Discrete Structure

<u>Chapter:4</u>

Counting and Advanced Counting

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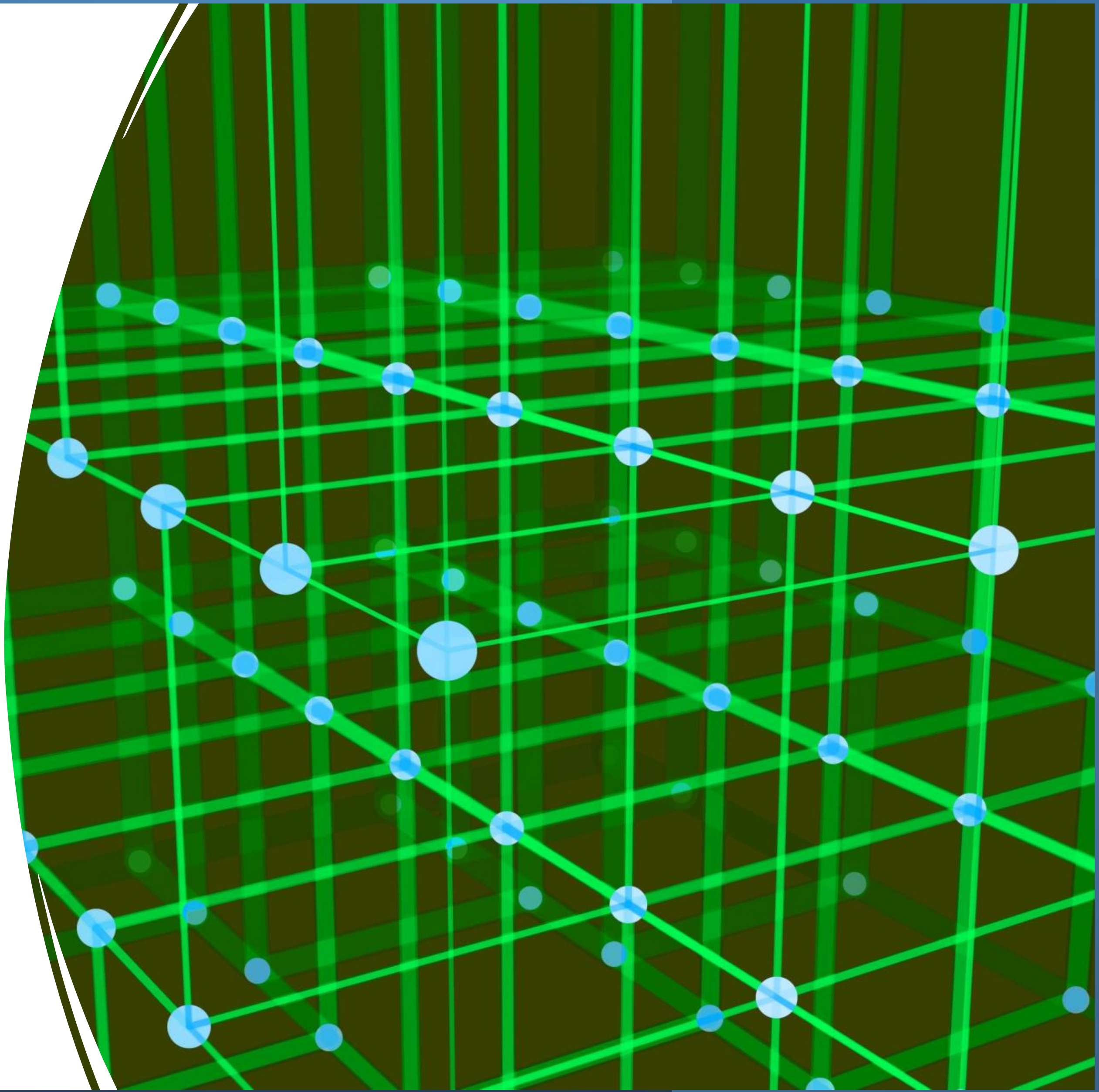
Note: This slides is only for theory containing definition and theorem. More numerical will be practice in classes



Product Rule

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

A task can be performed either in n_1 ways **AND** in n_2 ways, after the first task is complete, then there are $n_1 \cdot n_2$ ways to perform the task.

Product Rule

Example 1:

Suppose a department store has a selection of 3 different dairy milk, 2 different kit kat. how many ways to order the chocolate you have?

Let's start with our basic idea :

Suppose we have 3 different dairy milk (D1, D2 and D3) and 2 kit kat (K1 and K2)

We can order in the following ways:

(D1 K1),(D1 K2),(D2 K1),(D2 K2),(D3 K1),(D3 K2)

We have 6 different ways to order the product dairy milk and kit kat.

Hence: we can order n1 and n2 product in n1*n2 different ways

i.e, 3X2= 6 ways.

Product Rule

Example 2:

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution

The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600.

Product Rule

Problem 1:

How many different bit strings of length seven are there?

Problem 2:

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

Problem 3:

Suppose ABC bank contain 3-digit bank code 5-digit client code and 2-digit address code, then how many account numbers bank can open? (digit can be 0-9)

Sum Rule

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

A task can be performed either in n_1 ways **OR** in n_2 ways, where the two tasks cannot be performed simultaneously, then there are $n_1 + n_2$ ways to perform the task

Sum Rule

Example 1:

Suppose a department store has a selection of 3 different dairy milk, 2 different kit kat. If you are to select a taste, how many different choices of sweets you can choose from?

Let's start with our basic idea :

Suppose we have 3 different dairy milk (D1, D2 and D3) and 2 kit kat (K1 and K2)

We can have a test in following ways:

D1, D2,D3,K1,K2

We have 5 different ways to choose the taste from the product dairy milk and kit kat.

Hence: we can choose the taste from n1 and n2 product in n1+n2 different ways

i.e, 3+2= 5 ways.

Sum Rule

Example 3:

Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Solution:

There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are 37 + 83 = 120 possible ways to pick this representative.

Sum Rule

Example 3:

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution:

The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ *ways to choose a project.*

More Problems

Problem 1: *In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. (An alphanumeric character is either one of the 26 English letters or one of the 10 digits.) Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?*

Problem 2: *Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?*

Exercise

1. There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

2. A multiple-choice test contains 10 questions. There are four possible answers for each question.

a) In how many ways can a student answer the questions on the test if the student answers every question?

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

3) A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

4) How many different three-letter initials with none of the letters repeated can people have?

5) How many bit strings of length ten both begin and end with a 1?

6) How many bit strings of length n, where n is a positive integer, start and end with 1s?

7) How many strings are there of lowercase letters of length four or less, not counting the empty string?

8) How many strings are there of four lowercase letters that have the letter x in them?

The Subtraction Rule (Inclusion–Exclusion for Two Sets)

If a task can be done in either $n1$ ways or $n2$ ways, then the number of ways to do the task is $n1 + n2$ minus the number of ways to do the task that are common to the two different ways.

The subtraction rule is also known as the **principle of inclusion–exclusion**, specially when it is used to count the number of elements in the union of two sets.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Example

Problem 1: *How many bit strings of length eight either start with a 1 bit or end with the two bits 00?*

Bit strings starting with 1: To count bit strings that start with a 1 bit, the first bit is fixed, and the remaining 7 bits can be either 0 or 1, so there are 2^7 such strings.

Bit strings ending with 00: To count bit strings that end with 00, the last two bits are fixed, and the remaining 6 bits can be either 0 or 1, so there are 2^6 such strings.

Bit string start with 1 and end with 00: To count bit string start with 1 and end with 00, the first bit and last two bits are fixed, that the remaining 5 bits can be either 0 or 1 , so there are 2^6 such strings.

Now, by the definition of Subtract rule,

Total bit string of length eight start with 1 bit or end with 00 bits are: (2^7+ 2^6)- 2^5= 160 ways

Example

- ***Problem 2:*** *A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business ?*

Now , from the above questions let us assume,
A= 220, number of applicant majored in computer science
B= 147, number of applicant majored in Business
$|A \cap B| = 51$*, number of applicant majored in both.*
$|A \cup B| = ?$ *Number of applicant in either of the subject.*
$|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cup B| = 220 + 147 - 51 = 316$
Now we have,
Total number of applicant: 350,
Total applicant in either of the subject=316

Total applicant in neither of the subject: 350-316=34

The Division Rule

THE DIVISION RULE There are **n/d** ways to do a task if it can be done using a procedure that can be carried out in **n ways**, and for every way **w**, exactly **d** of the **n** ways correspond to way **w**.

Division Rule

How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

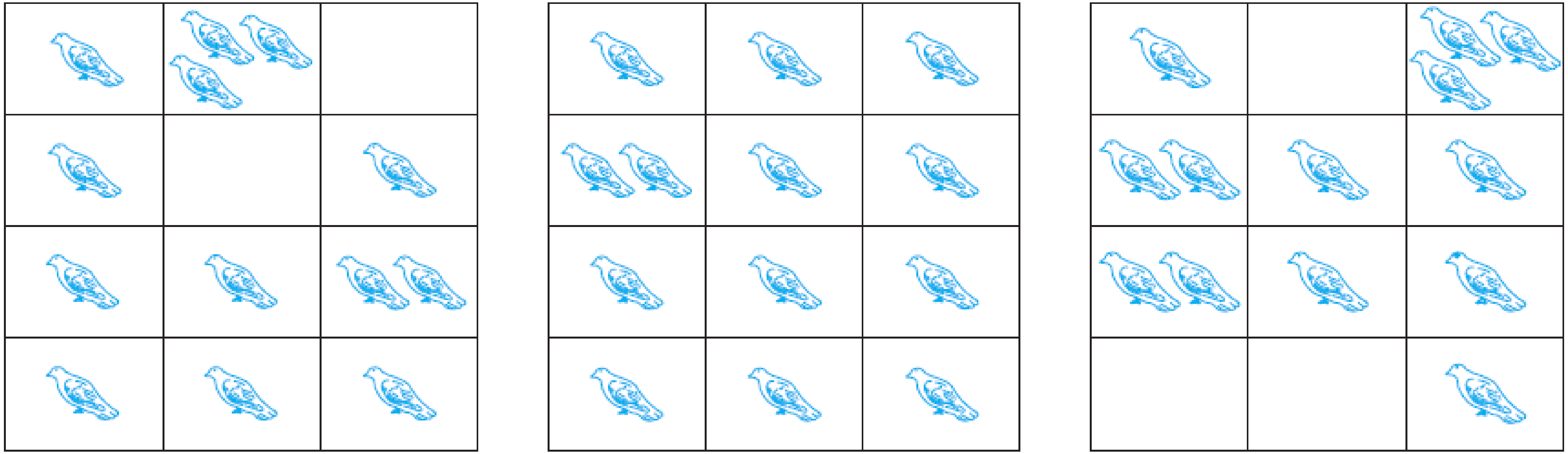
We arbitrarily select a seat at the table and label it seat 1. We number the rest of the seats in numerical order, proceeding clockwise around the table. Note that are four ways to select the person for seat 1, three ways to select the person for seat 2, two ways to select the person for seat 3, and one way to select the person for seat 4.

Thus, there are **4!=24 ways** to order the given four people for these seats. However, each of the four choices for **seat 1** leads to the same arrangement, as we distinguish two arrangements only when one of the people has a different immediate left or immediate right neighbor.

Because there are four ways to choose the person for **seat 1**, by the division rule there are **24/4 = 6** different seating arrangements of four people around the circular table.

The Pigeonhole Principle

THE PIGEONHOLE PRINCIPLE If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.



(a) (b) (c)

Problems:

If 5 color are used to paint 26 home. Show that at least 6 home of them will be of same color.

Solution:

|26/5 |=5

5 home will be each of 5 color.

Remainder 1 will be the home from one of the 5 color.

Hence, 6 home may have same color.

Generalized Pigeonhole principle

The pigeonhole principle states that there must be at least two objects in the same box when there are more objects than boxes.

THE GENERALIZED PIGEONHOLE PRINCIPLE If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

If n pigeonhole are occupied by kn+1 or more pigeons, then at least one pigeonhole is occupied by K+1 more pigeon.

Exercise:

Among 20 student in a class, at least how many of them were born in same month?

N=12 (i.e month)

Kn+1=20

1.66*12+1=20   (pick floor value of k)

i.e,

K+1 = 1+1=2

Hence at least 2 student were born in same month.

Exercise:

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

N=101

k+1=2

Kn+1=?

*1*101+1=102*

Problems

1. Ten people are swimming in the lake. Prove that at least two of them were born on the same day of the week.

2. Seventeen children are in an elevator. Prove that at least three of them were born on the same day of the week.

3. Sarah writes down random positive integers when she gets bored. Prove that if Sarah writes 1001 numbers, then there must be at least 2 with the same last three digits.

Problems

Twenty-five crates of apples are delivered to a store. The apples are of three different sorts, and al the apples in each crate are of the same sort. Show that among these crates there are at lease nine containing the same sort of apples.

Binomial Theorem

The binomial theorem gives the coefficients of the expansion of powers of binomial expressions.

A **binomial** expression is simply the sum of two terms

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n .$$

Explanation

$(a+b)^0=1$

$(a+b)^1=(a+b)$

$(a+b)^2=a^2b^0+2a^1b^1+a^0b^2$

$(a+b)^3=a^3b^0+3a^2b^1+a^1b^2+a^0b^3$

$(a+b)^4=a^4b^0+4a^3b^1+6a^2b^2+4a^1b^3+a^0b^4$

No of terms=power+1

$(a+b)^{5}= a^5b^0+a^4b^1+a^3b^2+a^2b^3+a^1b^4+a^0b^5$

$(a+b)^n= {}^nC_0a^n+{}^nc_1a^{n-1}b+ {}^nc_2a^{n-2}b^2+\ldots\ldots+ {}^nc_{n-1}ab^{n-1}+{}^nc_nb^n$

Expand $(x+y)^5$ using binomial theorem

$(x+y)^5 =$ $(x + y)^n = \Sigma\ [{}^n C_k] * x^{(n-k)} * y^k$, where the sum is taken from $k = 0$ to n, and $[{}^nC_k]$ represents the binomial coefficient, which is calculated as: $[{}^nC_k] = (n! / (k! * (n - k)!))$

Apply Binomial Theorem Formula

n=5 and k (0 to 5)

$x^{(5-0)} * y^0 = x^5$

$x^{(5-1)} * y^1 = x^4y$

$x^{(5-2)} * y^2 = x^3y^2$

$x^{(5-3)} * y^3 = x^2y^3$

$x^{(5-5)} * y^4 = xy^4$

$x^{(5-5)} * y^5 = y^5$

Calculate the binomial coefficients for k = 0 to 5:

$[{}^5 C_0] = 5! / (0! * 5!) = 1$

$[{}^5 C_1] = 5! / (1! * 4!) = 5$

$[{}^5 C_2] = 5! / (2! * 3!) = 10$

$[{}^5 C_3] = 5! / (3! * 2!) = 10$

$[{}^5 C_4] = 5! / (4! * 1!) =$

$[{}^5 C_5] = 5! / (5! * 0!) = 1$

Now add all the values which is your final answer

$x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

From the binomial theorem,

$$^{n}C_{r} = n!/r!(n!-r!)$$

We have $^{25}C_{13} = 25!/13!\ (25!-13!)$

$= 25!/13!12!$

$= 5{,}200{,}300$

Exercise:

1. *Find the coefficient of* x^5y^8 *in* $(x + y)^{13}$.
2. *Find the expansion of* $(x + y)^6$.
3. *What is the coefficient of* x^9 *in* $(2 - x)^{19}$?
4. *What is the coefficient of* x^8y^9 *in the expansion of* $(3x + 2y)^{17}$?
5. *What is the coefficient of* $x7$ *in* $(1 + x)11$?

PASCAL'S IDENTITY Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8} \\
\ldots
\end{array}$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\ldots
\end{array}$$

(b)

FIGURE 1 **Pascal's Triangle.**

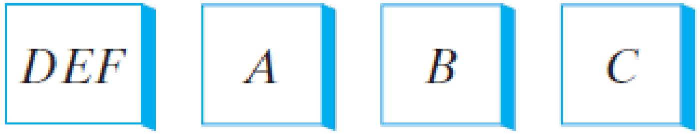
Permutations

- A *permutation* of n distinct elements $x_1, \ldots, x_n$ is an ordering of the n elements $x_1, \ldots, x_n$.
- There are six permutations of three elements. If the elements are denoted A, B, C, the six permutations are
- *ABC*, *ACB*, *BAC*, *BCA*, *CAB*, *CBA*.
- There are $n!$ permutations of n elements .
- The first element can be selected in n ways. Once the first element has been selected, the second element can be selected in $n - 1$ ways. Once the second element has been selected, the third element can be selected in $n - 2$ ways, and so on.
- $n(n - 1)(n - 2) \cdots 2 \cdot 1 = n!$

Permutations

How many permutations of the letters ABCDEF contain the substring DEF?

To guarantee the presence of the pattern *DEF* in the substring, these three letters must be kept together in this order. The remaining letters, *A*, *B*, and *C*, can be placed arbitrarily.



Thus, the number of permutations of the letters *ABCDEF* that contain the substring *DEF* is 4! = 24.

Permutations

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \dfrac{n!}{(n-r)!}$.

Permutations

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

We have,

$n=100, r=3$

$P(100,3)=n!/(n-r)!$

$=100!/(100-3)!$

$= 100!/97!$

$= 100 \times 99 \times 98 * 97!/97!$

$= 100 \times 99 \times 98$

$=970,200$

Permutations

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

We have,

n=8

r=3

P(n,r)=n!/(n-r)!

P(8,3)=8!/(8-3)!

=8!/5!

=8X7X6

336 possible ways

Permutations

Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

We have,

n=7 (must begin from one specified city, (8-1))

We have n! permutation for n elements.

i.E

7! =7X6X4X3X2X1

=5040 ways to choose the tour.

Permutations with Repetition

Counting permutations when repetition of elements is allowed can easily be done using the product rule

How many strings of length r can be formed from the uppercase letters of the English alphabet?

By the product rule, because there are 26 uppercase English letters, and because each letter can be used repeatedly, we see that there are 26^r strings of uppercase English letters of length r.

The number of r-permutations of a set of n objects with repetition allowed is n^r.

Combination

The combination is a way of selecting items from a collection, such that (unlike permutations) the order of selection does not matter.

Combination refers to the combination of n things taken k at a time without repetition.

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}.$$

Combination

How many poker hands of five cards can be dealt from a standard deck of 52 cards?

n=52

r=5

Combination ${}^{n}C_{r} = {}^{52}C_{5}$

$C(52, 5) = 52!/5!47!$

=2598960 ways

Combinations

A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

Solution: The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter. By Theorem 2, the number of such combinations is

$$C(30,6) = \frac{30!}{6!\,24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593{,}775.$$

Combinations

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution: By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By Theorem 2, the number of ways to select the committee is

$$C(9,3) \cdot C(11,4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27{,}720.$$

◀

Combination

In how many ways a committee consisting of 5 men and 3 women, can be chosen from 9 men and 12 women?

Choose 5 men out of 9 men = 9C5 ways = 126 ways

Choose 3 women out of 12 women = 12C3 ways = 220 ways

Total number of ways = (126 x 220)= 27720 ways

The committee can be chosen in 27720 ways.

Combinations

Suppose a box contains 2 white balls, 3 black balls and 4 red balls . How many ways 3 balls can be drawn for the box such that at least one ball black most be included?

Total ways to draw 3 balls without any restrictions = C(9, 3) = 84

Next, calculate the number of ways to draw 3 balls without any black balls. To do this, you'll consider only the white and red balls. You can choose 3 balls from the 6 non-black balls (2 white and 4 red) in C(6, 3) ways.

Ways to draw 3 non-black balls = C(6, 3) = 20

Now, use the principle of inclusion-exclusion to find the number of ways to draw 3 balls with at least one black ball:

Total ways - Ways to draw 3 non-black balls = 84 - 20 = 64 ways

So, there are 64 ways to draw 3 balls from the box such that at least one black ball must be included.

Combinations with Repetition

There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.

$C(n+r-1,r)= (n+r-1)!/r!\,(n-1)!$

How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

C(n+r-1,r)= (n+r-1)!/r! (n-1)!

C(3+4-1,4)=(3+4-1)!/4!(3-1)!

6!/4!2!

6.5.4!/4!2!

6.5/2.1

30/2

15

Solution: To solve this problem we list all the ways possible to select the fruit. There are 15 ways:

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange