only if it is possible to assign two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

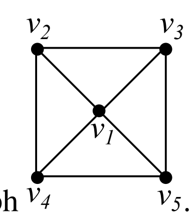
Complete Bipartite Graphs: The complete bipartite graph is a bipartite graph with the partition of the vertex set V into V_1 and V_2 such that every vertex in V_1 is joined to every vertex in V_2. It is denoted by $K_{m,n}$ where $m = |V_1|$ and $n = |V_2|$.
For example, the following graphs are complete bipartite graphs.



$K_{2,2}$ $K_{3,2}$ $K_{3,3}$

Subgraph of a graph: A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subset V$ and $F \subset E$.

For example 

is a subgraph of the graph 

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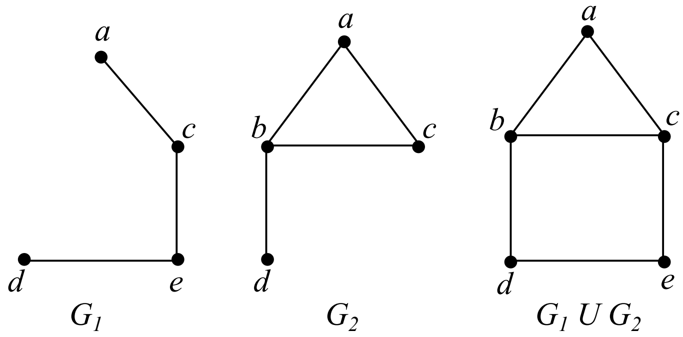
Union of graphs: The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. For example, the union of the graphs G_1 and G_2 below is the graph G.



5.1.3 Graph Representation

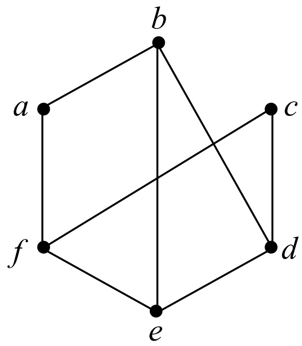
There are three ways to represent graphs:

(i) Adjacency Lists

(ii) Adjacency Matrix

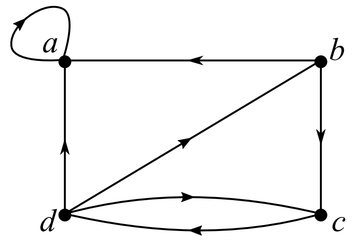
(iii) Incidence Matrix

Adjacency Lists: A graph with no multiple edges can be represented by using adjacency lists which specify the vertices that are adjacent to each vertex of the graph.
For example, the undirected graph below is represented using the adjacency list as follows:



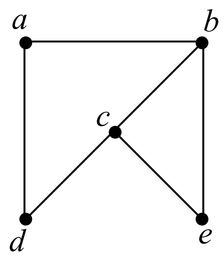
Vertex	Adjacent Vertices
a	b, f
b	a, e, d
c	f, d
d	e, b, c
e	f, b, d
f	a, c, e

Below, a directed graph is represented using the adjacency list:



Initial Vertex	Terminal Vertices
a	a
b	a, c
c	d
d	a, b, c

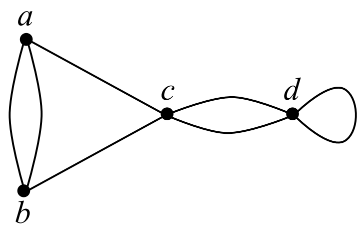
Adjacency Matrices: Let G be an undirected graph with n vertices $v_1, v_2, \cdots, v_n$. Then the adjacency matrix of G is an $n \times n$ matrix $A_G = [a_{ij}]_{n\times n}$ where a_{ij} is the number of edges joining vertices v_i and v_j.
For example, the adjacency matrix of the graph below



is

$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

The adjacency matrix of graph



is

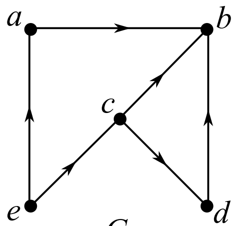
$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix} \end{array}$$

Remark:

1. The adjacency matrix of an undirected graph is always symmetric.
2. The sum of the rows and columns for each vertex are always equal and its value is the degree of that vertex except when that vertex has a loop on it.

Let G be a directed graph with n vertices $v_1, v_2, \cdots, v_n$. Then the adjacency matrix of G is the $n \times n$ matrix $A_G = [a_{ij}]_{n\times n}$ where a_{ij} is the number of edges with initial vertex v_i and the final vertex v_j.

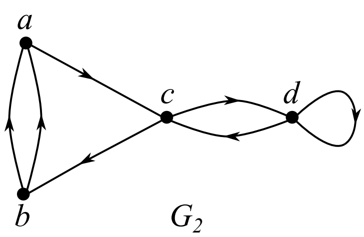
For example, the adjacency matrix of the directed graph G_1 is written as below:



G_1

$$A_{G_1} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

The adjacency matrix of the directed graph G_2 is written as below:



G_2

$$A_{G_2} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

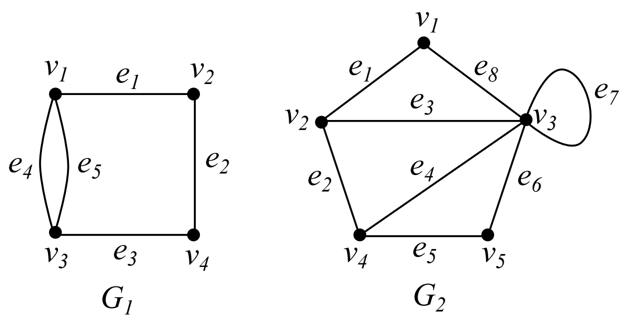
Remark:

1. The adjacency matrix of a directed graph is not necessarily symmetric.
2. The sum of the rows gives the outdegree and the sum of the columns gives the indegree of the corresponding vertex.

Incidence Matrices: Let G be an undirected graph with n vertices $v_1, v_2, \cdots, v_n$ and m edges $e_1, e_2, \cdots, e_m$. Then the incidence matrix of G is an $n \times m$ matrix $M = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 2 & \text{if edge } e_j \text{ is a loop on vertex } v_i \\ 1 & \text{if vertex } v_i \text{ is an end vertex of edge } e_j \\ 0 & \text{otherwise} \end{cases}$$

For example, the incidence matrices M_1 and M_2 of the graphs G_1 and G_2 are written below:



$$M_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array} \qquad M_2 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

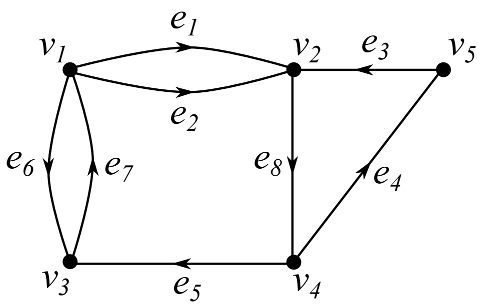
Remark:

1. The sum of numbers in each column is 2.
2. The sum of each row equals the degree of the corresponding vertex.
3. Parallel edges have identical columns in an incidence matrix.

Let G be a loopless directed graph with n vertices $v_1, v_2, \cdots, v_n$ and m edges $e_1, e_2, \cdots, e_m$. Then the incidence matrix of G is an $n \times m$ matrix $M = [a_{ij}]_{n \times m}$ where

$$a_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ is an initial vertex of edge } e_j \\ -1 & \text{if vertex } v_i \text{ is a terminal vertex of edge } e_j \\ 0 & \text{otherwise} \end{cases}$$

For example, the incidence matrix M_1 of the graph G_1 is written below:



$$M_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

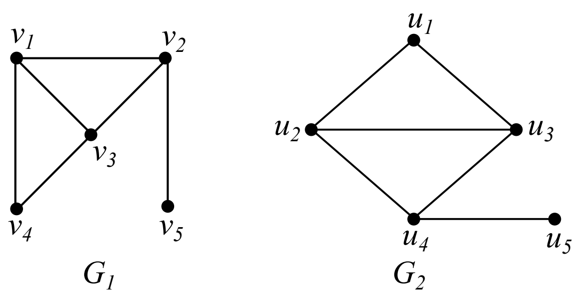
Remark:

1. Each column of an incidence matrix of a digraph has exactly one 1 and one -1.
2. The number of 1's in each row equals the outdegree and the number of -1's equals the indegree of the corresponding vertex.
3. Parallel edges have identical columns in an incidence matrix.

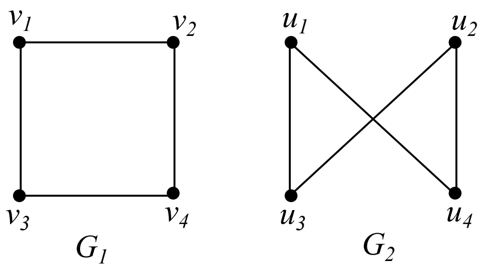
5.1.4 Graph Isomorphism

Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called **isomorphic** if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2 for all vertices a and b in V_1. Such a function f is called an **isomorphism**.

For example, consider the following graphs G_1 and G_2.



G_1 and G_2 are isomorphic graphs because there exists a one-to-one and onto function f with $f(v_1) = u_2$, $f(v_2) = u_4$, $f(v_3) = u_3$, $f(v_4) = u_1$ and $f(v_5) = u_5$ such that the vertices v_i and v_j are adjacent in G_1 if and only if the vertices $f(v_i)$ and $f(v_j)$ are adjacent in G_2.



Similarly, the above two graphs are also isomorphic because there is a one-to-one and onto function f defined as $f(v_1) = u_1$, $f(v_2) = u_4$, $f(v_3) = u_3$ and $f(v_4) = u_2$ such that two vertices are adjacent in G_1 if and only if the corresponding vertices are adjacent in G_2.

Invariant Property of a Graph: A property of a graph G is said to be an invariant property if every other graph isomorphic to G also has that property. Some invariant properties of graphs are:

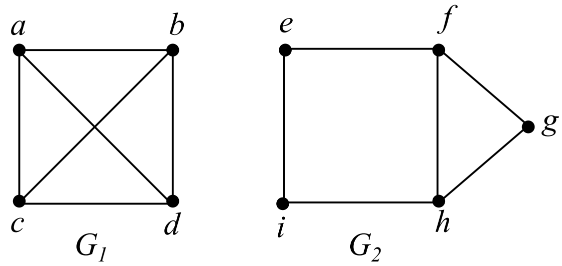
(i) The number of vertices: If two graphs G_1 and G_2 are isomorphic, then by definition of the graph isomorphism, we can conclude that the number of vertices in G_1 and G_2 must be same.

(ii) The number of edges: If two graphs G_1 and G_2 are isomorphic, then again by definition of the graph isomorphism, we can say that the number of edges in G_1 and G_2 must be equal.

(iii) Degree sequence of a graph: The degree sequence of a graph is the listing of the degrees of all the vertices of that graph written in descending order. If two graphs G_1 and G_2 are isomorphic, then both the graphs must have identical degree sequence.

Note: If two graphs G_1 and G_2 are isomorphic, then by definition of the invariant property, G_1 and G_2 both must have the same invariant properties. Hence, by contrapositive argument, we can say that if G_1 and G_2 differ in any one of the invariant properties, then G_1 and G_2 cannot be isomorphic.

Problems:
Determine whether the following graphs are isomorphic or not.

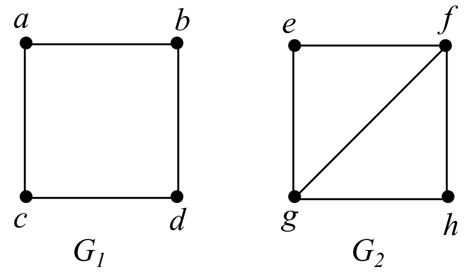
A. 

Solution: Here,

$$\text{number of vertices in } G_1 = 4$$
$$\text{number of vertices in } G_2 = 5.$$

Since G_1 and G_2 have different number of vertices, so G_1 and G_2 are not isomorphic.

B.

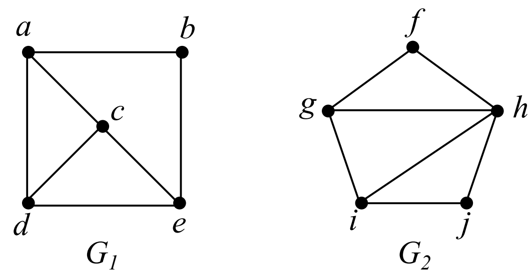


Solution: Here,

$$\text{number of vertices in } G_1 = 4$$
$$\text{number of vertices in } G_2 = 4$$
$$\text{number of edges in } G_1 = 4$$
$$\text{number of edges in } G_2 = 5.$$

Since G_1 and G_2 have different number of edges, so G_1 and G_2 are not isomorphic.

C.

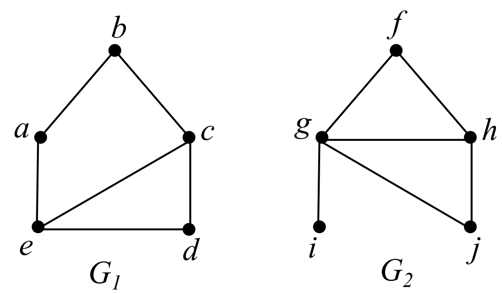


Solution: Here,

$$\text{number of vertices in } G_1 = 5$$
$$\text{number of vertices in } G_2 = 5$$
$$\text{number of edges in } G_1 = 7$$
$$\text{number of edges in } G_2 = 7$$
$$\text{degree sequence of } G_1 \text{ is } 3, 3, 3, 3, 2$$
$$\text{degree sequence of } G_2 \text{ is } 4, 3, 3, 2, 2.$$

Since G_1 and G_2 have different degree sequences, so G_1 and G_2 are not isomorphic.

D.



Solution: Here,

$$\text{number of vertices in } G_1 = 5$$
$$\text{number of vertices in } G_2 = 5$$
$$\text{number of edges in } G_1 = 6$$
$$\text{number of edges in } G_2 = 6$$
$$\text{degree sequence of } G_1 \text{ is } 3, 3, 2, 2, 2$$
$$\text{degree sequence of } G_2 \text{ is } 4, 3, 2, 2, 1.$$

Since G_1 and G_2 have different degree sequences, so G_1 and G_2 are not isomorphic.

5.1.5 Connectivity in Graphs

Paths and Circuits in Undirected Graphs:

Path: Let G be an undirected graph with vertices u and v. Then a path from u to v is a sequence of edges $e_1, e_2, \cdots, e_n$ of G such that e_1 is associated with $\{x_0, x_1\}$, e_2 is associated with $\{x_1, x_2\}$, $\cdots$, e_n is associated with $\{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$. The length of a path is defined to be the number of edges in that path.

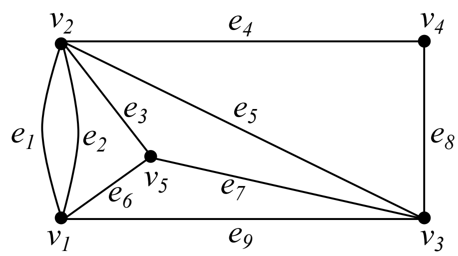
Circuit: A circuit is a path of length greater than zero that begins and ends at the same vertex.

Simple Path: A path is called a simple path if it does not contain the same edge more than once.

Simple Circuit: A circuit is called a simple circuit if it does not contain the same edge more than once.

Examples:

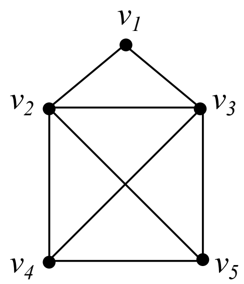
1. Let G be the following graph.



Then e_3, e_5, e_7, e_3, e_4 is a path of length 5 between the pair of vertices v_5, v_4 whereas e_3, e_5, e_8 is a simple path of length 3 between v_5 and v_4. Also, $e_1, e_5, e_9, e_2, e_5, e_7, e_6$ is a circuit that starts and ends in the vertex v_1 and e_1, e_3, e_7, e_9 is a simple circuit that also starts and ends in the vertex v_1.

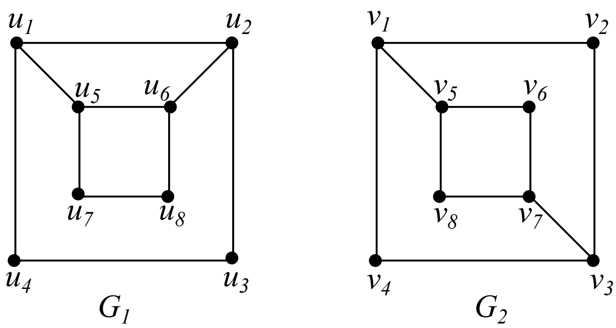
Note: When the graph G is simple, then paths or circuits in G can be specified by a sequence of vertices rather than edges as in the example below. This is because, in a simple graph, there can be at most one edge between any pair of vertices.

2. Let G be the simple graph as below:



Then $v_1, v_3, v_5, v_2, v_1, v_3$ is a path of length 5 between the vertices v_1 and v_3 and v_1, v_2, v_5, v_3 is a simple path of length 3 between v_1 and v_3. Also, $v_1, v_2, v_3, v_4, v_2, v_1$ is a circuit of length 5 that starts and ends in v_1 and $v_1, v_2, v_5, v_4, v_3, v_1$ is a simple circuit of length 5 that starts and ends in v_1.

One more invariant: The number of simple circuits of length k in a graph G is an invariant property of that graph. This invariant property can be used to show that two graphs G_1 and G_2 are not isomorphic.
For example, the following graphs G_1 and G_2 are not isomorphic because G_1 has three simple circuits of length 4 but G_2 has only two.



Paths and Circuits in Directed Graphs:

Path: Let G be a directed graph with vertices u and v. Then a path from u to v is a sequence of edges $e_1, e_2, \cdots, e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2), $\cdots$, e_n is associated with (x_{n-1}, x_n) where $x_0 = u$ and $x_n = v$. As for paths in undirected graphs, the number of edges in a path is called its length.

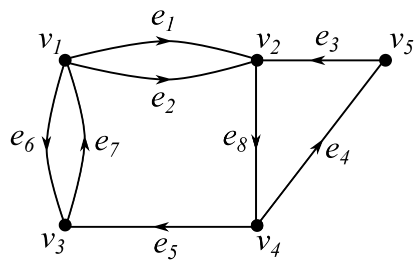
Circuit: A circuit is a path of length greater than zero that begins and ends at the same vertex.

Simple Path: A path is called a simple path if it does not contain the same edge more than once.

Simple Circuit: A circuit is called a simple circuit if it does not contain the same edge more than once.

Examples:

1. Let G be a directed graph as below:

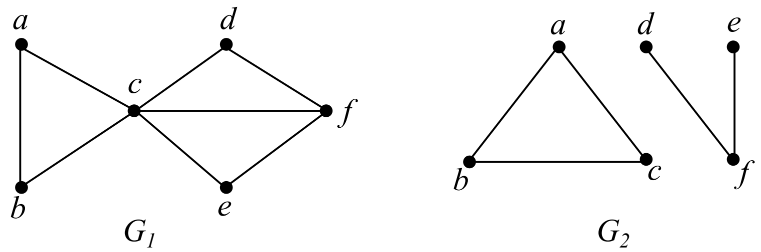


Then e_5, e_7, e_6, e_7, e_2 is a path of length 5 from vertex v_4 to vertex v_2 and e_7, e_1, e_8, e_4 is a simple path of length 4 from vertex v_3 to vertex v_5 but e_7, e_1, e_3 is not a path. Also, $e_1, e_8, e_5, e_7, e_6, e_7$ is a circuit of length 6 starting and ending in vertex v_1 whereas e_8, e_4, e_3 is a simple circuit of length 3 starting and ending in vertex v_2.

Note: When the graph G is a simple directed graph, then paths or circuits in G can be specified by a sequence of vertices rather than edges because in a simple digraph, there can be at most one edge between any pair of vertices.

Connectedness in Undirected Graphs:

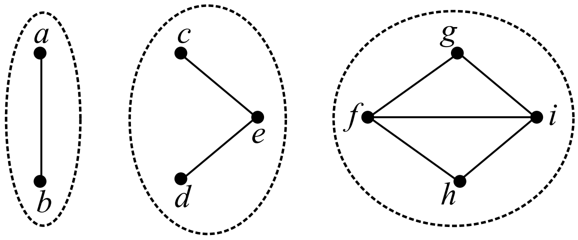
Connected Undirected Graphs: An undirected graph G is said to be connected if there is a path between every pair of distinct vertices in the graph. For example, G_1 is a connected graph and G_2 is not a connected graph.



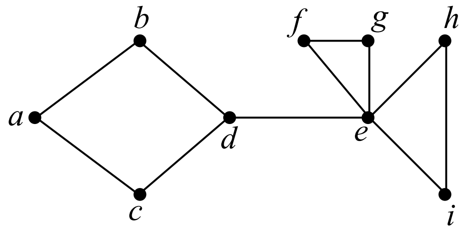
Theorem 1: There is a simple path between every pair of distinct vertices of a connected undirected graph.

Proof: Let u and v be two distinct vertices of the connected undirected graph G. Since G is connected, there is at least one path between u and v. Let $x_0, x_1, \cdots, x_n$ where $x_0 = u$ and $x_n = v$ be a path of least length. We claim that this path of least length is a simple path. To prove this, suppose that this path is not simple. Then $x_i = x_j$ for some i and j with $0 \leq i < j$. Then $x_0, x_1, \cdots, x_{i-1}, x_j, \cdots, x_n$ is a path from u to v obtained from the path $x_0, x_1, \cdots, x_n$ by removing the vertices $x_i, x_{i+1}, \cdots, x_{j-1}$. So the path $x_0, x_1, \cdots, x_{i-1}, x_j, \cdots, x_n$ is a path of shorter length than the path $x_0, x_1, \cdots, x_n$ i.e., $x_0, x_1, \cdots, x_n$ is not a path of shortest length. Hence (by indirect proof method) the path $x_0, x_1, \cdots, x_n$ from u to v must be a simple path. □

Connected Components: A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G. In other words, a maximal connected subgraph of G is called its connected component.
For example, the graph below has three connected components:

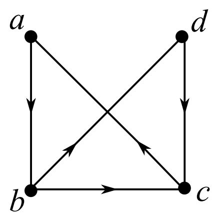


Cut Vertices and Cut Edges: A vertex in G is said to be a cut vertex if removing that vertex and all the edges incident on it produces a subgraph of G with more connected components than in G.
An edge in G is said to be a cut edge or bridge if removing that edge produces a disconnected subgraph of G.
For example, in the graph below, d and e are cut vertices and $\{d, e\}$ is a cut edge.

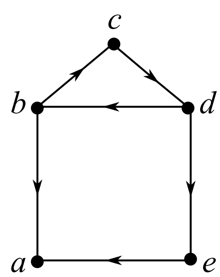


Connectedness in Directed Graphs:

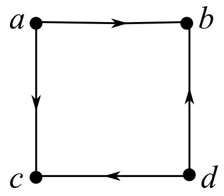
Strongly Connected Directed Graphs: A directed graph is said to be strongly connected if there is a path from a to b and from b to a for every pair of vertices a, b in the graph.
For example, the following directed graph is strongly connected.



Unilaterally Connected Digraph: A directed graph in which there exists a directed path either from a to b or from b to a for any pair of distinct vertices a and b is called a unilaterally connected digraph. Note that every strongly connected digraph is unilaterally connected as well. For example, the following digraph is a unilaterally connected digraph. It is not a strongly connected digraph because it does not have a path from a to b.



Weakly Connected Directed Graphs: A directed graph is said to be weakly connected if there is a path between every two vertices in the underlying undirected graph i.e., if the underlying undirected graph is connected.
For example, the following graph is weakly connected.

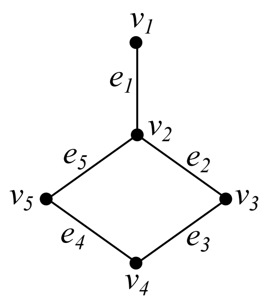


Clearly, the underlying undirected graph of this digraph is connected so it is weakly connected. However, this graph is not unilaterally connected because there is no path either from a to d or from d to a.

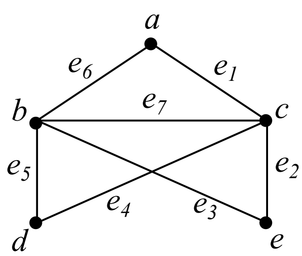
5.1.6 Euler and Hamiltonian Paths and Circuits

Euler Paths and Circuits:

Euler Path: An Euler path in a graph G is a simple path that contains every edge of G. For example, in the graph G below, e, d, b, a, c, d is an Euler path.

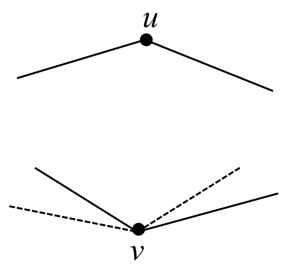


Euler Circuit: An Euler circuit in a graph G is a simple circuit that contains every edge of G. For example, the graph below has an Euler circuit a, b, c, d, b, e, c, a.



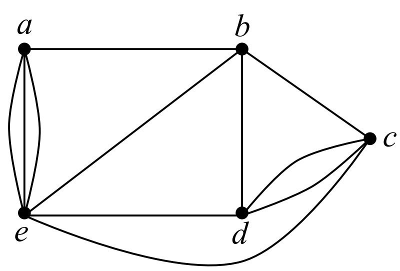
Theorem 1: A connected multigraph has an Euler circuit if and only if each of its vertices has even degree.

Proof: Suppose that a connected multigraph G has an Euler circuit. We need to prove that each of the vertices of G has even degree. Now suppose that the Euler circuit in G start and ends at a vertex u. If v is any vertex of G different from u then v must be on the Euler circuit because G is connected and the Euler circuit contains every edge of G. Moreover, each time v occurs in the Euler circuit, it enters and leaves v by different edges because each edge of G occurs only once in the Euler circuit. Thus each occurrence of v in the Euler circuit adds two to $deg(v)$ i.e., $deg(v)$ is even. Finally, since the Euler circuit must end at u, the first and last edges add two to $deg(u)$ as will any intermediate occurrence if u in the Euler circuit. So $deg(u)$ is also even.



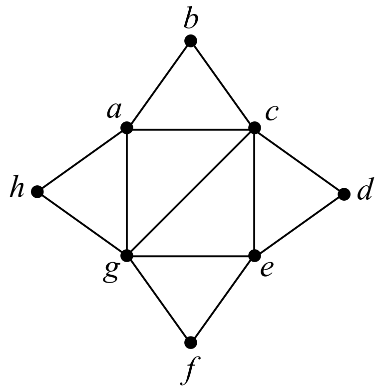
Conversely suppose that a connected multigraph G has all its vertices of even degree. We need to show that G has an Euler circuit. So let u be any arbitrary vertex of G. Starting from u, we construct a simple path that is as long as possible. Since every vertex is of even degree, we can exit from every vertex we enter so the path can only stop at vertex u. We then have a simple circuit in G. If this simple circuit contains all the edges of G, then this is an Euler circuit as required. If not, then consider a subgraph H of G obtained by removing all the edges in the circuit and the resulting isolated vertices if any. Since both G and the simple circuit have all their vertices of even degree, so the degrees of the vertices of H are also even. Also, since G is connected, H has at least one vertex in common with the simple circuit that was removed. Let v be that vertex. Starting at the vertex v, we can again construct a new simple path. Since all the vertices of H are of even degree, this path must terminate at vertex v thus forming a simple circuit. Now this simple circuit can be combined with the previous simple circuit to obtain a larger simple circuit that starts and ends at vertex u. This process is continued until one obtains a simple circuit that contains all the edges of G. We would then have an Euler circuit in G. □

Problems:
Which of the following graphs have an Euler circuit?
(a)



Solution: Here $\deg(a) = \deg(b) = \deg(c) = \deg(d) = 4$ and $\deg(e) = 6$. So the degree of all the vertices are even and hence the graph must have an Euler circuit. e.g., $abcdeaebdcea$

(b)



Solution: Here $\deg(b) = \deg(d) = \deg(f) = \deg(h) = 2$, $\deg(a) = \deg(e) = 4$ and $\deg(c) = \deg(g) = 5$. So this graph has vertices of odd degree and hence it cannot have an Euler circuit.

Algorithm for constructing Euler circuits:

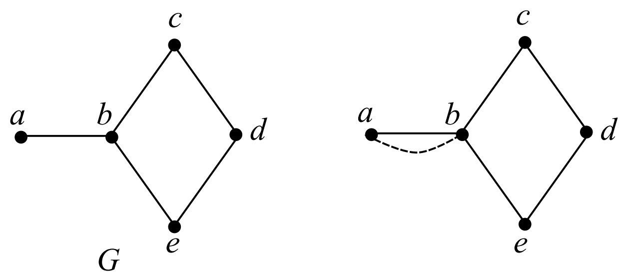
PROCEDURE Euler (G: connected multigraph with all vertices of even degree)

> *Circuit*:= a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex
> *H*:= G with edges of circuit removed
> WHILE H has edges
>
> BEGIN
>
> *Subcircuit*:= a circuit in H beginning at a vertex in H that also is an endpoint of an edge of *Circuit*
> *H*:= *H* with edges of *Subcircuit* and all isolated vertices removed
> *Circuit*:= *Circuit* with *Subcircuit* inserted at the appropriate vertex
>
> END

Circuit is the required Euler circuit.

Theorem 2: A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Proof: Let G be a connected multigraph that doesn't have an Euler circuit but has an Euler path from a to b. If we join the vertices a and b with the edge $\{a, b\}$, then this Euler path becomes an Euler circuit and the degrees of a and b increases by one whereas the degrees of other vertices remains unchanged. By Theorem 1, the degree of all vertices of this newly formed graph must be even i.e., the degree of a and b in this new graph must be even i.e., the degree of a and b in the original graph must have been odd and the degree of all other vertices must be even. So G has exactly two verices a and b of odd degree.



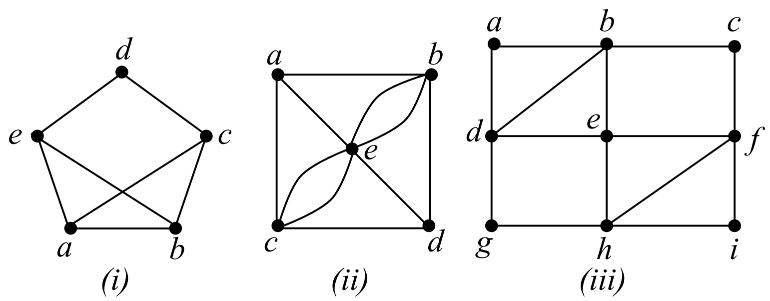
Conversely suppose that a connected multigraph G has exactly two vertices a and b of odd degree. If the vertices a and b are joined by another edge $\{a, b\}$ then this new graph has all the vertices of even degree and so by Theorem 1, there must be an Euler circuit in this new graph. If the newly added edge $\{a, b\}$ is removed from this Euler circuit, then it becomes an Euler path from vertex a to vertex b in the original graph G. □

Theorem 3: A connected multigraph has an Euler path if and only if it has either no vertices of odd degree or exactly two vertices of odd degree.

Proof: Let G be a connected multigraph which has an Euler path. If this Euler path is an Euler circuit as well, then by Theorem 1, G has no vertices of odd degree. If this Euler path is not an Euler circuit, then by Theorem 2, G has exactly 2 vertices of odd degree.

Conversely suppose that a connected multigraph G has no vertices of odd degree. Then all the vertices of G has even degree and so by Theorem 1, G must have an Euler circuit and hence an Euler path. If G has exactly two vertices of odd degree, then G must have an Euler path by Theorem 2. □

Problems:
Which of the following graphs has an Euler path?



Solution:

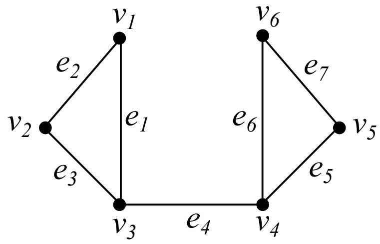
(i) Here $\deg(a) = \deg(b) = \deg(c) = \deg(e) = 3$. So this graph does not have an Euler path because it has more that two vertices of odd degree.

(ii) This graph has exactly two vertices, a and d, of odd degree. So it has an Euler path.

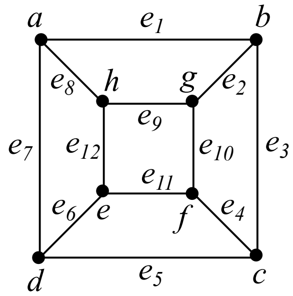
(iii) Here, $\deg(a) = \deg(c) = \deg(i) = \deg(g) = 2$ and $\deg(b) = \deg(f) = \deg(h) = \deg(d) = \deg(e) = 4$. So this graph has no vertices of odd degree and therefore it has an Euler path.

Hamiltonian Paths and Circuits:

Hamiltonian Path: A Hamiltonian path in a graph G is a simple path that passes through every vertex of G exactly once.
For example, $v_1v_2v_3v_4v_5v_6$ is a Hamiltonian path in the graph below.



Hamiltonian Circuit: A Hamiltonian circuit in a graph G is a simple circuit that passes through every vertex of G exactly once except the starting vertex which must also be the final vertex.
For example, in the graph below, $abgfcdeha$ is a Hamiltonian circuit.



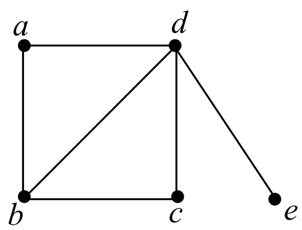
Proposition 1: The complete graph K_n has a Hamiltonian circuit whenever $n \geq 3$.

Proof: Let K_n, $n \geq 3$ be a complete graph of n vertices $v_1, v_2, \cdots, v_n$. Since there is an edge joining v_i and v_j for each distinct i and j, so the cycle $v_1v_2 \cdots v_nv_1$ is a subgraph of K_n. Clearly, this cycle is a Hamiltonian circuit of K_n. □

Proposition 2: A graph with vertex of degree one cannot have a Hamiltonian circuit.

Proof: Suppose u is a vertex of degree one. Then whenever a path reaches this vertex, there is no way to leave this vertex through a different edge. So this graph cannot have a Hamiltonian circuit. □

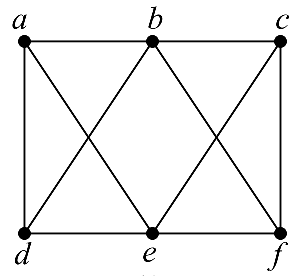
For example, the following graph below cannot have a Hamiltonian circuit.



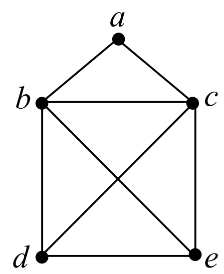
Dirac's Theorem: (Statement only) If G is a simple graph with n vertices, $n \geq 3$, such that the degree of every vertex in G is at least $\left\lceil \frac{n}{2} \right\rceil$, then G has a Hamiltonian circuit.

Ore's Theorem: (Statement only) If G is a simple graph with n vertices, $n \geq 3$, such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamiltonian circuit.

Problems:
Determine whether the following graphs have Hamiltonian circuits:



(i)



(ii)

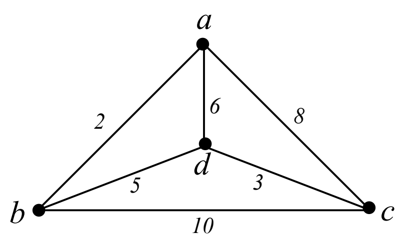
Solution:

(i) In this graph, the number of vertices $n = 6$ and so $\left\lceil \frac{n}{2} \right\rceil = 3$. Now $\deg(a) = \deg(c) = \deg(f) = \deg(d) = 3$ and $\deg(b) = \deg(e) = 4$. So the degree of all the vertices is at least $\left\lceil \frac{n}{2} \right\rceil = 3$ and so by Dirac's Theorem, this graph must have a Hamiltonian circuit.

(ii) Here, the number of vertices $n = 5$. Now the nonadjacent pairs of vertices in this graph are $\{a, d\}$ and $\{a, e\}$. So $\deg(a) + \deg(d) = 2 + 3 = 5$, $\deg(a) + \deg(e) = 2 + 3 = 5$ i.e., $\deg(u) + \deg(v) \geq n$ for all nonadjacent pairs of vertices u and v in G. So by Ore's Theorem, G must have a Hamiltonian circuit.

5.1.7 Shortest Path Algorithm

Weighted graph: Graphs that have a number assigned to each edge are called weighted graphs. For example,

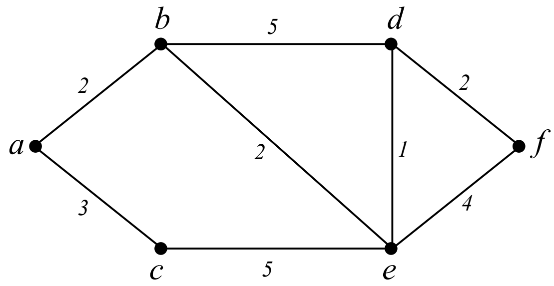


The number assigned to an edge e is called its weight and is denoted by $w(e)$. So in the above figure, $w(a, b) = 2$, $w(c, d) = 3$ etc.

Length of a path: The length of a path in a weighted graph is the sum of all the weights of the edges in that path. For example, in the previous weighted graph, the length of the path a, d, b, c is $6 + 5 + 10 = 21$.

Weighted graphs can arise in many situations while modeling real-world problems such as finding the path of shortest distance from one city to another while traveling or finding the best route to transfer data from one computer to another in a computer network.

Shortest-Path Problem: Given a weighted graph G, the problem of finding a path of least length between two of its vertices is called the shortest-path problem.
For example, in the graph below, we can see that the shortest path from a to f is a, b, e, d, f whose length is 7.



Dijkstra's Algorithm for solving shortest-path problem:
Dijkstra's algorithm is used for finding a path of shortest length between two given vertices in a weighted connected simple graph where all the weights are positive.

Dijkstra's Algorithm:
PROCEDURE Dijkstra (G: weighted connected simple graph, with all weights positive)
$\{G$ has vertices $a = v_0, v_1, \cdots, v_n = z$ and weights $w(v_i, v_j)$ where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in $G\}$

FOR $i := 1$ TO n

$L(v_i) := \infty$

$L(a) := 0$
$S := \emptyset$
$\{$The labels are now initialized so that the label of a is 0 and all other labels are ∞ and S is the empty set.$\}$
WHILE $z \notin S$

BEGIN

$u :=$ a vertex not in S with $L(u)$ minimal.
$\{$If there is more than one vertex with same minimum label $L(u)$, then select one arbitrarily.$\}$ $S := S \cup \{u\}$
FOR all vertices v not in S and adjacent to u
IF $L(u) + w(u, v) < L(v)$ THEN $L(v) = L(u) + w(u, v)$
$\{$This adds a vertex to S with minimal label and updates the labels of the vertices not in S.$\}$

END

{Length of the shortest path from a to z is $L(z)$.}

Description of Dijkstra's Algorithm:
Suppose we are given a weighted connected simple graph G with the weight of all the edges positive. Let a and z be two vertices in G and suppose that we have to find the path of shortest length from a to z.
Dijkstra's algorithm to find this path of shortest length proceeds iteratively by first finding the length of a shortest path from a to the first vertex, then the length of a shortest path from a to the second vertex, and so on, until the length of a shortest path from a to z is found. For this, the algorithm labels each vertex v of G by $L(v)$ which denotes the length of path from a to v. It also maintains a set S of vertices whose final shortest path length from the vertex a have already been determined.
The algorithm proceeds stepwise as follows:

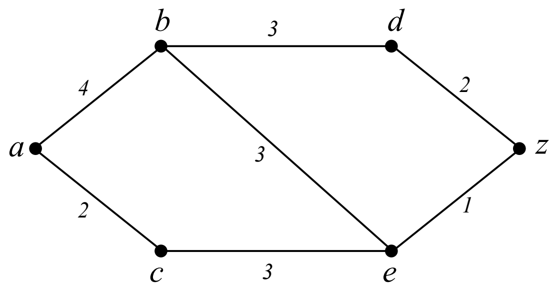
STEP 1: Set $L(a) = 0$ and $L(v) = \infty$ for all other vertices.

STEP 2: If z is selected, then stop; otherwise proceed to next step.

STEP 3: Let u be an unselected vertex such that $L(u)$ is the minimum. (If there are more than one such vertices, select one arbitrarily.) Select u and for each unselected vertex v adjacent to u, if $L(u) + w(u, v) < L(v)$, then change the value of $L(v)$ to $L(u) + w(u, v)$, otherwise don't change the value of $L(v)$. Proceed to STEP 2.

The algorithm terminates when z is a selected and the value of $L(z)$ at that time is the length of a shortest-path from a to z.

Examples:
(i) Find the path of shortest length from a to z in the following weighted connected simple graph:

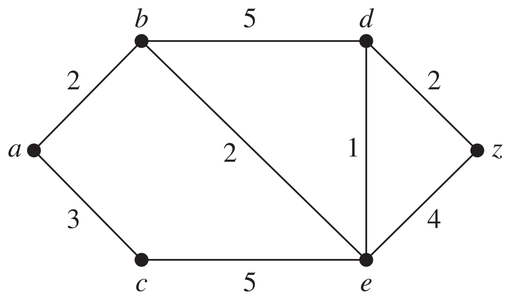


Solution:

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(z)$
$\boxed{0}$	∞	∞	∞	∞	∞
$\boxed{0}$	4 (a)	$\boxed{2}$ (a)	∞	∞	∞
$\boxed{0}$	$\boxed{4}$ (a)	$\boxed{2}$ (a)	∞	5 (a,c)	∞
$\boxed{0}$	$\boxed{4}$ (a)	$\boxed{2}$ (a)	7 (a,b)	$\boxed{5}$ (a,c)	∞
$\boxed{0}$	$\boxed{4}$ (a)	$\boxed{2}$ (a)	7 (a,b)	$\boxed{5}$ (a,c)	$\boxed{6}$ (a,c,e)

Hence the shortest path from a to z is $acez$ with length 6.

(ii) Find the path of shortest length from a to all other vertices in the following weighted connected simple graph:



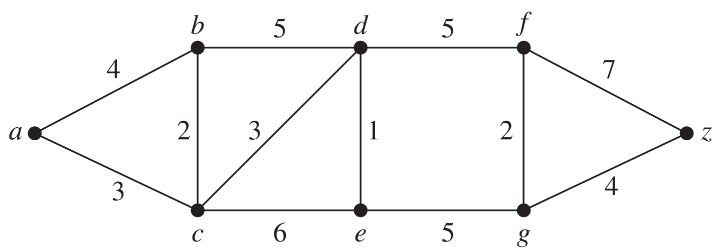
Solution:

$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(z)$
$\boxed{0}$	∞	∞	∞	∞	∞
$\boxed{0}$	$\boxed{2}$ (a)	3 (a)	∞	∞	∞
$\boxed{0}$	$\boxed{2}$ (a)	$\boxed{3}$ (a)	7 (a,b)	4 (a,b)	∞
$\boxed{0}$	$\boxed{2}$ (a)	$\boxed{3}$ (a)	7 (a,b)	$\boxed{4}$ (a,b)	∞
$\boxed{0}$	$\boxed{2}$ (a)	$\boxed{3}$ (a)	$\boxed{5}$ (a,b,e)	$\boxed{4}$ (a,b)	8 (a,b,e)
$\boxed{0}$	$\boxed{2}$ (a)	$\boxed{3}$ (a)	$\boxed{5}$ (a,b,e)	$\boxed{4}$ (a,b)	$\boxed{7}$ (a,b,e,d)

Hence the shortest path from

a to b is ab with length 2
a to c is ac with length 3
a to d is $abed$ with length 5
a to e is abe with length 4
a to z is $abedz$ with length 7.

(iii)



Solution:

5.1.8 Traveling Salesman Problem

Traveling Salesman Problem (TSP): Given n number of cities and the distance between each pair of those cities, the traveling salesman problem is to find a path that the salesman should travel so as to visit every city precisely once and return home, with the minimum distance traveled. Therefore in graph-theoretic terms, the traveling salesman problem is equivalent to finding a Hamiltonian circuit that has minimum total weight in a weighted complete undirected graph.

Solving TSP: The total number of different Hamiltonian circuits in a complete undirected graph of n vertices is $\frac{(n-1)!}{2}$. So, theoretically, the TSP can always be solved by finding these $\frac{(n-1)!}{2}$ different Hamiltonian circuits, finding the total weight of each of those circuits and then choosing a circuit with the least weight.

But practically, this method is very time consuming because for large n, finding $\frac{(n-1)!}{2}$ different Hamiltonian circuits is very inefficient. For example, if $n = 25$, then

$$\frac{(n-1)!}{2} = \frac{24!}{2} \approx 3.1 \times 10^{23}.$$

Assuming that it take just one nanosecond (10^{-9} second) to examine each Hamiltonian circuit, a total of approximately ten million years would be required to find a minimum-length Hamiltonian circuit.
Therefore TSP are practically solved using approximation algorithms which do not necessarily produce the exact solution to the problem but instead produce a solution that is close to the exact solution in a reasonable period of time.

5.1.9 Graph Coloring

Planar Graph: A graph is said to be planar if it can be drawn on a plane in such a way that no edges cross one another except at common vertices. Such a drawing is called a plane representation of graph.
For example, the complete graph K_4 in figure 1 below is a planar graph because it can be drawn as in figure 2 such that no edges intersect.

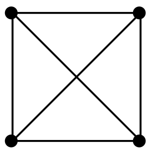


Fig. 1

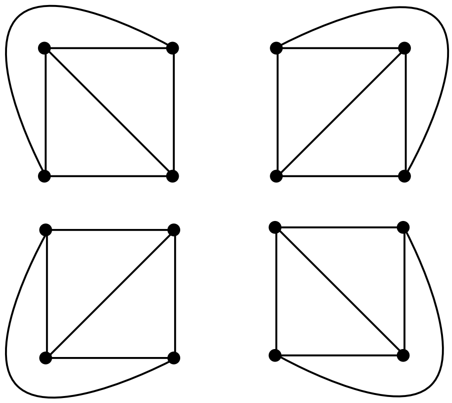
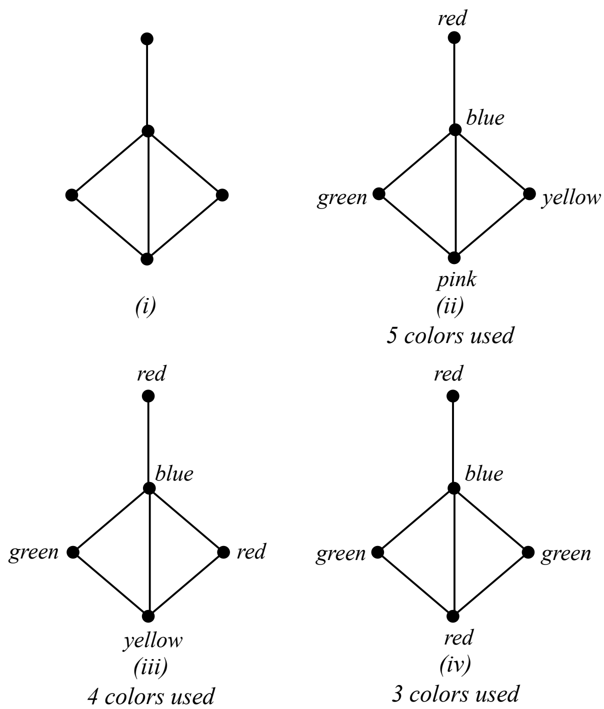


Fig. 2
(planar representations of Fig. 1)

However, the complete graph K_5 is not a planar graph. The complete bipartite graph $K_{3,3}$ is also not a planar graph.

Graph Coloring: The assignment of a color to each vertex of a simple graph so that no two adjacent vertices are assigned the same color is called the graph coloring or coloring of the graph.
For example, in the figures below, (ii), (iii) and (iv) are the graph colorings of the graph is figure (i).



Chromatic Number: The chromatic number of a graph G, denoted by $\chi(G)$, is the least number of colors needed for a coloring of the graph.
For example, the chromatic number of the above graph is 3.

Chromatic number of some common graphs:

1. $\chi(K_n)$: Each vertex of a complete graph K_n is connected with every other vertex. So if a color is used for one vertex of the graph, that color cannot be reused for any other vertex. So we need at least n colors to properly color K_n. Hence $\chi(K_n) = n$.

2. $\chi(C_n)$: A cycle C_n needs either two colors or three colors to color its vertices depending upon whether n is even or odd. So

$$\chi(C_n)=\begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

3. $\chi(W_n)$: A wheel graph W_n has one more vertex that C_n and this vertex is connected with all the other vertices. So

$$\chi(W_n)=\begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$$

4. χ(Bipartite graph): If the vertex set V of a bipartite graph G is partitioned into two subsets U and W, then the vertices in U can be given the same color since none of the vertices in U are adjacent. Similarly the vertices in W can be given the same color. Therefore $\chi(G) = 2$ for any bipartite graph G.

Applications of Graph Coloring

Scheduling Exams:
Graph coloring can be used to schedule the exams so that no student has two exams at the same time and also the exams are completed in the minimum amount of time possible. For this we proceed as follows:

(1) Represent the courses by vertices.

(2) If there is a common student in the courses, then join the corresponding vertices by an edge.

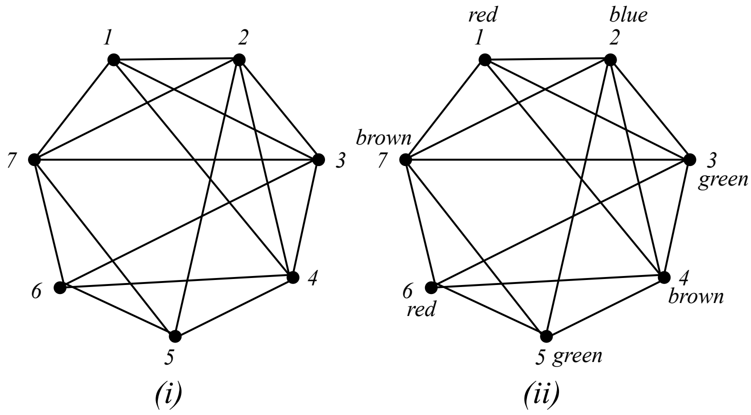
(3) Find a coloring of this associated graph.

(4) Schedule the exam such that each time slot for the exam is represented by a different color.

Examples:

1. Suppose that there are seven courses which are numbered as $1, 2, 3, 4, 5, 6, 7$ and suppose that the following courses have common students:
$\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 7\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{2, 7\}$, $\{3, 4\}$, $\{3, 6\}$, $\{3, 7\}$, $\{4, 5\}$, $\{4, 6\}$, $\{5, 6\}$, $\{5, 7\}$, $\{6, 7\}$.
Make an optimum schedule for the exam of these subjects.

   Solution: The graph associated with the subjects is as in figure (i) below:



The coloring of the graph with minimum number of colors is shown in figure (ii).

Since 4 different colors are used, the exam can be scheduled using 4 time slots as follows:

Time Slots	Courses
I (red)	1, 6
II (blue)	2
III (green)	3, 5
IV (brown)	4, 7

2. Schedule the exam for ten subjects $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ assuming that the pairs of subjects $\{1, 2\}$, $\{1, 5\}$, $\{1, 8\}$, $\{2, 4\}$, $\{2, 9\}$, $\{2, 7\}$, $\{3, 6\}$, $\{3, 7\}$, $\{3, 10\}$, $\{4, 8\}$, $\{4, 3\}$, $\{4, 10\}$, $\{5, 6\}$, $\{5, 7\}$ have common students.

3. Schedule the exams for Math115, Math116, Math185, Math195, CS101, CS102, CS273, and CS473, using the fewest number of different time slots, if there are no students taking the following pairs of subjects but there are students in every other combination of courses:
{Math115, CS473}, {Math115, Math116}, {Math115, Math185}, {Math116, CS 473}, {Math185, Math195}, {Math195, CS101}, {Math195, CS102}.