Unit 1

Logic and Proofs

1.1 Logic

1.1.1 Propositional Logic

Definition (Proposition):

A proposition is a declarative sentence that is either true or false but not both.

For example,

(i) Kathmandu is the capital city of Nepal.

(ii) $2 + 2 = 4$.

(iii) The Earth is flat.

(iv) $2 + 3 = 6$.

All the sentences above are propositions. The first two are true whereas the last two are false propositions.

Note that we need not have the knowledge of truth or falsity of that sentence for it to be a proposition. The fact that it must be one or the other is enough. For example,

(v) It rained in Pokhara yesterday.

(vi) $\pi + e$ is an irrational number.

are both propositions. In sentence (v), we may not know whether or not it rained in Pokhara yesterday but we can easily determine the truth or falsity of this sentence (just call a friend in Pokhara!). However, nobody knows whether $\pi + e$ is an irrational number or not (it is still an open problem as of now). Even then, (vi) is a proposition because it must be either true or false and not both. The following sentences are not propositions:

(vii) What time is it?

(viii) $2x = 4$.

Sentence (vii) is not a proposition because it is an interrogative sentence, not a declarative sentence. Sentence (viii) is also not a proposition. Although this is a declarative sentence, we cannot determine whether it is a true sentence or a false sentence no matter how hard we try because x is a variable i.e., its value is not fixed. The truthness or falseness of this sentence depends upon the value that the variable x might take. If $x = 2$, then (viii) is true whereas for all other values of x, it is false. Therefore one way to change sentences of type (viii) into propositions is to specify the value of the variable x.

Small-case letters such as p, q, $r \cdots$ are commonly used to denote arbitrary propositions.

Definition (Truth Value):

The truth or falsity of a given proposition is called its truth value . If a proposition is true, then its truth value is 'True' which is symbolically denoted by 'T' and if a proposition is false, then its truth value is 'False' which is symbolically denoted by 'F'.

For example, the truth values of propositions (i)-(iv) above are 'T', 'T', 'F' and 'F' respectively.

Sometimes 1 and 0 are used instead of 'T' and 'F' to denote the truth values of a proposition. However, we shall always use 'T' and 'F' in this book.

Definition (Compound Proposition):

A proposition formed by combining one or more existing propositions is called a compound proposition . The propositions participating to form a compound proposition are called component propositions and the elements used to combine these component propositions are called logical operators or connectives .

Definition (Truth Table):

A compound proposition is first of all a proposition and so it must be either true or false. But the truthness or falseness of a compound proposition depends upon the truthness or falseness of its component propositions. A table which gives the truth value of a compound proposition in terms of the truth values of its component propositions is called a truth table .

Types of logical operators:

(i) **Negation**: If p is any proposition, then its negation is the proposition "It is not the case that p." which is denoted by $\neg p$. It is read as "not p". The proposition $\neg p$ is a proposition that is true when p is false and false when p is true.

Truth table of Negation

p	$\neg p$
T	F
F	T

For example,

(a) If p is the proposition "The Earth is flat", then $\neg p$ is the proposition "It is not the case that the Earth is flat" or simply "The Earth is not flat".

(b) If p is the proposition "$2+2=4$", then $\neg p$ is the proposition "$2+2\neq 4$".

(ii) **Conjunction**: Given two propositions p and q, their conjunction is the proposition which is true when both p and q are true and false otherwise. The conjunction of p and q is denoted by $p \wedge q$ and is read as "p and q".

Truth table of Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

For example,

(a) If p is the proposition "It is cold" and q is the proposition "It is raining" then $p \wedge q$ is the proposition "It is cold and it is raining" or simply "It is cold and raining".

(b) If p is the proposition "$3+7=10$" and q is the proposition "2 is an odd number" then $p \wedge q$ is the proposition "$3+7=10$ and 2 is an odd number".

(iii) **Disjunction**: Given two propositions p and q, their disjunction is the proposition which is false when both p and q are false and true otherwise. The disjunction of p and q is denoted by $p \vee q$ and is read as "p or q".

Truth table of Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

For example,

(a) If p is the proposition "The Earth is flat" and q is the proposition "$2+2=4$" then $p \vee q$ is the proposition "The Earth is flat or $2+2=4$".

(b) If p is the proposition "Today is Friday" and q is the proposition "Tomorrow is holiday" then $p \vee q$ is the proposition "Today is Friday or tomorrow is holiday".

(iv) **Exclusive Or**: Given two propositions p and q, their exclusive or is the proposition which is true when p and q have opposite truth values and false otherwise. The exclusive or of p and q is denoted by $p \oplus q$ and is read as "p xor q".

Truth table of Exclusive Or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

(v) **Implication**: Given propositions p and q, their implication "$p \to q$" is the proposition that is false when p is true and q is false and true otherwise. In the implication "$p \to q$", the proposition p is called the hypothesis or antecedent and the proposition q is called the conclusion or consequent .

We read "$p \to q$" as "if p, then q", or "p implies q", or "p only if q", or "q if p" or "p is sufficient for q" or "q is necessary for p".

Truth table of Implication

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

For example,

(a) If p is the proposition "I am hungry" and q is the proposition "I will eat" then $p \to q$ is the proposition "If I am hungry then I will eat".

(b) If p is the proposition "$2 + 3 = 7$" and q is the proposition "The Earth is flat" then $p \to q$ is the proposition "If $2 + 3 = 7$ then the Earth is flat".

Note that the use of the implication "if p then q" in Mathematics is quite different from its use in everyday English. Normally, when "if p then q" is used in conversation, we expect some kind of cause-and-effect relationship to exist between the propositions p and q as in (a) above. But no such requirement is necessary in Mathematics as example (b) above shows. All we need for a valid "if p then q" proposition is that both p and q are propositions regardless of the existence or nonexistence of any kind of cause-and-effect relationship between them.

Converse, Inverse and Contrapositive of an implication:

If $p \to q$ is some implication, then we define the converse , inverse and contrapositive implications of $p \to q$ as follows:

$$\text{converse: } q \to p$$
$$\text{inverse: } \neg p \to \neg q$$
$$\text{contrapositive: } \neg q \to \neg p$$

For example, in the implication

$$p \to q\text{: If } 2+3=7 \text{ then the Earth is flat.}$$

we have,

$$q \to p\text{: If the Earth is flat then } 2+3=7.$$
$$\neg p \to \neg q\text{: If } 2+3 \neq 7 \text{ then the Earth is not flat.}$$
$$\neg q \to \neg p\text{: If the Earth is not flat then } 2+3 \neq 7.$$

Truth table of Implication, its Converse, Inverse and Contrapositive

p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

(vi) **Biconditional**: Given propositions p and q, their biconditional "$p \leftrightarrow q$" is the proposition that is true when both p and q have the same truth values and false otherwise.

We read "$p \leftrightarrow q$" as "p if and only if q" or "p iff q" or "p is necessary and sufficient for q".

Truth table of Biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

For example,

(a) If p is the proposition "I will eat" and q is the proposition "I am hungry" then $p \leftrightarrow q$ is the proposition "I will eat if and only if I am hungry."

(b) If p is the proposition "The alarm goes off" and q is the proposition "Smoke is detected" then $p \leftrightarrow q$ is the proposition "The alarm goes off if and only if smoke is detected."

Expressing statements in propositional logic:
Propositions which are written in sentential forms can be expressed in symbolic forms and conversely, propositions given in symbolic forms can be written in sentential forms by identifying the component propositions and the logical operators used. Some examples are given below. By convention, we denote the positive assertion of a sentence by a symbol.

Express the following sentences in logical or symbolic form:

1. I will not go to the movie and I will study a novel.

   Solution: Let p denote the proposition "I will go to the movie" and q denote the proposition "I will study a novel". Then the given proposition can be written symbolically as $\neg p \wedge q$.

2. If Santosh is not with his friends then Santosh is playing tennis.

   Solution: Let p denote the proposition "Santosh is with his friends" and q denote the proposition "Santosh is playing tennis". Then the given proposition can be written in the form $\neg p \to q$.

3. You can access the Internet from campus only if you are a computer science major or you are not a freshman.

   Solution: Let p denote the proposition "You can access the Internet from campus", q denote the proposition "You are a computer science major" and r denote the proposition "You are a freshman". Then the given proposition can be symbolically written in the form $p \to (q \vee \neg r)$.

4. Getting an A on the final exam and doing every assignment is sufficient for getting an A in this class.

   Solution: Let p denote the proposition "You get an A on the final exam", q denote the proposition "You do every assignment" and r denote the proposition "You get an A in this class". Then the given proposition can be written symbolically as $(p \wedge q) \to r$.

Below we translate the propositions written in symbolic form into English sentences.

1. Let p denote the proposition "You have fever", q denote the proposition "You miss the final examination" and r denote the proposition "You pass the course". Then the proposition $(p \to \neg r) \vee (q \to \neg r)$ translates into "If you have fever, then you will not pass the course or if you miss the final examination, then you will not pass the course."

1.1.2 Propositional Equivalences

Definition (Tautology):

A compound proposition which is always true regardless of the truth values of its component propositions is called a tautology .

For example, $p \vee \neg p$ and $\neg p \vee (p \vee q)$ are tautologies. We can verify this using the truth tables of these propositions as follows:

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

p	q	$\neg p$	$p \vee q$	$\neg p \vee (p \vee q)$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

Definition (Contradiction):

A compound proposition which is always false regardless of the truth values of its component propositions is called a contradiction .

For example, $p \wedge \neg p$ and $(p \wedge q) \wedge \neg p$ are contradictions as the following truth tables show.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

p	q	$\neg p$	$p \wedge q$	$(p \wedge q) \wedge \neg p$
T	T	F	T	F
T	F	F	F	F
F	T	T	F	F
F	F	T	F	F

Definition (Contingency):

A compound proposition that is neither a tautology nor a contradiction is called a contingency .

For example, $\neg p \vee \neg q$, $p \to \neg q$ etc. are contingencies as can be easily verified by forming their truth tables as above.

Definition (Logically Equivalent Propositions):

Two compound propositions P and Q are said to be logically equivalent if they have the same truth values for all the possible combination of truth values assigned to their components. This means that it is impossible for logically equivalent propositions to have different truth values and hence $P \leftrightarrow Q$ must always be true, that is, $P \leftrightarrow Q$ must be a tautology.

We use the notation "$P \equiv Q$" to denote the logical equivalence of P and Q. Note that $\equiv$ is not a logical operator that connects the propositions P and Q. It simply denotes a relationship between propositions P and Q which, in this case, is P and Q having the same truth values.

For example, $p \to q \equiv \neg q \to \neg p$ (an implication is logically equivalent to its contrapositive) and $q \to p \equiv \neg p \to \neg q$ (a converse of an implication is logically equivalent to its inverse) as shown in the previous section using the truth table. Below, we give some more examples:

$\underline{p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)}$:

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Since the columns for $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ are identical, so these two propositions are equivalent.

$\underline{p \to q \equiv \neg p \vee q}$:

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$\neg(\neg p) \equiv p$:

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

$\neg(p \wedge q) \equiv \neg p \vee \neg q$:

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Similarly $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$:

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Similarly $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

Standard logical equivalences: The most commonly used logical equivalents are listed below. Here the proposition T denotes a tautology and the proposition F denoted a contradiction.

1. Identity laws: $p \wedge T \equiv p$, $p \vee F \equiv p$
2. Domination laws: $p \vee T \equiv T$, $p \wedge F \equiv F$
3. Idempotent laws: $p \vee p \equiv p$, $p \wedge p \equiv p$
4. Double negation law: $\neg(\neg p) \equiv p$
5. Commutative laws: $p \vee q \equiv q \vee p$, $p \wedge q \equiv q \wedge p$
6. Associative laws: $p \vee (q \vee r) \equiv (p \vee q) \vee r$, $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
7. Distributive laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8. de Morgan's laws: $\neg(p \vee q) \equiv \neg p \wedge \neg q$, $\neg(p \wedge q) \equiv \neg p \vee \neg q$

9. Absorption laws: $p \vee (p \wedge q) \equiv p, p \wedge (p \vee q) \equiv p$
10. Negation laws: $p \vee \neg p \equiv T, p \wedge \neg p \equiv F$
11. $p \rightarrow q \equiv \neg q \rightarrow \neg p$
12. $p \rightarrow q \equiv \neg p \vee q$
13. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
14. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

We know that one method of showing logical equivalences between propositions is by the construction of truth tables as above. However, this method is extremely inefficient even when it involves moderately large number of propositions. When only three propositions p, q, r are involved, the truth table will have 8 rows. If four propositions are involved then 16 rows are required and in general 2^n rows are required in the table when n propositions are involved. So it is desirable to have a better method for establishing the logical equivalence of propositions which overcomes this inefficiency. One such method is the use of standard logical equivalents listed above. We can establish logical equivalence by developing a series of logical equivalences as shown below. The main idea behind this is that the logical equivalence is a transitive relation i.e. if $P \equiv Q$ and $Q \equiv R$ then $P \equiv R$ as well.

1. Show that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$.

Solution:

$$
\begin{array}{lll}
 & L.H.S & \\
\equiv & \neg(p \vee (\neg p \wedge q)) & \\
\equiv & \neg p \wedge \neg(\neg p \wedge q) & \{\text{by de Morgan's law}\} \\
\equiv & \neg p \wedge (\neg(\neg p) \vee \neg q) & \{\text{by de Morgan's law}\} \\
\equiv & \neg p \wedge (p \vee \neg q) & \{\text{by double negation law}\} \\
\equiv & (\neg p \wedge p) \vee (\neg p \wedge \neg q) & \{\text{by distributive law}\} \\
\equiv & F \vee (\neg p \wedge \neg q) & \{\text{by negation law}\} \\
\equiv & \neg p \wedge \neg q & \{\text{by identity law}\} \\
\equiv & R.H.S &
\end{array}
$$

2. Show that $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \equiv r$.

Solution:

$$
\begin{array}{lll}
 & L.H.S & \\
\equiv & (\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) & \\
\equiv & (\neg p \wedge (\neg q \wedge r)) \vee ((q \vee p) \wedge r) & \{\text{by distributive law}\} \\
\equiv & ((\neg p \wedge \neg q) \wedge r) \vee ((q \vee p) \wedge r) & \{\text{by associative law}\} \\
\equiv & (\neg(p \vee q) \wedge r) \vee ((q \vee p) \wedge r) & \{\text{by de Morgan's law}\} \\
\equiv & (\neg(p \vee q) \wedge r) \vee ((p \vee q) \wedge r) & \{\text{by commutative law}\} \\
\equiv & (\neg(p \vee q) \vee (p \vee q)) \wedge r & \{\text{by distributive law}\} \\
\equiv & T \wedge r & \{\text{by negation law}\} \\
\equiv & r & \{\text{ by identity law}\} \\
\equiv & R.H.S &
\end{array}
$$

3. Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

We need to show that $(p \wedge q) \rightarrow (p \vee q) \equiv T$.

$$\begin{aligned}
& L.H.S \\
\equiv\ & (p \wedge q) \rightarrow (p \vee q) \\
\equiv\ & \neg(p \wedge q) \vee (p \vee q) && \{\text{since } p \rightarrow q \equiv \neg p \vee q\} \\
\equiv\ & (\neg p \vee \neg q) \vee (p \vee q) && \{\text{by de Morgan's law}\} \\
\equiv\ & (\neg p \vee p) \vee (\neg q \vee q) && \{\text{by associative and commutative laws}\} \\
\equiv\ & T \vee T && \{\text{by negation law}\} \\
\equiv\ & T && \{\text{by domination law}\} \\
\equiv\ & R.H.S
\end{aligned}$$

Definition (Dual of a Compound Proposition):

The dual of a compound proposition P that contains only the logical operators $\vee$, $\wedge$, and $\neg$ is the proposition obtained by replacing each $\vee$ by $\wedge$, $\wedge$ by $\vee$, T by F and F by T. The dual of any compound proposition P is denoted by P^*. For example,

1. The dual of the compound proposition $p \wedge \neg q \wedge \neg r$ is $p \vee \neg q \vee \neg r$.
2. The dual of the compound proposition $(p \vee F) \wedge (q \vee T)$ is $(p \wedge T) \vee (q \wedge F)$.

It can be proven that if P and Q are logically equivalent compound propositions formed by the use of logical operators $\vee, \wedge$ and $\neg$, then their dual propositions P^* and Q^* must also be logically equivalent. A look at the first ten standard logical equivalents above (except the double negation law) will support this fact.

1.1.3 Predicates and Quantifiers

Consider a sentence such as "x is a positive number". As we mentioned before, sentences of this kind are not propositions because it does not have any truth value. Its truthness or falseness depends upon the value of x which is not fixed yet. However, we can study the structure of sentences of this kind. This sentence in particular consists of two parts, the **subject** "x" and the property "is a positive number" that the subject x can have. Such kind of property is known as the **predicate** . We can denote the sentence "x is a positive number" by $P(x)$ where P denotes the predicate "is a positive number" and x is the variable denoting the subject. This expression $P(x)$ is known as a **propositional function** . A propositional function $P(x)$ becomes a proposition when we assign some value to the variable x. For example, when $x = 2$, then $P(x)$ becomes $P(2)$ which is the proposition "2 is a positive number" with truth value 'T'.

A propositional function may have more than one variable as well. For example, we can denote the sentence "x is greater that y" by $Q(x, y)$ which is a propositional function containing two subjects x and y related by the predicate "is greater than". Again, as in one-variable case, we can assign some values to the variables x and y to change $Q(x, y)$ into a proposition. In this example, if $x = 2$ and $y = 3$, then $Q(x, y)$ changes into $Q(2, 3)$ which is a (false) proposition.

Quantification:

Apart from assigning some value to the variable x, there are other ways of changing a propositional function into a proposition. These are called **quantification** methods which are of two types: universal quantification and existential quantification.

1. **Universal Quantification**: Universal quantification of a propositional function $P(x)$ is the proposition "For all values of x, $P(x)$" or "For every value of x, $P(x)$" which is a proposition that is true when $P(x)$ is true for every x and false when $P(x)$ is false for at least one x. Universally quantified propositions are written symbolically as $\forall x P(x)$. Here the symbol $\forall$ is known as the **universal quantifier** and is read as "for all" or "for every".

   For example,

   a. If $P(x)$ is the propositional function "x is greater than 3", then its universal quantification is the proposition $\forall x P(x)$ which is "For all values of x, x is greater than 3" which is a false proposition.

   b. If $P(x)$ is the propositional function $x^2 \geq 0$, then its universal quantification is $\forall x P(x)$ which is "For all values of x, $x^2 \geq 0$" which is a true proposition.

2. **Existential Quantification**: Existential quantification of a propositional function $P(x)$ is the proposition "There exists a value of x such that $P(x)$" which is a proposition that is true if there is at least one x for which $P(x)$ is true and false if $P(x)$ is false for all x. Existentially quantified propositions are written symbolically as $\exists x P(x)$. Here the symbol $\exists$ is knowns as the **existential quantifier** and is read as "there exists".

   For example,

   c. If $P(x)$ is the propositional function "x is greater than 3", then its existential quantification is the proposition $\exists x P(x)$ read as "There exists a value of x such that x is greater than 3" which is a true proposition.

   d. If $P(x)$ is the propositional function $x^2 + 1 = 0$, then its existential quantification is the proposition $\exists x P(x)$ read as "There exists a value of x such that $x^2 + 1 = 0$" which is a false proposition.

Universe of Discourse:

When a proposition is made from a propositional function by means of quantification, its truth value depends not only upon the type of quantification used but also upon the allowed values that the variable or subject can take. The collection of all such values that a variable is allowed to take is called its **universe of discourse** or **domain of discourse** .
For example, if $P(x)$ is "x is less than 5", then for the universe of discourses

$$U_1 = \{-1, 0, 1, 2, 4\},$$

$$U_2 = \{3, -2, 7, 8\},$$
$$U_3 = \{15, 20, 24\}$$

the proposition $\forall x P(x)$ is true for U_1 but false for U_2 and U_3. Similarly, the proposition $\exists x P(x)$ is true for U_1 and U_2 but false for U_3. Therefore whenever quantifiers are used, we must specify along with it the universe of discourse of the variables, to determine the truth values of the quantified propositions. So in examples a. to d. above, we can take the universe of discourse of x to be the set of all real numbers. Note that, had we taken the universe of discourse of x as the set of all complex numbers, then proposition b. would have been false because there exists a complex number i such that $i^2 = -1 < 0$ and proposition d. would have been true because again there exists a complex number i such that $i^2 + 1 = -1 + 1 = 0$.

Definition (Free and Bound Variables):

When a quantifier is used on a variable within its scope (see definition below) or when we assign a value to this variable, then we say that this occurrence of the variable is bound . An occurrence of the variable that is not bound by a quantifier or not set equal to a particular value is said to be free .

For example, in $\exists x Q(x, y)$, x is a bound variable and y is a free variable whereas in $\forall x \exists y Q(x, y)$, both x and y are bound variables.

All the variables that occur in a propositional function must be bound to turn it into a proposition. This can be done using a combination of universal quantifiers, existential quantifiers and value assignments.

Definition (Scope of Quantifier):

The scope of a quantifier occurring in a logical expression is that part of logical expression to which that quantifier is applied. Usually, the scope of a quantifier is the smallest logical expression that follows immediately after the quantifier.

For example, in the expression $\exists x P(x) \vee Q(x)$, the scope of the quantifier $\exists x$ is limited to $P(x)$. So the variable x in $P(x)$ is a bound variable whereas the x in $Q(x)$ is a free variable. We can use parentheses within a logical expression to change the scope of a quantifier. Had we intended to bound x occurring in both $P(x)$ and $Q(x)$ by the same quantifier $\exists$, then the logical expression would have been $\exists x(P(x) \vee Q(x))$. Similarly, in the expression $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$, the scope of the quantifier $\exists x$ is limited to $P(x) \wedge Q(x)$ and the scope of the quantifier $\forall x$ is limited to $R(x)$. So every x is within the scope of some quantifiers and therefore all x are bound variables in this case.

Nested Quantifiers: One can make a proposition out of a propositional function containing more than one subjects using quantifiers to bind the variables. When more than one quantifiers are used such that one is within the scope of another, then they are called nested quantifiers . For example, in $\forall x \exists y(x > y)$, the quantifiers $\forall x$ and $\exists y$ are nested.

When nested quantifiers are used, the order in which the quantifiers are used is very important. For example, in $\forall x \forall y Q(x, y)$, the order of the quantifiers can be changed to make it $\forall y \forall x Q(x, y)$ without changing the meaning of the proposition i.e. $\forall x \forall y Q(x, y) \equiv \forall y \forall x Q(x, y)$. Similarly, $\exists x \exists y Q(x, y) \equiv \exists y \exists x Q(x, y)$. However, $\forall x \exists y Q(x, y)$ and $\exists y \forall x Q(x, y)$ are not the same propositions. For example, consider the propositions $\forall x \exists y(x > y)$ and $\exists y \forall x(x > y)$.

The first proposition says that for all values of x, there exists a value of y (which may depend upon x) such that x is greater than y. This is a true proposition. However, the second proposition claims that there exists a value of y such that for all value of x, x is greater than y i.e., it claims the existence of a number y which is smaller than any other number. This is clearly false.

Negation of quantified propositions:

Negation of proposition (quantified or not) is a proposition that has just the opposite truth value. Consider the quantified proposition $\forall x P(x)$. Its negation is the proposition "It is not the case that $\forall x P(x)$" or equivalently "There exists a value of x such that $\neg P(x)$" which is the quantified proposition $\exists x \neg P(x)$. For example, taking the universe of discourse of x as the set of all real numbers, if $P(x)$ is the proposition $\forall x(x > 3)$, then its negation is the proposition $\exists x \neg(x > 3)$ or equivalently, $\exists x(x \leq 3)$. It is clear that $\forall x(x > 3)$ is a false proposition and so its negation is true.
Similarly, the negation of a quantified proposition $\exists x P(x)$ is the proposition " It is not the case that $\exists x P(x)$" or equivalently "For all values of x, $\neg P(x)$". For example, if $P(x)$ is the proposition $\exists x(x^2 = 2)$ then its negation is the proposition $\forall x \neg(x^2 = 2)$ or equivalently, $\forall x(x^2 \neq 2)$. Since $\exists x(x^2 = 2)$ is a true proposition, so its negation must be false.

Therefore

$$\neg \forall x P(x) \equiv \exists x \neg P(x) \text{ e.g. } \neg \forall x(x > 3) \equiv \exists x(x \leq 3)$$
$$\neg \exists x P(x) \equiv \forall x \neg P(x) \text{ e.g. } \neg \exists x(x^2 = 2) \equiv \forall x(x^2 \neq 2)$$

Similar rules of negation exists for propositions with nested quantifiers:

$$\neg \forall x \forall y Q(x, y) \equiv \exists x \exists y \neg Q(x, y) \text{ e.g. } \neg \forall x \forall y(x > y) \equiv \exists x \exists y(x \leq y)$$
$$\neg \forall x \exists y Q(x, y) \equiv \exists x \forall y \neg Q(x, y) \text{ e.g. } \neg \forall x \exists y(x > y) \equiv \exists x \forall y(x \leq y)$$
$$\neg \exists x \forall y Q(x, y) \equiv \forall x \exists y \neg Q(x, y) \text{ e.g. } \neg \exists x \forall y(x > y) \equiv \forall x \exists y(x \leq y)$$
$$\neg \exists x \exists y Q(x, y) \equiv \forall x \forall y \neg Q(x, y) \text{ e.g. } \neg \exists x \exists y(x > y) \equiv \forall x \forall y(x \leq y)$$

Translating quantified propositions into logical expressions:

Translate the following sentences into logical form:

1. Every student in this class has studied calculus.

   Solution: Let $C(x)$ denote the propositional function "x has studied calculus." and let the universe of discourse for x consist of all the students in the class. Then the given sentence can be written in the form

   > "For every student x in this class, x has studied calculus".

   Symbolically, we can write $\forall x C(x)$.

   Alternate solution: Let $T(x)$ denote the propositional function "x is a student of this class." and $C(x)$ denote the propositional function "x has studied calculus."and let the universe of discourse for x consist of all the human beings. Then the given sentence can be written in the form

"For every x, if x is a student of this class then x has studied calculus".

Symbolically, we can write $\forall x(T(x) \rightarrow C(x))$.

2. No one is perfect.

Solution: Let $P(x)$ denote the propositional function "x is perfect." and let the universe of discourse for x consist of all the human beings in this world. Then the given sentence can be written in the following form:

"For every x, x is not perfect."

Symbolically, we can write $\forall x \neg P(x)$.

3. Some student in this class has a dog but not a cat.

Solution: Let $D(x)$ and $C(x)$ denote the propositional functions "x has a dog." and "x has a cat." respectively and let the universe of discourse for x consist of all the students in the class. Then the given sentence can be written in the following form:

"There exists a student x in this class such that x has a dog and x does not have a cat."

Symbolically, we can write $\exists x(D(x) \wedge \neg C(x))$.

4. Not everyone is perfect.

Solution: Let $P(x)$ denote the propositional function "x is perfect." and let the universe of discourse for x consist of all the human beings in this world. Then the given sentence can be written in the following form:

"There exists x such that x is not perfect."

Symbolically, we can write $\exists x \neg P(x)$.

5. If a person is female and is a parent, then this person is someone's mother.

Solution: Let

$$F(x) : x \text{ is a female.}$$

$$P(x) : x \text{ is a parent.}$$

and

$$M(x, y) : x \text{ is a mother of } y.$$

Then the given sentence can be written as

"For all x, if x is a female and x is a parent, then there exists some y such that x is a mother of y."

where the universe of discourse of x and y consists of all the people in the world. Symbolically, this can be written as

$$\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y)).$$

6. Everyone has exactly one best friend.

Solution: Let

$$B(x, y) : y \text{ is a best friend of } x.$$

and the universe of discourse consist of all the people in this world. Then the given sentence can be written as

> "For all x, there exists a y such that y is a best friend of x and for all z, if z is not y, then z is not a best friend of x."

Symbolically, this can be written as

$$\forall x \exists y (B(x, y) \wedge \forall z ((z \neq y) \to \neg B(x, z))).$$

7. There is a woman who has taken a flight on every airline in the world.

Solution: Let

$$P(w, f) : w \text{ has taken a flight } f.$$

$$Q(f, a) : f \text{ is a flight on airlines } a.$$

and the universe of discourse of w be the set of all the women, for f all the flights and for a, all the airlines in the world. Then the given sentence can be written as

> "There is a woman w such that for every airlines a, there exists a flight f such that f is a flight on airlines a and woman w has taken on flight f."

Symbolically,

$$\exists w \forall a \exists f (Q(f, a) \wedge P(w, f)).$$

8. Let

$$F(x, y) : x \text{ can fool } y.$$

Then translate following into symbolic form.

a. Everybody can fool Fred.

Solution: For all x, x can fool Fred. Symbolically, $\forall x F(x, \text{Fred})$.

b. Evelyn can fool everyone.

Solution: For all y, Evelyn can fool y. Symbolically, $\forall y F(\text{Evelyn}, y)$.

c. Everybody can fool somebody.

Solution: For all x, there exists y such that x can fool y. Symbolically, $\forall x \exists y F(x, y)$.

d. There is no one who can fool everybody.

Solution: For all x, there exists y such that x cannot fool y. Symbolically, $\forall x \exists y \neg F(x, y)$.

e. Everyone can be fooled by somebody.

Solution: For all y, there exists x such that x can fool y. Symbolically, $\forall y \exists x F(x, y)$.

f. No one can fool both Fred and Jerry.

Solution: For all x, x cannot fool Fred and x cannot fool Jerry. Symbolically, $\forall x (\neg F(x, \text{Fred}) \wedge \neg F(x, \text{Jerry}))$.

g. Nancy can fool exactly two people.
Solution: $\exists x \exists y (x \neq y \wedge F(\text{Nancy}, x) \wedge F(\text{Nancy}, y) \wedge \forall z (((z \neq x) \wedge (z \neq y)) \rightarrow \neg F(\text{Nancy}, z)))$.

h. There is exactly one person whom everybody can fool.
Solution: $\exists y \forall x (F(x, y) \wedge \forall z ((z \neq y) \rightarrow \exists w \neg F(w, z)))$.

i. No one can fool himself or herself.
Solution: $\forall x \neg F(x, x)$.

j. There is someone who can fool exactly one person besides himself or herself.
Solution: $\exists x \exists y ((x \neq y) \wedge F(x, y) \wedge \forall z ((x \neq z) \wedge (y \neq z)) \rightarrow \neg F(x, z))$.

Translate from symbolic form to language form:

9. Let $H(x)$:= x is a man. and $M(x)$:= x is mortal. Then translate $\forall x (H(x) \rightarrow M(x))$ into language form.

Solution: We can write the above logical expression in the language form as

"For all x, if x is a man then x is mortal."

or simply as

"All men are mortal."

1.1.4 Rules of Inference

Definition (Argument):

An argument is a sequence of two or more propositions where the last proposition is called a conclusion and the set of proposition before that is called premises. The conclusion of an argument usually begins with the word 'Therefore' and is separated from the set of premises by a horizontal line. For example,

$$\begin{array}{c} \text{It is hot or it is snowing.} \\ \text{It is not hot.} \\ \hline \therefore \text{ It is snowing.} \end{array}$$

is an argument where the first two propositions are the premises and the last proposition is the conclusion.

Definition (Valid Argument):

An argument is said to be valid if we can derive the truthness of the conclusion based upon the truthness of the set of premises. The premises (or the conclusions as well) in an argument may actually be true or not but this makes no difference for it to be valid. Valid arguments are used based upon the *assumption* that the premises are true, not upon whether the premises are actually true or not.

For example, the argument given in the above example is valid. If we assume that the first premise is true then at least one of the propositions “It is hot.” or “It is snowing.” must be true. If we assume that the second premise is also true, then “It is hot.” must be false and therefore “It is snowing.” must be true as desired. So this is a valid argument. Whether “It is hot” or “It is snowing” are actually true or not makes no difference in the validity of this argument.

Definition (Argument Form and Valid Argument Form):

An argument form is an argument in which proposition variables $p, q, r, \cdots$ etc are used in place of actual propositions. An argument form is said to be valid if the argument resulting from the replacement of proposition variables by any set of particular propositions results in a valid argument.

For example,

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

is an argument form. It is in fact a valid argument form.

How to show validness of an argument form?
Suppose that an argument form is given involving a set of premises $p_1, p_2, \cdots, p_n$ and a conclusion q. To show the validity of this argument form is to show that the truthness of all $p_1, \cdots, p_n$ implies the truthness of q. But truthness of all propositions $p_1, \cdots, p_n$ is equivalent to their conjunction $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ being true and therefore we need to show the truthness of q assuming the truthness of $p_1 \wedge p_2 \wedge \cdots \wedge p_n$. This can be done if it is shown that it is impossible for $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ to be true and q to be false i.e., the implication $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$ can never be false and therefore $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$ is a tautology.

For example, the validness of the argument form in previous example can be proved by showing that the implication $[(p \vee q) \wedge \neg q] \to q$ is a tautology which is easily done using a truth table.

Examples:

1. Determine whether $\neg r$ is a valid conclusion from the premises $\neg p$, $p \to q$ and $q \to r$.

   Solution: To determine whether $\neg r$ is a valid conclusion from the premises $\neg p$, $p \to q$ and $q \to r$ or not, we check whether $[\neg p \wedge (p \to q) \wedge (q \to r)] \to \neg r$ is a tautology or not using truth table as follows where we denote by A the proposition $\neg p \wedge (p \to q) \wedge (q \to r)$:

p	q	r	$\neg p$	$\neg r$	$p \to q$	$q \to r$	A	$A \to \neg r$
T	T	T	F	F	T	T	F	T
T	T	F	F	T	T	F	F	T
T	F	T	F	F	F	T	F	T
T	F	F	F	T	F	T	F	T
F	T	T	T	F	T	T	T	F
F	T	F	T	T	T	F	F	T
F	F	T	T	F	T	T	T	F
F	F	F	T	T	T	T	T	T

From the above truth table, we can see that $[\neg p \wedge (p \to q) \wedge (q \to r)] \to \neg r$ is not a tautology. Hence $\neg r$ is not a valid conclusion from the premises $\neg p$, $p \to q$ and $q \to r$.

Rules of Inference for Propositional Logic

As explained above, checking the validity of an argument form is just a matter of checking whether the implication formed by taking as hypothesis the conjunction of the set of all the premises and as conclusion the conclusion of the argument form, is a tautology or not. And this can be done easily by forming the truth table of that implication as shown in the examples above. However, this method becomes very inefficient when the number of different propositions involved in the argument form is even moderately large. So it is desirable to have a more efficient way to prove validity of argument forms. One such method is to first establish the validity of relatively simple argument forms, called the **rules of inference** and then use these rules to prove the validity of more complicated argument forms.

There are basically eight commonly used rules of inference when only simple unquantified propositions are involved.

1. **Addition:** If the premise p is assumed to be true, then the addition rule of inference infers that the conclusion $p \vee q$ must also be true for any proposition q. This follows from the tautology of the implication $p \to (p \vee q)$ and is written in the following way:

$$\frac{p}{\therefore p \vee q}$$

   For example, if “It is raining” is true, then “It is raining or it is snowing” is true as well.

2. **Simplification:** If the premise $p \wedge q$ is assumed to be true, then the simplification rule infers that the conclusion p must also be true. This follows from the tautology of the implication $(p \wedge q) \to p$ and is written as

$$\frac{p \wedge q}{\therefore p}$$

   For example, if “It is raining and it is snowing” is true then “It is raining” must be true.

3. **Conjunction:** If p and q are two true premises, then the conjunction rule says that the conclusion $p \wedge q$ must also be true. This follows from the tautology $((p) \wedge (q)) \to (p \wedge q)$. This is written as

$$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

   For example, if “It is raining” and “It is snowing” are both true, then “It is raining and it is snowing” is also true.

4. **Modus Ponens:** If the premises p and $p \to q$ are assumed to be true, then the rule of modus ponens infers that the conclusion q must also be true. This follows from the tautology of the implication $[p \wedge (p \to q)] \to q$. This rule is written as

$$\begin{array}{c} p \\ p \to q \\ \hline \therefore q \end{array}$$

For example, suppose that the premises "It is hot" and "If it is hot, then we will go swimming" are both true. Then the conclusion "We will go swimming" will also be true.

5. **Modus Tollens:** If a proposition q is false i.e., its negation $\neg q$ is true and the implication $p \to q$ is true, then modus tollens rule says that p must be false as well i.e., $\neg p$ is true. This follows from the tautology of the implication $[\neg q \wedge (p \to q)] \to \neg p$ and is written as

$$\begin{array}{c} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$$

For example, suppose that the proposition "We will go swimming" is false and the implication "If it is hot, then we will go swimming" is true. Then "It is hot" must be false.

6. **Hypothetical Syllogism:** If the implications $p \to q$ and $q \to r$ are both true premises, then the hypothetical syllogism rule gives us that $p \to r$ is a true conclusion as well. This follows from the tautology of the implication $[(p \to q) \wedge (q \to r)] \to (p \to r)$ and is written as

$$\begin{array}{c} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

For example, if the implications "If it is hot, then we will go swimming" and "If we go swimming, then we will get tired" are both true, then the implication "If it is hot, then we will get tired" is true as well.

7. **Disjunctive Syllogism:** If the compound proposition $p \vee q$ and the proposition $\neg p$ are both true as premises, then the disjunctive syllogism rule gives that q is a true conclusion as well. This follows from the tautology of the implication $[(p \vee q) \wedge \neg p] \to q$ and is written as

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

For example, if "It is raining or it is hot" and "It is not raining" are true, then "It is hot" must be true.

8. **Resolution:** If the compound proposition $p \vee q$ and $\neg p \vee r$ are both true premises, then the rule of resolution says that the conclusion $q \vee r$ must be true. This follows from the tautology of the implication $[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$ and is written as

$$\begin{array}{c} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

The final disjunction in the resolution rule, $q \vee r$ is called the resolvent.

For example, if "It is hot or it is raining" and "It is not hot or the sun is shining" are true, then "It is raining or the sun is shining" must be true.

Examples of the application of rules of inference:

Below, we use the above eight rules of inference to draw valid conclusions from a set of premises. Note that we can use the logical equivalents at any step in our derivations because truth of a proposition clearly implies the truth of its logically equivalent proposition.

1. Show that $\neg q$ is a valid conclusion from the premises $q \to r$, $r \to s$, and $\neg s$.

Proof:

	Step	Reason
1.	$\neg s$	Hypothesis
2.	$r \to s$	Hypothesis
3.	$\neg r$	Modus tollens using steps 1 and 2
4.	$q \to r$	Hypothesis
5.	$\neg q$	Modus tollens using steps 3 and 4

2. Show that s is a valid conclusion from the premises $p \to (r \to s)$, $\neg r \to \neg p$ and p.

Proof:

	Step	Reason
1.	p	Hypothesis
2.	$\neg r \to \neg p$	Hypothesis
3.	$p \to r$	Since $p \to q \equiv \neg q \to \neg p$
4.	r	Modus ponens using steps 1 and 3
5.	$p \to (r \to s)$	Hypothesis
6.	$r \to s$	Modus ponens using steps 1 and 5
7.	s	Modus ponens using steps 4 and 6

3. Construct an argument using rules of inference to show that the following hypotheses

> If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on.
> If the sailing race is held, then the trophy will be awarded.
> The trophy was not awarded.

imply the conclusion

It rained.

Solution: Let,

p:= It rained.
q:= It is foggy.
r:= Sailing race will be held.
s:= Lifesaving demonstration will go on.
t:= Trophy will be awarded.

Then we need to show that p is a valid conclusion from the hypotheses $(\neg p \vee \neg q) \rightarrow (r \wedge s)$, $r \rightarrow t$ and $\neg t$.

	Step	Reason
1.	$r \rightarrow t$	Hypothesis
2.	$\neg t$	Hypothesis
3.	$\neg r$	Modus tollens using steps 1 and 2
4.	$\neg r \vee \neg s$	Addition using step 3
5.	$\neg(r \wedge s)$	Using de Morgan's rule in step 4
6.	$(\neg p \vee \neg q) \rightarrow (r \wedge s)$	Hypothesis
7.	$\neg(\neg p \vee \neg q)$	Modus tollens using steps 5 and 6
8.	$p \wedge q$	Using de Morgan's rule and double negation in step 7
9.	p	Simplification using step 8.

4. Show by using resolution law that $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

Proof:

	Step	Reason
1.	$(p \wedge q) \vee r$	Hypothesis
2.	$(p \vee r) \wedge (q \vee r)$	Using distributive law in step 1
3.	$p \vee r$	Simplification using step 2
4.	$r \vee p$	Commutative law in step 3
5.	$r \rightarrow s$	Hypothesis
6.	$\neg r \vee s$	From step 5
7.	$p \vee s$	Resolution using steps 4 and 6

Definition (Fallacy):

An invalid argument is known as a fallacy . So in a fallacy, the truth of all the premises does not necessarily imply the truth of the conclusion i.e., the implication formed by the premises and conclusion is not a tautology but a contingency. The conclusion may be true in some cases and false in others. Some fallacies look so convincing as a valid argument that it is easy to be deceived. They are as follows:

Types of fallacies:

1. **Fallacy of affirming the conclusion:** If we derive that p is true when both $p \to q$ and q are true, then such type of reasoning is called the fallacy of affirming the conclusion . Such type of reasoning is incorrect because $[(p \to q) \wedge q] \to p$ is a contingency, not a tautology.

   For example, following is an incorrect argument based upon the fallacy of affirming the conclusion:

   If the Buddha was born in Kathmandu, then the Buddha was born in Nepal.
   The Buddha was born in Nepal.
   Therefore, the Buddha was born in Kathmandu.

2. **Fallacy of denying the hypothesis:** If we derive that $\neg q$ is true when both $p \to q$ and $\neg p$ are true, then such type of reasoning is called the fallacy of denying the hypothesis . Such type of reasoning is incorrect because $[(p \to q) \wedge \neg p] \to \neg q$ is a contingency, not a tautology.

   For example, following is a faulty argument based upon the fallacy of denying the hypothesis:

   If the Buddha was born in Kathmandu, then the Buddha was born in Nepal.
   The Buddha was not born in Kathmandu.
   Therefore, the Buddha was not born in Nepal.

Rules of Inference for Quantified Statements

When we are dealing with quantified propositions, the previous eight rules of inference together with the following additional four rules of inference are also used.

1. **Universal instantiation**: If the quantified proposition $\forall x P(x)$ is true as a premise, then the rule of universal instantiation says that $P(c)$ is a true conclusion where c is an arbitrarily selected member of the universe of discourse for x. The validity of this rule is clear from the meaning of the proposition $\forall x P(x)$ itself. We can express this rule symbolically as

$$\frac{\forall x P(x)}{\therefore P(c) \text{ for any arbitrary } c}$$

   For example, if we suppose that "All men are mortal" is true, then we can conclude that "Newton is mortal" is also true where Newton is an arbitrarily selected member of the universe of discourse of all men.

2. **Universal generalization**: If a proposition $P(c)$ is true for any member c of some set D, then the rule of universal generalization says that the universally quantified proposition $\forall x P(x)$ is true where the universe of discourse of x can be taken as the set D. Therefore, to show that $\forall x P(x)$ is true, it is sufficient to show that $P(c)$ is true for an arbitrary element c of the universe of discourse of x. We express this rule symbolically as

$$\frac{P(c) \text{ for any arbitrary } c}{\therefore \forall x P(x)}$$

For example, $a^2 \geq 0$ for any real number a. Therefore, we can say that $\forall x(x^2 \geq 0)$ is true where the universe of discourse of x is the set of all real numbers.

3. **Existential instantiation**: If the quantified proposition $\exists x P(x)$ is true as a premise, then the rule of existential instantiation says that $P(c)$ must be true for some member c in the universe of discourse of x. Again the validity of this rule is obvious from the meaning of the quantified proposition $\exists x P(x)$ itself. We can express this rule symbolically as

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

For example, $\exists x(x^2 < 5)$ is true where the universe of discourse of the variable x is the set of all integers $\mathbb{Z}$. Therefore $c^2 < 5$ must be true for some integer c, in particular for integers $-2, -1, 0, 1, 2$.

4. **Existential generalization**: If $P(c)$ is true for some element c, then the rule of existential generalization says that the existentially quantified proposition $\exists x P(x)$ is also true where the universe of discourse of x is any set containing the element c. We can express this rule symbolically as

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

For example, there is an integer $c = 2$ such that $2c = 2^c$. Hence $\exists x(2x = 2^x)$ is true where the universe of discourse of x can be taken as the set of all integers.

Examples of the application of rules of inference for quantified propositions:

1. Show that $M(s)$ is a valid conclusion from $\forall x(H(x) \to M(x))$ and $H(s)$.

Proof:

	Step	Reason
1.	$\forall x(H(x) \to M(x))$	Premise
2.	$H(s)$	Premise
3.	$H(s) \to M(s)$	Universal instantiation from 1
4.	$M(s)$	Modus ponens using steps 2 and 3

2. Show that $\forall x(P(x) \to R(x))$ is a valid conclusion from $\forall x(P(x) \to Q(x))$ and $\forall x(Q(x) \to R(x))$.

Proof:

	Step	Reason
1.	$\forall x(P(x) \to Q(x))$	Premise
2.	$P(c) \to Q(c)$	Universal instantiation from 1; c arbitrary
3.	$\forall x(Q(x) \to R(x))$	Premise
4.	$Q(c) \to R(c)$	Universal instantiation from 3; c arbitrary
5.	$P(c) \to R(c)$	Hypothetical syllogism using 2 and 4; c arbitrary
6.	$\forall x(P(x) \to R(x))$	Universal generalization from 5

3. Show that $\exists x P(x) \land \exists x Q(x)$ is a valid conclusion from $\exists x(P(x) \land Q(x))$.

Proof:

	Step	Reason
1.	$\exists x(P(x) \land Q(x))$	Premise
2.	$P(c) \land Q(c)$	Existential instantiation from 1; c fixed
3.	$P(c)$	Simplification using 2
4.	$Q(c)$	Simplification using 2
5.	$\exists x P(x)$	Existential generalization from 3
6.	$\exists x Q(x)$	Existential generalization from 4
7.	$\exists x P(x) \land \exists x Q(x)$	Conjunction using steps 5 and 6

4. Show that $\exists x(P(x) \land \neg B(x))$ is a valid conclusion from $\exists x(C(x) \land \neg B(x))$ and $\forall x(C(x) \to P(x))$.

Proof:

	Step	Reason
1.	$\exists x(C(x) \land \neg B(x))$	Premise
2.	$C(a) \land \neg B(a)$	Existential instantiation from 1; a fixed
3.	$C(a)$	Simplification using 2
4.	$\forall x(C(x) \to P(x))$	Premise
5.	$C(a) \to P(a)$	Universal instantiation from 4
6.	$P(a)$	Modus ponens using 3 and 5
7.	$\neg B(a)$	Simplification using 2
8.	$P(a) \land \neg B(a)$	Conjunction using 6 and 7
9.	$\exists x(P(x) \land \neg B(x))$	Existential generalization from 8

1.2 Proof Methods

A theorem is any proposition that can be proven to be true. A proposition that can be proved as an immediate consequence of some another theorem is called a corollary of that theorem. Most of the theorems can be stated in the following form:

$\forall x(P(x) \to Q(x))$: For all x, if $P(x)$ then $Q(x)$.

For example, consider the famous trigonometrical identity $\sin^2\theta + \cos^2\theta = 1$. This can be stated as

"For all θ, if θ is a real number then $\sin^2\theta + \cos^2\theta = 1$."

Similarly, the equally famous Pythagorean Theorem can be stated as

"For all real numbers a, b and c, if a is the length of the base, b the length of the perpendicular and c the length of the hypotenuse of some right-angled triangle, then $a^2 + b^2 = c^2$."

Therefore proving a theorem of this type is to show that the proposition $\forall x(P(x) \to Q(x))$ is true. For this, we take an arbitrary element c and prove that $P(c) \to Q(c)$ is true. So by proving a theorem, we are essentially showing the truthness of a proposition of the kind $p \to q$. We now study some methods by which this can be done.

1.2.1 Methods of Proof

Vacuous proof: In the implication $p \to q$, if we can show that the hypothesis p is false, then regardless of the truth value of q, the implication $p \to q$ will be true. Such kind of proof of an implication $p \to q$ is called a vacuous proof .

Example: Prove that an empty set is a subset of every set A.

Proof: We have to show that $\emptyset \subseteq A$ for any set A. For this, we have to show that if $x \in \emptyset$ then $x \in A$. Since $\emptyset$ is an empty set, so there are no elements in $\emptyset$, i.e., the hypothesis $x \in \emptyset$ is false. So the implication "if $x \in \emptyset$, then $x \in A$" is true, i.e., $\emptyset \subseteq A$. $\square$

Trivial proof: In the implication $p \to q$, if we can show that the consequence q is true, then regardless of the truth value of p, the implication $p \to q$ is true. Such kind of proof of an implication $p \to q$ is called a trivial proof .

Example: Prove that if a is a rational number, then $(3 - \log_{10} 1000)a = 0$.

Proof: Since $\log_{10} 1000 = 3$, so $3 - \log_{10} 1000 = 0$ and hence $(3 - \log_{10} 1000)a = 0$ for any real number a i.e., the consequence $(3 - \log_{10} 1000)a = 0$ is true. So the implication "if a is a rational number, then $(3 - \log_{10} 1000)a = 0$" is true. $\square$

Direct proof: In the implication $p \to q$, if the hypothesis p is false, then the implication is true, which was the vacuous method of proof. However, if p is true, then the implication $p \to q$ may not necessarily be true. It may be true or false which depends upon the truth value of q. But if we can show that q is also true then the implication $p \to q$ must be true. That is, if we assume that p is true and then show that this leads to truthness of q as well, then the implication $p \to q$ must be true. Such kind of proof of an implication $p \to q$ is called a direct proof .

Example: Prove that if a and b are odd integers, then $a + b$ is an even integer.

Proof: Suppose a and b are odd integers. We know that an odd integer can be written as $2k + 1$ where k is some integer. So we can write $a = 2m + 1$ and $b = 2n + 1$ for some integers m and n. Therefore

$$a + b = (2m + 1) + (2n + 1) = 2(m + n + 1) = 2l$$

where $l = m + n + 1$. However, if an integer can be written in the form $2l$ for some integer l, then such an integer must be even. Therefore $a + b$ is an even integer. $\square$

Indirect proof (Proof by Contraposition): The implication $p \to q$ is logically equivalent to its contrapositive implication $\neg q \to \neg p$. Hence proving the implication $p \to q$ is the same as proving its contrapositive implication $\neg q \to \neg p$. To prove $\neg q \to \neg p$, we use the direct method i.e., we prove $\neg p$ is true by assuming that $\neg q$ is true. Such method of proving an implication $p \to q$ is called an indirect proof .

Example: Prove that if $3n + 2$ is odd then n is odd.

Proof: We shall prove the equivalent contrapositive implication "if n is even then $3n + 2$ is even". Now if n is even, then $n = 2k$ for some integer k. So

$$3n + 2 = 3 \times 2k + 2 = 2(3k + 1).$$

So $3n + 2$ is an integer which is a multiple of 2. Therefore, $3n + 2$ is an even integer. □

Proof by contradiction: The implication $p \to q$ is logically equivalent to $\neg p \vee q$. So to prove that $p \to q$ is true, it is sufficient to show that $\neg p \vee q$ is true, i.e., $\neg(\neg p \vee q)$ is false. But $\neg(\neg p \vee q) \equiv p \wedge \neg q$. So to show that $p \to q$ is true, it is sufficient to show that $p \wedge \neg q$ is false i.e. $p \wedge \neg q \equiv F$. Such method of proof is known as the proof by contradiction .

Example: In a room of eight people, show that two or more people have birthdays in the same weekday.

Proof: We have to prove the implication "If a room has eight people, then two or more people have birthdays in the same weekday." Suppose that the hypothesis is true and the conclusion is false i.e., a room has eight people but no two people have birthdays on the same weekday. So each of the eight people must have birthdays on separate weekdays and so there must be at least eight weekdays. This is a contradiction because it is a well-known fact that there are only seven weekdays. Therefore it is false that a room has eight people and no two of them have birthdays on the same weekday, that is, if a room has eight people then at least two of them must have birthdays on the same weekday. □

Proof by cases: Suppose we have to prove an implication of the form $(p_1 \vee p_2) \to q$ in which the hypothesis itself is a disjunction of two separate propositions. From the logical equivalence $(p_1 \vee p_2) \to q \equiv (p_1 \to q) \wedge (p_2 \to q)$ we can instead prove $(p_1 \to q) \wedge (p_2 \to q)$ for which we prove the two separate implications $p_1 \to q$ and $p_2 \to q$ using any suitable method among previous ones. Such method of proving is known as proof by cases .

Note that we can use the same strategy as above to prove an extension of above implication in the form of $(p_1 \vee p_2 \vee \cdots \vee p_n) \to q$ for $n > 2$. Since $(p_1 \vee p_2 \vee \cdots \vee p_n) \to q \equiv (p_1 \to q) \wedge (p_2 \to q) \wedge \cdots \wedge (p_n \to q)$, so the implication $(p_1 \vee p_2 \vee \cdots \vee p_n) \to q$ can be proved by proving separately the implications $p_1 \to q, p_2 \to q, \cdots, p_n \to q$.

Example: If an integer n is such that $n - 1$ or $n - 2$ is divisible by 3 then $n^2 - 1$ is divisible by 3.

Proof: For an integer n, we have to prove that if $n - 1$ is divisible by 3 or $n - 2$ is divisible by 3 then $n^2 - 1$ is also divisible by 3. We shall prove each case separately.
First we prove that if $n - 1$ is divisible by 3, then $n^2 - 1$ is also divisible by 3. A number is divisible by 3 if and only if it can be written as $3k$ for some integer k. So if $n - 1$ is divisible by 3, then $n - 1 = 3k$ for some integer k. Therefore

$$n^2 - 1 = (n - 1)(n + 1) = 3k(n + 1).$$

It follows that $n^2 - 1$ is divisible by 3 because $k(n - 1)$ is an integer.
We now prove that if $n - 2$ is divisible by 3, then $n^2 - 1$ is also divisible by 3. If $n - 2$ is divisible by 3, then $n - 2 = 3k$ for some integer k. Therefore,

$$n^2 - 1 = (n - 1)(n + 1) = (n - 1)[(n - 2) + 3]$$

$$= (n-1)(3k+3) = 3(n-1)(k+1).$$

So it follows that $n^2 - 1$ is divisible by 3 because $(n-1)(k+1)$ is an integer. □

Existence proof: Suppose we have to prove that $\exists x P(x)$ is true. For this, we have to show the existence of an element c in the universe of discourse of x such that $P(c)$ is true. This method of proving $\exists x P(x)$ true is known as existence proof .

Existence proofs can be of two types: constructive and nonconstructive. In constructive existence proofs, one actually finds an element c for which $P(c)$ is true as in example below:

Example: Show that there exists a positive integer which is equal to the sum of its positive divisors less than itself.

Proof: We know that the positive divisors less than itself of the integer 6 are 1, 2 and 3. Also, $1 + 2 + 3 = 6$. So there exists a positive integer 6 which is equal to the sum of its positive divisors less than itself. □

In the above example, we proved $\exists x P(x)$ true by actually finding an element c for which $P(c)$ was true. In nonconstructive existence proof however, we show that $\exists x P(x)$ is true by some method other than actually finding an exact element c for which $P(c)$ is true.

Example: Prove that there exists irrational numbers x and y such that x^y is a rational number.

Proof: We know that $\sqrt{2}$ is an irrational number. Let $x = y = \sqrt{2}$. Then $x^y = \sqrt{2}^{\sqrt{2}}$. If $\sqrt{2}^{\sqrt{2}}$ is rational, then our assertion is proved. If not, i.e., if $\sqrt{2}^{\sqrt{2}}$ is irrational, then let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. Then

$$x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$$

which is a rational number. So either the pair

$$x = \sqrt{2}, y = \sqrt{2}$$

or the pair

$$x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}$$

is such that x^y is rational. □

Proof by counterexample: Proof by counterexample is a proof method to show that $\forall x P(x)$ is false. For this it is sufficient to find an element c in the universe of discourse of x such that $P(c)$ is false. Such an element c for which $P(c)$ is false is called a counterexample.

Example: Show that "Every positive integer is the sum of the squares of three integers" is false.

Proof: Consider the integer 7. The only squares less than 7 are 0^2, 1^2 and 2^2 and non of them combine to give a sum of 7. So 7 cannot be written as a sum of the squares of three integers. Hence "Every positive integer can be written as a sum of squares of three integers" is false. □

1.2.2 Mistakes in Proof

1. **Fallacies:** The proof may use invalid arguments such as fallacy of affirming the conclusion or fallacy of denying the hypothesis.

2. **Canceling the zeros:** If both sides of an equation involves zeros, then those two zeros cannot be canceled, otherwise false results can be obtained. e.g. canceling 0's in $1 \times 0 = 2 \times 0$ we get $1 = 2$.

3. **Circular reasoning:** If the steps of a proof are based on the truth of the statement that is being proved then it is called circular reasoning. e.g., to prove that n is an even integer, if we start by supposing that n is divisible by 2 then it is circular reasoning.