Discrete Structure

Chapter:3

Induction and Recursion

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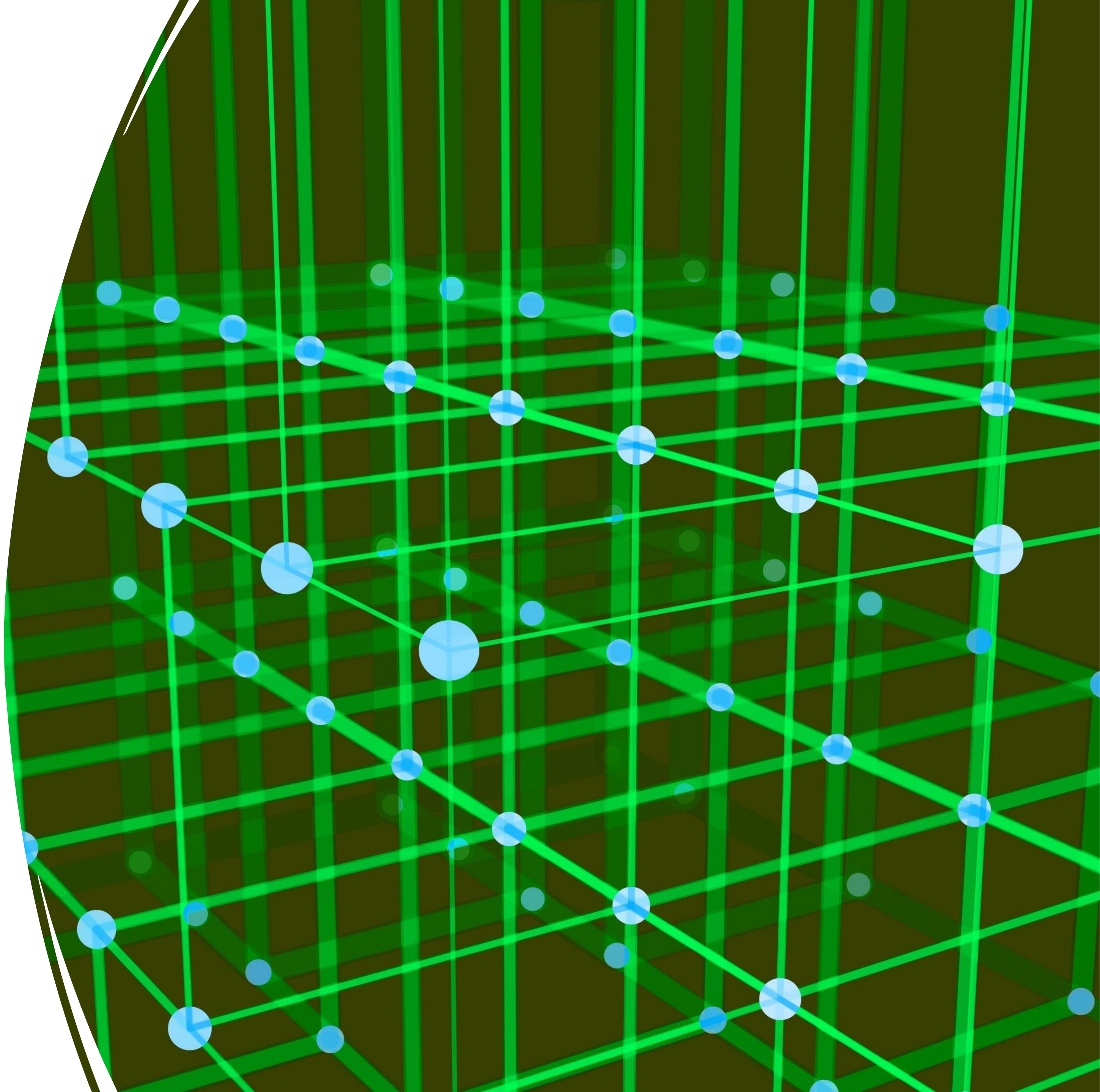
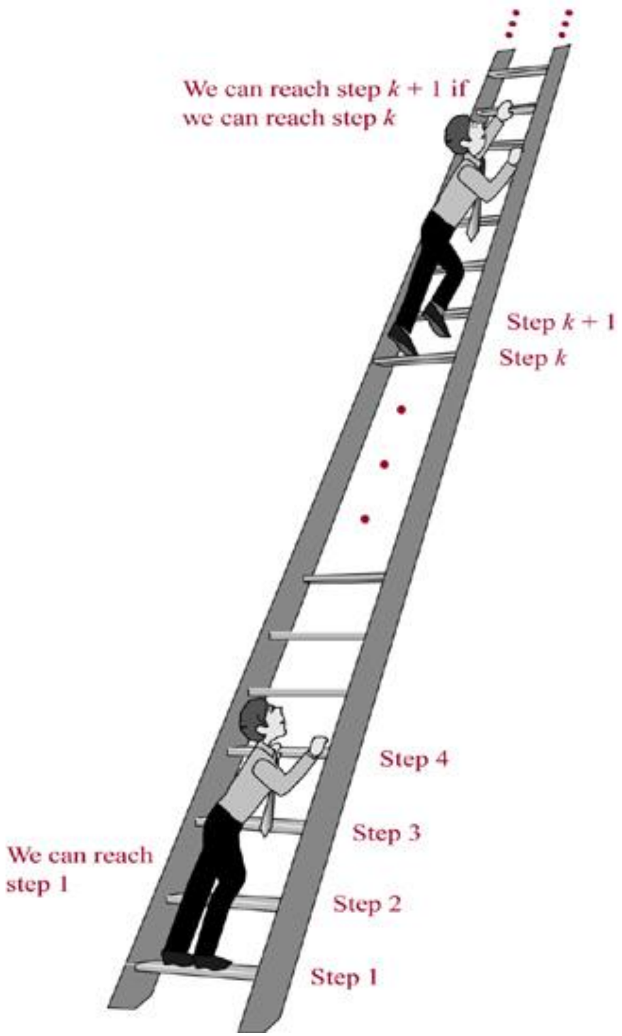
Note: This slides is only for theory containing definition and theorem. More numerical will be practice in classes



Mathematical Induction



- Want to know whether we can reach *every* step of this ladder
  - We can reach *first* rung of the ladder
  - If we can reach *a particular* run of the ladder, then we can reach *the next* run
- **Mathematical induction**: show that p(n) is true for **every** positive integer n

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

Mathematical Induction

- Two steps
  - **Basis step**: show that p(1) is true
  - **Inductive step**: show that for all positive integers k, if p(k) is true, then p(k+1) is true. That is, we show p(k)→p(k+1) for all positive integers k
- The assumption p(k) is true is called the **inductive hypothesis**
- Proof technique:

$$[p(1) \wedge \forall k(p(k) \rightarrow p(k+1))] \rightarrow \forall n p(n)$$

Example-1: (**Proving Summation Formulae)**

Show that 1+2+… +n=n(n+1)/2, if n is a positive integer

Let p(n) be the proposition that 1+2+… +n=n(n+1)/2

Basis step: p(1) is true, because 1=1*(1+1)/2

Inductive step: Assume p(k) is true for an arbitrary k. That is, 1+2+…+k=k(k+1)/2

We must show that 1+2+…+(k+1)=(k+1)(k+2)/2

From p(k), 1+2+…+k+(k+1)=k(k+1)/2+(k+1)=(k+1)(k+2)/2
which means p(k+1) is true

We have completed the basic and inductive steps, so by mathematical induction we know that p(n) is true for all positive integers n. That is 1+2+…+n=n(n+1)/2

Example-2

Show that $1+3+5+\ldots+(2n-1)=n^2$, if n is a positive integer

- Let p(n) denote the proposition
- Basic step: $p(1)=1^2=1$
- Inductive steps: Assume that p(k) is true, i.e., $1+3+5+\ldots+(2k-1)=k^2$

  We must show $1+3+5+\ldots+(2k+1)=(k+1)^2$ is true for p(k+1)

  Thus, $1+3+5+\ldots+(2k-1)+(2k+1)=k^2+2k+1=(k+1)^2$ which means p(k+1) is true

  (Note p(k+1) means $1+3+5+\ldots+(2k+1)=(k+1)^2$)
- We have completed both the basis and inductive steps. That is, we have shown p(1) is true and $p(k) \rightarrow p(k+1)$
- Consequently, p(n) is true for all positive integers n

Example-3

Use mathematical induction to show that: $\boldsymbol{1+2+2^2+\ldots+2^n=2^{n+1}-1}$***,*** *for all non-negative integers*

Let p(n) be the proposition: $1+2+2^2+\ldots+2^n=2^{n+1}-1$

Basis step: $p(0)=2^{0+1}-1=1$

Inductive step: Assume p(k) is true, i.e., $1+2+2^2+\ldots+2^k=2^{k+1}-1$

It follows

$(1+2+2^2+\ldots+2^k)+2^{k+1}=(2^{k+1}-1)+2^{k+1}=2*2^{k+1}-1=2^{k+2}-1$ which means p(k+1): $1+2+2^2+\ldots+2^{k+1}=2^{k+2}-1$ is true

We have completed both the basis and inductive steps. By induction, we show that $1+2+2^2+\ldots+2^n=2^{n+1}-1$

Example 4: (Geometric Progressions)

Sums of Geometric Progressions Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term a and common ratio r:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1} \qquad \text{when } r \neq 1,$$

where n is a nonnegative integer.

Solution: To prove this formula using mathematical induction, let $P(n)$ be the statement that the sum of the first $n+1$ terms of a geometric progression in this formula is correct.

BASIS STEP: $P(0)$ is true, because

$$\frac{ar^{0+1} - a}{r - 1} = \frac{ar - a}{r - 1} = \frac{a(r-1)}{r - 1} = a.$$

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, where k is an arbitrary nonnegative integer. That is, $P(k)$ is the statement that

$$a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1} - a}{r - 1}.$$

To complete the inductive step we must show that if $P(k)$ is true, then $P(k+1)$ is also true. To show that this is the case, we first add ar^{k+1} to both sides of the equality asserted by $P(k)$. We find that

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} \overset{\text{IH}}{=} \frac{ar^{k+1} - a}{r - 1} + ar^{k+1}.$$

Rewriting the right-hand side of this equation shows that

$$\frac{ar^{k+1} - a}{r - 1} + ar^{k+1} = \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+2} - ar^{k+1}}{r - 1}$$
$$= \frac{ar^{k+2} - a}{r - 1}.$$

Combining these last two equations gives

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1}.$$

This shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ must also be true. This completes the inductive argument.

We have completed the basis step and the inductive step, so by mathematical induction $P(n)$ is true for all nonnegative integers n. This shows that the formula for the sum of the terms of a geometric series is correct. ◄

Example 5: (*PROVING INEQUALITIES*)
Use mathematical induction to prove the inequality: *$n < 2^n$, for all positive integer.*

Let $P(n)$ be the proposition that $n < 2^n$.

BASIS STEP: $P(1)$ is true, because $1 < 2^1 = 2$. This completes the basis step.

INDUCTIVE STEP: We first assume the inductive hypothesis that $P(k)$ is true for an arbitrary positive integer k. That is, the inductive hypothesis p(k) is the statement that $k < 2^k$.To complete the inductive step, we need to show that if $P(k)$ is true, then $P(k + 1)$, which is the statement

that $k + 1 < 2^k+1$, is true. That is, we need to show that if $k < 2^k$, then $k + 1 < 2^k+1$. To show that this conditional statement is true for the positive integer k, we first add 1 to both sides of $k < 2^k$, and then note that $1 \leq 2^k$. This tells us that

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2k = 2^{k+1}$$

This shows that $P(k + 1)$ is true, namely, that $k + 1 < 2^k+1$, based on the assumption that $P(k)$ is true. The induction step is complete.

Therefore, because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $n < 2^n$ is true for all positive integers n.

Example 6: (*PROVING INEQUALITIES*)

Use mathematical induction to prove that $2^n < n!$ for every integer n with $n \geq 4$. (Note that this inequality is false for $n = 1, 2$, and 3.)

Solution: Let $P(n)$ be the proposition that $2^n < n!$.

BASIS STEP: To prove the inequality for $n \geq 4$ requires that the basis step be $P(4)$. Note that $P(4)$ is true, because $2^4 = 16 < 24 = 4!$

INDUCTIVE STEP: For the inductive step, we assume that $P(k)$ is true for an arbitrary integer k with $k \geq 4$. That is, we assume that $2^k < k!$ for the positive integer k with $k \geq 4$. We must show that under this hypothesis, $P(k+1)$ is also true. That is, we must show that if $2^k < k!$ for an arbitrary positive integer k where $k \geq 4$, then $2^{k+1} < (k+1)!$. We have

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k && \text{by definition of exponent} \\ &< 2 \cdot k! && \text{by the inductive hypothesis} \\ &< (k+1)k! && \text{because } 2 < k+1 \\ &= (k+1)! && \text{by definition of factorial function.} \end{aligned}$$

This shows that $P(k+1)$ is true when $P(k)$ is true. This completes the inductive step of the proof.

We have completed the basis step and the inductive step. Hence, by mathematical induction $P(n)$ is true for all integers n with $n \geq 4$. That is, we have proved that $2^n < n!$ is true for all integers n with $n \geq 4$. ◀

Example 7: (Proving divisibility results):

Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

let $P(n)$ denote the proposition: $n^3 - n$ is divisible by 3.

- ***Basic Step:*** The statement $P(1)$ is true because $1^3 - 1 = 0$ is divisible by 3. This completes the basis step.
- ***Inductive Step:*** For inductive hypothesis, we assume that p(k) is true; that is we assume that k^3-k is divisible by 3 for any arbitrary positive integer k. To complete this inductive state we must show that when we assume the inductive hypothesis, it follows that $P(k + 1)$,

Inductive step: *k^3-k → Inductive Hypothesis*

- The statement that $(k + 1)^3 - (k + 1)$ is divisible by 3, is also true. That is, we must show that $(k + 1)^3 - (k + 1)$ is divisible by 3. Note that

$(k + 1)^3 - (k + 1)$

$(k^3 + 3k^2 + 3k + 1) - (k + 1)$

$k^3 + 3k^2 + 3k + 1 - k - 1$

$k^3 + 3k^2 + 3k - k$

$(k^3-k)+3(k^2+k)$

Using th inductive hypothesis, we conclude that k^3-K is divisible by 3, We have complete both basic steps and inductive steps, by using mathematical induction we know that n^3-n is divisible by 3 for all positive interge for n.

Practice

1. Use induction to show that $n<2^n$ for $n>0$.
2. Use induction to show that $2^n<n!$ for $n \geq 4$.
3. Show that n^3-n is divisible by 3 when n is positive.
4. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ whenever n is a positive integer.
5. Using mathematical induction prove that $n(n^2+5)$ is an integer divisible by 6 for all positive integer.
6. Using mathematical induction show that: $1^2+2^2+3^2+ \ldots\ldots +n^2=\frac{n(n+1)(2n+1)}{6}$
7. Show by mathematical induction: $1^2+3^2+5^2+(2n-1)^2= \frac{n(2n+1)(2n-1)}{3}$

Strong induction and well-ordering

- **Strong induction**: To prove p(n) is true for all positive integers n, where p(n) is a propositional function, we complete two steps
- Basis step: we verify that the proposition p(1) is true
- Inductive step: we show that the conditional statement $(p(1) \wedge p(2) \wedge \ldots \wedge p(k)) \rightarrow p(k+1)$ is true for all positive integers k

Strong induction

- Can use all k statements, p(1), p(2), …, p(k) to prove p(k+1) rather than just p(k)
- Mathematical induction and strong induction are equivalent
- Any proof using mathematical induction can also be considered to be a proof by strong induction (induction → strong induction)
- It is more awkward to convert a proof by strong induction to one with mathematical induction (strong induction → induction)

Strong induction

- Also called the **second principle of mathematical induction** or **complete induction**
- The principle of mathematical induction is called **incomplete induction**, a term that is somewhat misleading as there is nothing incomplete
- Analogy:
  - If we can reach the first step
  - For every integer k, if we can reach all the first k steps, then we can reach the k+1 step

Example

- Suppose we can reach the 1st and 2nd rungs of an infinite ladder
- We know that if we can reach a rung, then we can reach two rungs higher
- Can we prove that we can reach every rung using the principle of mathematical induction? or strong induction?

Example – mathematical induction

- Basis step: we verify we can reach the 1^{st} rung
- Attempted inductive step: the inductive hypothesis is that we can reach the k-th rung
- To complete the inductive step, we need to show that we can reach k+1-th rung based on the hypothesis
- However, no obvious way to complete this inductive step (because we do not know from the given information that we can reach the k+1-th rung from the k-th rung)

Example – strong induction

- Basis step: we verify we can reach the 1st rung
- Inductive step: the inductive hypothesis states that we can reach <u>each of the first k rungs</u>
- To complete the inductive step, we need to show that we can reach k+1-th rung
- We know that we can reach 2nd rung.
- We note that we can reach the (k+1)-th rung from (k-1)-th rung we can climb 2 rungs from a rung that we already reach
- This completes the inductive step and finishes the proof by strong induction

Which one to use

- Try to prove with mathematical induction first
- Unless you can clearly see the use of strong induction for proof

Recursive Definitions

The function that uses the previous term to find the next term in the sequence is called a recursive function.

We use two steps to define a function with the set of nonnegative integers as its domain:

Basis step: Specify the value of the function at zero.

Recursive step: Give a rule for finding its value at an integer from its values at smaller integers.

Suppose that *f* is defined recursively by

$f(0) = 3,$

$f(n + 1) = 2f(n) + 3.$

Find $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

$$f(1) = 2f(0) + 3 = 2 \cdot 3 + 3 = 9,$$

$$f(2) = 2f(1) + 3 = 2 \cdot 9 + 3 = 21,$$

$$f(3) = 2f(2) + 3 = 2 \cdot 21 + 3 = 45,$$

$$f(4) = 2f(3) + 3 = 2 \cdot 45 + 3 = 93.$$

Recursive Example

Q: Give a recursive definition of a^n, where a is a nonzero real number and n is a nonnegative integer.

Answer: The recursive definition contains two parts. First a^0 is specified, namely, $a^0 = 1$. Then the rule for finding a^{n+1} from an, namely, $a^{n+1} = a \cdot a^n$, for $n = 0, 1, 2, 3, \ldots,$ is given. These two equations uniquely define a^n for all nonnegative integers n.

Q: Give a recursive definition for Fibonacci sequence.

Answer: The recursive definition contains two parts. First base case where f(0), f(1) are the starting point of the sequence. Then, for n greater than 1 ,the n^{th} Fibonacci number is calculated by adding two previous Fibonacci numbers i.e, f(n-1)+f(n-2). The equation f(n)=f(n-1)+f(n+2) define the Fibonacci series.

Factorial: The factorial of a non-negative integer n is defined recursively as follows:

Base Case: n! = 1 if n = 0

Recursive Step: n! = n * (n-1)! for n > 0

Recursive Algorithm

An algorithm is called ***recursive*** if it solves a problem by reducing it to an instance of the same problem with smaller input.

Give a recursive algorithm for computing *n*!, where *n* is a nonnegative integer.

```
factorial(int n) {
    // Base case: factorial of 0 is 1
    if (n == 0) {
        return 1;
    }
    // Recursive case: factorial of n is n times factorial of (n-1)
    else {
        return n * factorial(n - 1);
    }
}
```

Write a recursive algorithm for computing a^n, where a is non zero real number n is a nonnegative integer.

```
Power_number(int n, double a)
{
// base case a^0 =1 for any non zero a.
If(n==0)
return 1;
}
// recursive case
{
return a* power(a,n-1)
}
```

Work

Write a recursive algorithm for Fibonacci series.

Write a recursive algorithm for Tower of Hanoi.

Write a recursive algorithm to check for palindrome.