Set - A

Basic Mathematics

Group-A [10×2 = 20]

1) Solve the following differential equation

$$\frac{dy}{dx} = 3x^2 + 5x + 6$$

Solution

Given,

$$\frac{dy}{dx} = 3x^2 + 5x + 6$$

or, $dy = (3x^2 + 5x + 6)\, dx$

Integrating on both sides,

$$\int dy = \int (3x^2 + 5x + 6)\, dx$$

or, $\int dy = 3\int x^2\, dx + 5\int x\, dx + 6\int dx$

or, $y = 3 \times \frac{x^{2+1}}{2+1} + 5 \times \frac{x^{1+1}}{1+1} + 6 \times x + c$

$\left[\because \int dx = x + c,\ \int x^n\, dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1\right]$

or, $y = \cancel{\frac{3}{4}}\ \frac{3}{3} \times x^3 + \frac{5}{2} x^2 + 6x + c$

$\therefore y = x^3 + \frac{5}{2} x^2 + 6x + c$ is the required solution.

2) Write down the order and degree of the differential equation

$$\frac{d^2y}{dx^2} + \frac{5dy}{dx} - 6 = 0$$

Its order is : 2

Its degree is : 1

3) Find the area bounded by the curve $y = x^2 + x - 2$, x-axis and the ~~co~~ ordinates at $x = 1$ and $x = 2$.

Solution

Given,

$y = x^2 + x - 2$, $x = 1$, $x = 2$ i.e. $b = 2$ & $a = 1$

We know

Required area $= \int_a^b y\,dx$

$$= \int_1^2 (x^2 + x - 2)\,dx$$

$$= \int_1^2 x^2\,dx + \int_1^2 x\,dx - 2\int_1^2 dx$$

$$= \frac{x^{2+1}}{2+1}\Big|_1^2 + \frac{x^{1+1}}{1+1}\Big|_1^2 - 2\times x\Big|_1^2$$

$$= \frac{1}{3}x^3\Big|_1^2 + \frac{1}{2}x^2\Big|_1^2 - 2x\Big|_1^2$$

$$= \frac{1}{3}(2^3 - 1^3) + \frac{1}{2}(2^2 - 1^2) - 2\times(2-1)$$

$$= \frac{7}{3} + \frac{3}{2} - 2$$

$$= \frac{11}{6} \text{ sq. units}$$

Therefore required area is $\frac{11}{6}$ sq. units

4) Evaluate:

$$\lim_{x \to \infty} \left(\frac{4x^2 + 5x - 6}{3x^2 - 2x + 8} \right)$$

Given,

$$\lim_{x \to \infty} \left[\frac{4x^2 + 5x - 6}{3x^2 - 2x + 8} \right]$$

Dividing numerator & denominator by x^2

$$= \lim_{x \to \infty} \left[\frac{4 + \frac{5}{x} - \frac{6}{x^2}}{3 - \frac{2}{x} + \frac{8}{x^2}} \right]$$

$$= \frac{\lim_{x \to \infty} \left[4 + \frac{5}{x} - \frac{6}{x^2} \right]}{\lim_{x \to \infty} \left[3 - \frac{2}{x} + \frac{8}{x^2} \right]}$$

$$= \frac{4 + \frac{5}{\infty} - \frac{6}{\infty^2}}{3 - \frac{2}{\infty} + \frac{8}{\infty^2}}$$

$$= \frac{4 + \frac{5}{\infty} - \frac{6}{\infty}}{3 - \frac{2}{\infty} + \frac{8}{\infty}} \quad [\because \infty^2 = \infty]$$

$$= \frac{4+0-0}{3-0+0} \quad \left[\because \frac{\text{finite value}}{\infty} = 0\right]$$

$$= \frac{4}{3}$$

$$\therefore \lim_{x\to\infty}\left[\frac{4x^2+5x-6}{3x^2-2x+8}\right] = \frac{4}{3}$$

5) Integrate $\frac{x-2}{x+5}$ with respect to x.

Given,

$$\int \frac{x-2}{x+5}\,dx$$

$$= \int \frac{x+5-7}{x+5}\,dx$$

$$= \int \left(\frac{x+5}{x+5} - \frac{7}{x+5}\right) dx$$

$$= \int \left(1 - \frac{7}{x+5}\right) dx$$

$$= 1\int dx - 7\int \frac{1}{x+5}\,dx$$

$$= x - 7\log(x+5) + c \quad \left[\because \int dx = x+c,\ \int \frac{f'(x)}{f(x)} = \log x + c\right]$$

$$\therefore \int \frac{x-2}{x+5} = x - 7\log(x+5) + c$$

6) Find the critical point with

$y = x^3 - 3x^2 + 8x - 7$

Solution

Given,

$y = x^3 - 3x^2 + 8x - 7$

Differentiating both sides with respect to x,

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^3 - 3x^2 + 8x - 7)$$

$$\text{or, } \frac{dy}{dx} = \frac{d}{dx}(x^3) - \frac{d}{dx}(3x^2) + \frac{d}{dx}(8x) - \frac{d}{dx}(7)$$

$$\text{or, } \frac{dy}{dx} = 3x^2 - 6x + 8 - 0 \quad \left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

$$\text{or, } \frac{dy}{dx} = 3x^2 - 6x + 8$$

We know,

At the critical point, $\frac{dy}{dx} = 0$,

i.e. $3x^2 - 6x + 8 = 0$

$$\therefore x = 1 + \frac{\sqrt{15}}{3} i, \; 1 - \frac{\sqrt{15}}{3} i$$

7) If $f(x) = \frac{1+x}{1-x}$, find $\frac{f(x)\,f(x^2)}{1+[f(x)]^2}$.

Given,

$$f(x) = \frac{1+x}{1-x}$$

Then,

$$f(x^2) = \frac{1+x^2}{1-x^2}$$

$$\frac{f(x)\,f(x^2)}{1+[f(x)]^2} = \frac{\frac{1+x}{1-x} \times \frac{1+x^2}{1-x^2}}{1+\left[\frac{1+x}{1-x}\right]^2}$$

$$= \frac{\frac{1+x}{1-x} \times \frac{1+x^2}{(1+x)(1-x)}}{1+\frac{(1+x)^2}{(1-x)^2}}$$

$$= \frac{\frac{(1+x)(1+x^2)}{(1-x)(1+x)(1-x)}}{\frac{(1-x)^2+(1+x)^2}{(1-x)^2}}$$

$$= \frac{\frac{1+x^2}{(1-x)^2}}{\frac{1-2x+x^2+1+2x+x^2}{(1-x)^2}}$$

$$= \frac{1+x^2}{2+2x^2}$$

$$= \frac{1+x^2}{2(1+x^2)} = \frac{1}{2}$$

$= \frac{1}{2}$

$\therefore \frac{f(x)f(x^2)}{1+[f(x)]^2} = \frac{1}{2}$

8) What are the cube roots of unity?

$\Rightarrow$ The cube roots of unity are 1, $\frac{-1+\sqrt{3}i}{2}$, $\frac{-1-\sqrt{3}i}{2}$

i.e. 1, ω and ω^2.

9) Rewrite $|2x-5| \leq 3$ without using absolute value sign.

The given inequality is

$|2x-5| \leq 3$

$\Rightarrow -3 \leq 2x-5 \leq 3$

$\Rightarrow -3+5 \leq 2x-5+5 \leq 3+5$

$\Rightarrow 2 \leq 2x \leq 8$

$\Rightarrow \frac{2}{2} \leq \frac{2x}{2} \leq \frac{8}{2}$

$\Rightarrow 1 \leq x \leq 4$

10) If $z_1 = 2+3i$ and $z_2 = 4-5i$, find $\overline{z_1+z_2}$

Given,

$z_1 = 2+3i$

$z_2 = 4-5i$

Then,

$z_1+z_2 = 2+3i+4-5i$

$\therefore z_1+z_2 = 6-2i$

Then, $\overline{z_1+z_2} = \overline{6-2i}$

$\therefore \overline{z_1+z_2} = 6+2i$ $[\because \overline{x+iy} = x-iy]$

Group-B [$6 \times 5 = 30$]

11) Solve the differential equation :

$$(x^2-1)\frac{dy}{dx} + 2xy = 4x^2$$

The given differential equation is

$$(x^2-1)\frac{dy}{dx} + \cancel{2xy}\ 2xy = 4x^2$$

or, $$\frac{dy}{dx} + \frac{2x}{(x^2-1)}y = \frac{4x^2}{(x^2-1)} \quad \text{--- (1)}$$

Comparing (1) with $\frac{dy}{dx} + Py = Q$ we get,

$$P = \frac{2x}{x^2-1}, \quad Q = \frac{4x^2}{x^2-1}$$

we know,

Integrating Factor (I.F) $= e^{\int p\,dx}$

$$= e^{\int \frac{2x}{x^2-1} dx}$$

$$= e^{\log(x^2-1)}$$

$\therefore$ IF $= x^2-1$ [$\because e^{\log x} = x$]

Now, the required solution becomes,

$$y \times IF = \int (Q \times IF) + c$$

or, $y \times (x^2-1) = \int \frac{4x^2}{(x^2-1)} \times (x^2-1) + c$

or, $y \times (x^2-1) = \int 4x^2 + c$

or, $y \times (x^2-1) = 4 \times \frac{x^{2+1}}{2+1} + c$

or, $y \times (x^2-1) = \frac{4}{3} x^3 + c$ is the required solution.

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12) If the marginal revenue function for output x is given by $\frac{6}{(x+2)^2} - 5$, find the demand function $P(x)$.

Given,

$$MR(x) = \frac{6}{(x+2)^2} - 5$$

We know,

$$TR(x) = \int MR(x)\, dx$$

$$= \int \left[\frac{6}{(x+2)^2} - 5\right] dx$$

$$= \int \frac{6}{(x+2)^2}\, dx - 5 \int dx$$

$$= 6\int \frac{1}{(x+2)^2}\,dx - 5x + c \quad [\because \int dx = x + c]$$

$$= 6\int (x+2)^{-2}\,dx - 5x + c$$

$$= 6 \times \frac{(x+2)^{-2+1}}{1(-2+1)} - 5x + c$$

$$= -\frac{6}{(x+2)} - 5x + c$$

~~When $x = 0$,~~

$$\therefore TR(x) = -\frac{6}{(x+2)} - 5x + c \quad \text{--- (1)}$$

When $x = 0$,

$$TR(0) = -\frac{6}{0+2} - 5 \times 0 + c$$

or, $0 = -3 - 0 + c \quad [\because TR(0) = 0]$

$\therefore c = 3$

Now, (1) becomes,

$$TR(x) = -\frac{6}{(x+2)} - 5x + 3$$

Finally,

$$P(x) = \frac{TR(x)}{x}$$

$$= \frac{-\frac{6}{(x+2)} - 5x + 3}{x}$$

$$= -\frac{6}{x(x+2)} - 5 + \frac{3}{x}$$

$$= -\frac{6}{x(x+2)} + \frac{3}{x} - 5$$

$$= \frac{-6 + 3(x+2)}{x(x+2)} - 5$$

$$= \frac{-6 + 3x + 6}{x(x+2)} - 5$$

$$= \frac{3x}{x(x+2)} - 5$$

$$= \frac{3}{(x+2)} - 5$$

$$\therefore P(x) = \frac{3}{(x+2)} - 5$$

13) Integrate the following :

a) $\int \frac{dx}{\sqrt{x+1} - \sqrt{x+2}}$

$= \int \frac{dx}{\sqrt{x+1} - \sqrt{x+2}} \times \frac{\sqrt{x+1} + \sqrt{x+2}}{\sqrt{x+1} + \sqrt{x+2}}$

$= \int \frac{\sqrt{x+1} + \sqrt{x+2}}{(\sqrt{x+1})^2 - (\sqrt{x+2})^2} \times dx$

$= \int \frac{\sqrt{x+1} + \sqrt{x+2}}{x+1-x-2} \, dx$

$= -1 \int \{(x+1)^{1/2} + (x+2)^{1/2}\} \, dx$

$= -\int (x+1)^{1/2} dx - \int (x+2)^{1/2} dx$

$= -\frac{(x+1)^{\frac{1}{2}+1}}{1(\frac{1}{2}+1)} - \frac{(x+2)^{\frac{1}{2}+1}}{1(\frac{1}{2}+1)} + C$

$\left[\because \int (ax+b)^n = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \; n \neq -1\right]$

$= -\frac{2}{3}(x+1)^{3/2} - \frac{2}{3}(x+2)^{3/2} + C$

b) $\int \frac{dx}{x+\sqrt{x}}$

$= \int \frac{dx}{\sqrt{x}.\sqrt{x}+\sqrt{x}}$

$= \int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

Put $\sqrt{x}+1 = t$

Differentiating both sides with respect to x,

$\frac{d}{dx}(\sqrt{x}+1) = \frac{d}{dx}(t)$

or, $\frac{d}{dx}(x^{1/2}) + \frac{d}{dx}(1) = \frac{dt}{dx}$

or, $\frac{1}{2}x^{-\frac{1}{2}} + 0 = \frac{dt}{dx}$

or, $\frac{1}{2\sqrt{x}} = \frac{dt}{dx}$

$\therefore \frac{dx}{2\sqrt{x}} = dt$

Then,

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)}$$

$$= \int \frac{1}{(\sqrt{x}+1)} \cdot \frac{dx}{\sqrt{x}}$$

$$= 2 \int \frac{1}{(\sqrt{x}+1)} \cdot \frac{dx}{2\sqrt{x}}$$

$$\cancel{= 2 \int (t^{-1})\, dt}$$

$$\cancel{\left[\because \sqrt{x}+1 = t,\ \frac{dx}{2\sqrt{x}} = dt\right]}$$

$$\cancel{= 2 \times t^{-1+1}}$$

$$= 2 \cdot \int \frac{1}{t}\, dt$$

$$\left[\because \sqrt{x}+1 = t,\ \frac{dx}{2\sqrt{x}} = dt\right]$$

$$= 2 \cdot \log(t) + C \quad \left[\because \int \frac{dx}{x} = \log x + c\right]$$

$$= 2 \cdot \log(\sqrt{x}+1) + C$$

$$\therefore \int \frac{dx}{x+\sqrt{x}} = 2 \cdot \log(\sqrt{x}+1) + C$$

14) Find $\frac{dy}{dx}$ when,

a) $y = \frac{x^2+x-1}{x^2+3x-7}$

Given,

$$y = \frac{x^2+x-1}{x^2+3x-7}$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}(y) = \frac{d}{dx}\left[\frac{x^2+x-1}{x^2+3x-7}\right]$$

$$= \frac{(x^2+3x-7) \cdot \frac{d}{dx}(x^2+x-1) - (x^2+x-1) \cdot \frac{d}{dx}(x^2+3x-7)}{(x^2+3x-7)^2}$$

$$= \frac{(x^2+3x-7)(2x+1) - (x^2+x-1)(2x+3)}{(x^2+3x-7)^2}$$

$$= \frac{(2x^3+6x^2-14x+x^2+3x-7) - (2x^3+2x^2-2x+3x^2+3x-3)}{(x^2+3x-7)^2}$$

$$= \frac{(2x^3+7x^2-11x-7) - (2x^3+5x^2+x-3)}{(x^2+3x-7)^2}$$

$$= \frac{\cancel{2x^3}+7x^2-11x-7-\cancel{2x^3}-5x^2-x+3}{(x^2+3x-7)^2}$$

$$= \cancel{\frac{2x^2 - 12x + 3}{}}$$

$$= \frac{2x^2 - 12x - 4}{(x^2+3x-7)^2}$$

$$= \frac{2(x^2 - 6x - 2)}{(x^2+3x-7)^2}$$

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b) $x = z^2 + 2$, $y = z^3 + 2z + 1$

Given,

$$x = z^2 + 2$$

~~Differentiation~~

Differentiating both sides with respect to 'z'

$$\frac{d}{dz}(x) = \frac{d}{dz}(z^2 + 2)$$

$$\text{or, } \frac{dx}{dz} = 2z + 0$$

$$\text{or, } \frac{dx}{dz} = 2z$$

$$\therefore \frac{dz}{dx} = \frac{1}{2z} \quad \text{——— (1)}$$

Also,

$$y = z^3 + 2z + 1$$

Differentiating both sides with respect to 'z'

$$\frac{d}{dz}(y) = \frac{d}{dz}(z^3 + 2z + 1)$$

$$\text{or, } \frac{dy}{dz} = 3z^2 + 2 + 0$$

$$\therefore \frac{dy}{dz} = 3z^2 + 2$$

Now,

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

$$= (3z^2+2) \times \frac{1}{2z}$$

$$\therefore \frac{dy}{dx} = \frac{3z^2+2}{2z}$$

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15) A function $y = f(x)$ is defined by

$$f(x) = \begin{cases} Kx+3, & x > 3 \\ 3x, & x < 3 \\ 9, & x = 3 \end{cases}$$

If it is continuous at $x=3$, find K.

Given,

$$y = f(x) = \begin{cases} Kx+3, & x > 3 \\ 3x, & x < 3 \\ 9, & x = 3 \end{cases}$$

Since, the above function is continuous at $x=3$,

Left Hand Limit = Right Hand Limit = Functional value

or $\lim_{x \to 3^-} [f(x)] = \lim_{x \to 3^+} [f(x)] = \cancel{f(x)}\ f(3)$

or, $\lim_{x\to 3^-}(3x) = \lim_{x\to 3^+}(kx+3) = 9$

or, $3\times 3 = k\times 3+3 = 9$

or, $9 = 3k+3 = 9$

<u>Taking 1st & 2nd</u>

$3k+3 = 9$

or, $3k = 6$

$\therefore k = 2$

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16) Evaluate:

$$\lim_{x\to 1} \frac{x-\sqrt{2-2x^2}}{2x-\sqrt{2+2x^2}}$$

$$= \lim_{x\to 1}\left[\frac{x-\sqrt{2-2x^2}}{2x-\sqrt{2+2x^2}} \times \frac{x+\sqrt{2-2x^2}}{x+\sqrt{2-2x^2}} \times \frac{2x+\sqrt{2+2x^2}}{2x+\sqrt{2+2x^2}}\right]$$

$$= \lim_{x\to 1}\left[\frac{(x)^2-(\sqrt{2-2x^2})^2}{(2x)^2-(\sqrt{2+2x^2})^2} \times \frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-2x^2}}\right]$$

$$= \lim_{x\to 1}\left[\frac{x^2-2+2x^2}{4x^2-2-2x^2} \times \frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-2x^2}}\right]$$

$$= \lim_{x\to 1}\left[\frac{3x^2-2}{2x^2-2} \times \frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-2x^2}}\right]$$

$$= \frac{3\times(1)^2-2}{2\times(1)^2-2} \times \frac{2\times 1+\sqrt{2+2\times(1)^2}}{1+\sqrt{2-2\times(1)^2}}$$

$$= \frac{3-2}{2-2} \times \frac{2+2}{1+0}$$

$= \frac{4}{0}$ (undefined)

~~Hen~~ Hence, limit ~~cannot be determined~~ doesnot exist.

17. a) Find the square root of $5+12i$

Solution

Let $\sqrt{5+12i} = x+iy$ where $a=5$, & $b=12$

we know that,

$$x^2 = \frac{\sqrt{a^2+b^2}+a}{2}$$

$$\text{or } x^2 = \frac{\sqrt{5^2+12^2}+5}{2}$$

$$\text{or } x^2 = \frac{13+5}{2}$$

$$\text{or } x^2 = \frac{18}{2}$$

$$\text{or } x^2 = 9$$

$$\therefore x = \pm 3$$

Also,

$$y^2 = \frac{\sqrt{a^2+b^2}-a}{2}$$

$$\text{or, } y^2 = \frac{\sqrt{5^2+12^2}-5}{2}$$

$$\text{or, } y^2 = \frac{13-5}{2}$$

$$\text{or } y^2 = \frac{8}{2}$$

$$\text{or } y^2 = 4$$

$$\therefore y = \pm 2$$

Since 'b' is positive, x and y must have same sign,

~~$x+iy$~~

When $x=3, y=2$

$x+iy = 3+2i$

When $x=-3, y=-2$

$x+iy = -3-2i$

$\therefore x+iy = -(3+2i)$

Therefore, $\sqrt{5+12i} = \pm(3+2i)$

17. b) Express $-1+i\sqrt{3}$ in polar form.

Solution

Let, $-1+i\sqrt{3} = x+iy$ where $x=-1, y=\sqrt{3}$

We know,

$\tan\theta = \frac{y}{x}$

or, $\tan\theta = \frac{\sqrt{3}}{-1}$

or, $\tan\theta = -\sqrt{3}$

or, $\tan\theta = \tan 120°$

$\therefore \theta = 120°$

Also,

$r = \sqrt{x^2+y^2}$

$= \sqrt{(-1)^2+(\sqrt{3})^2}$

$\therefore r = 2$

Now,

$-1 + i\sqrt{3} = 2(\cos 120^\circ + i \sin 120^\circ)$ which is in polar form

Group-E [3 x 10 = 30]

18) In a group of 200 students, 100 are interested in music, 70 are interested in photography, 40 like swimming, 40 are interested in music and photography, 30 are interested in music and swimming, 20 are interested in photography and swimming and 10 are interested in all three activities. Find the number of students that are interested in

a) exactly two activities
b) at least one activities
c) None of the activities

Solution

Let the set of students interested in music be 'M', photography be 'P', swimming be 'S' and all three be '$M \cap P \cap S$'.

Then the above information can be written as:

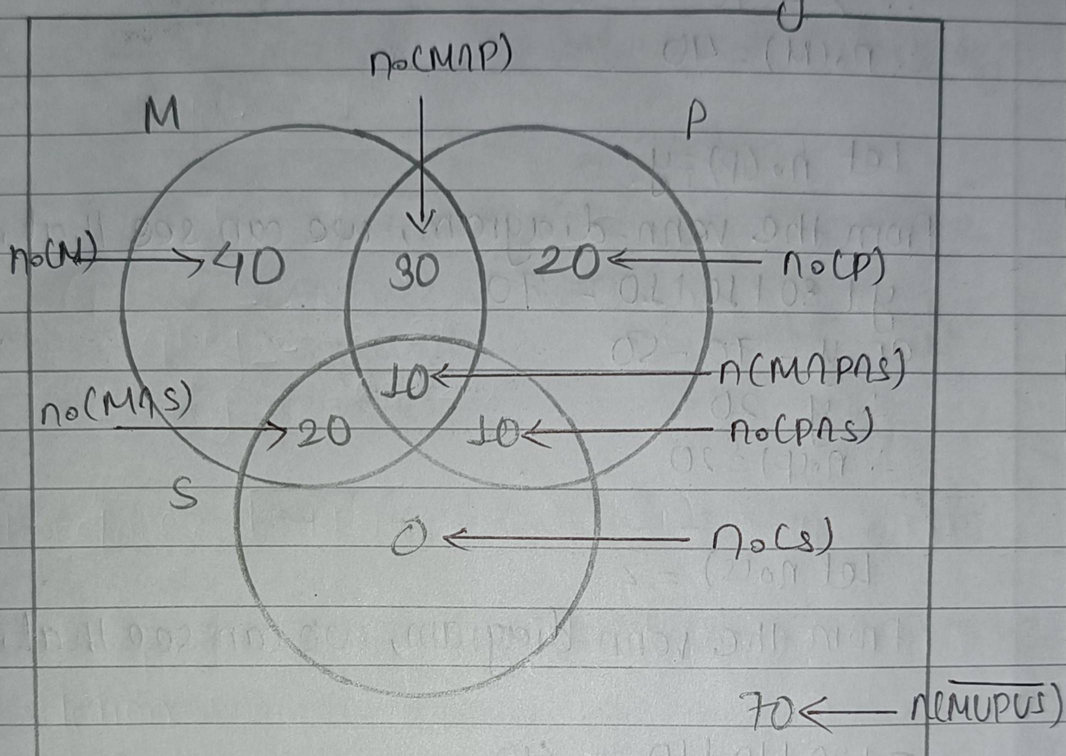
$n(U) = 200$
$n(M) = 100$
$n(P) = 70$
$n(S) = 40$
$n(M \cap P) = 40$

$n(M \cap S) = 30$

$n(P \cap S) = 20$

$n(M \cap P \cap S) = 10$

Showing the above information in a venn-diagram, as



We know,

$n_o(M \cap S) = n(M \cap S) - n(M \cap P \cap S)$

$= 30 - 10$

$\therefore n_o(M \cap S) = 20$

$n_o(M \cap P) = n(M \cap P) - n(M \cap P \cap S)$

$= \cancel{30}\ 40 - 10$

$\therefore n_o(M \cap P) = 30$

$n_o(P \cap S) = n(P \cap S) - n(M \cap P \cap S)$

$= 20 - 10$

$\therefore n_o(P \cap S) = 10$

Let $n_o(M) = x$

From the venn-diagram, we can see that,

$x + 20 + 10 + 30 = 100$

or $x = 100 - 60$

$\therefore x = 40$

$\therefore n_o(M) = 40$

Let $n_o(P) = y$

From the venn-diagram, we can see that,

$y + 30 + 10 + 10 = 70$

or $y = 70 - 50$

$\therefore y = 20$

$\therefore n_o(P) = 20$

Let $n_o(S) = z$

From the venn-diagram, we can see that,

$z + 20 + 10 + 10 = 40$

or, $z = 40 - 40$

$\therefore z = 0$

$\therefore n_o(S) = 0$

Also we know,

$n(\overline{M \cup P \cup S}) = n(U) - n(M \cup P \cup S)$

$= n(U) - [n(M) + n(P) + n(S) - n(M \cap P) - n(P \cap S) - n(M \cap S) + n(M \cap P \cap S)]$

$= 200 - [100 + 70 + 40 - 40 - 20 - 30 + 10]$

$= 200 - 130$

$= 70$

$\therefore n(\overline{M \cup P \cup S}) = 70$

Then,

a) Number of students interested in exactly two activities $= n_o(M \cap P) + n_o(M \cap S) + n_o(P \cap S)$

$= 30 + 20 + 10$

$= 60$

Therefore 60 students are interested in exactly two activities.

b) Number of students interested in at least one activities $= n(M \cup P \cup S) = 130$

Therefore 130 students are interested in at least one activities.

c) Number of students interested in none of the activities $= n(\overline{M \cup P \cup S}) = 70$

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19) The marginal revenue and the marginal cost are $MR = 15 - 9x + 6x^2$ and $MC = 10 - 24x - 2x^2$ respectively. If the fixed cost is Rs. 5,000, find

a) the total cost function

b) the total revenue function

c) demand function

d) total profit (or loss) when $x = 100$

e) Average cost function.

Solution

Given,

$MR(x) = 15 - 9x + 6x^2$

$MC(x) = 10 - 24x - 2x^2$

We know,

b) $TR(x) = \int MR(x)\,dx$

$= \int (15 - 9x + 6x^2)\,dx$

$= 15\int dx - 9\int x\,dx + 6\int x^2\,dx$

$= 15 \times x - 9 \times \frac{x^{1+1}}{1+1} + 6 \times \frac{x^{2+1}}{2+1} + C$

$= 15x - \frac{9}{2}x^2 + \frac{6}{3}x^3 + C$

$\therefore TR(x) = 15x - \frac{9}{2}x^2 + 2x^3 + C \quad \text{---} \ (1)$

When $x = 0$,

$TR(0) = 15 \times 0 - \frac{9}{2} \times 0^2 + 2 \times 0^3 + C$

or, $0 = 0 - 0 + 0 + C$ [$\because TR(0) = 0$]

$\therefore C = 0$

Now, (1) becomes,

$TR(x) = 15x - \frac{9}{2}x^2 + 2x^3 + 0$

i.e. $TR(x) = 15x - \frac{9}{2}x^2 + 2x^3$

Therefore, the total revenue function is $15x - \frac{9}{2}x^2 + 2x^3$.

a) $TC(x) = \int MC(x)\, dx$

$$= \int (10 - 24x - 2x^2)\, dx$$
$$= 10\int dx - 24\int x\, dx - 2\int x^2\, dx$$
$$= 10 \times x - 24 \times \frac{x^{1+1}}{1+1} - 2 \times \frac{x^{2+1}}{2+1} + K$$

$\therefore TC(x) = 10x - 12x^2 - \frac{2}{3}x^3 + K$ —— (11)

When $x = 0$,

$TC(0) = 10 \times 0 - 12 \times 0^2 - \frac{2}{3} \times 0^3 + K$

or, $5000 = 0 - 0 - 0 + K$ [$\because TC(0) = 5000$]

$\therefore K = 5000$

Now, (11) becomes

$TC(x) = 10x - 12x^2 - \frac{2}{3}x^3 + 5000$

Therefore the total cost function is $10x - 12x^2 - \frac{2}{3}x^3 + 5000$.

c) We know,

Demand function i.e. $P(x) = \frac{TR(x)}{x}$

$$= \frac{15x - \frac{9}{2}x^2 + 2x^3}{x}$$

$$= 15 - \frac{9}{2}x + 2x^2$$

Therefore, the demand function is $15 - \frac{9}{2}x + 2x^2$

d) we know,

$\pi(x) = TR(x) - TC(x)$

$= 15x - \frac{9}{2}x^2 + 2x^3 - 10x + 12x^2 + \frac{2}{3}x^3 - 5000$

$= \frac{15}{2}x^2 + 5x + \frac{8}{3}x^3 - 5000$

$\therefore \pi(x) = \frac{8}{3}x^3 + \frac{15}{2}x^2 + 5x - 5000$

when $x = 100$,

$\pi(100) = \frac{8}{3} \times (100)^3 + \frac{15}{2} \times (100)^2 + 5 \times 100 - 5000$

$= 2666666.67 + 75000 + 500 - 5000$

$\therefore \pi(100) = 2737166.67$

Therefore, the total profit is Rs 2737166.67 when $x = 100$.

e) Average cost $= \frac{TC(x)}{x}$

$$= \frac{10x - 12x^2 - \frac{2}{3}x^3 + 5000}{x}$$

$\therefore AC(x) = 10 - 12x - \frac{2}{3}x^2 + \frac{5000}{x}$

Therefore, the average cost function is

$10 - 12x - \frac{2}{3}x^2 + \frac{5000}{x}$

20) In a perfect competition, the demand and supply functions of a commodity are $P_d = 40 - x^2$ and $P_s = 3x^2 + 8x + 8$, find the consumer's surplus (CS), producers surplus (PS) and the Total surplus (TS).

Solution

Given,

$P_d(x) = 40 - x^2$ —— (I)

$P_s(x) = 3x^2 + 8x + 8$ —— (II)

We know, At perfect Competition / Equilibrium

$P_d = P_s$

or, $40 - x^2 = 3x^2 + 8x + 8$

or, $3x^2 + 8x + 8 - 40 + x^2 = 0$

or, $4x^2 + 8x - 32 = 0$

or, $4(x^2 + 2x - 8) = 0$

or, $x^2 + 2x - 8 = 0$

or, $x^2 + 4x - 2x - 8 = 0$

or, $x(x+4) - 2(x+4) = 0$

or, $(x+4)(x-2) = 0$

Either

$x + 4 = 0$

$\therefore x = -4$ (Rejected)

or,

$x - 2 = 0$

$\therefore x = 2$

$\therefore x = 2$

Substituting the value of x in (I)

$P = 40 - 2^2$

$= 40 - 4$

$\therefore P = 36$

Now,

Consumer's surplus $= \int_0^x P_d(x)\,dx - P \times x$

$= \int_0^2 (40 - x^2)\,dx - 36 \times 2$

$= 40\int_0^2 dx - \int_0^2 x^2\,dx - 72$

$= 40 \times x|_0^2 - \frac{1}{3} \times x^3|_0^2 - 72$

$= 40 \times (2-0) - \frac{1}{3} \times (2^3 - 0^3) - 72$

$= 80 - \frac{8}{3} - 72$

$\therefore CS = \frac{16}{3}$

Also,

Producer's surplus (PS) $= P \times x - \int_0^x P_s(x)\,dx$

$= 36 \times 2 - \int_0^2 (3x^2 + 8x + 8)\,dx$

$= 72 - 3\int_0^2 x^2 - 8\int_0^2 x - 8\int_0^2 dx$

$= 72 - 3 \times \frac{x^{2+1}}{2+1}\Big|_0^2 - 8 \times \frac{x^{1+1}}{1+1}\Big|_0^2 - 8 \times x|_0^2$

$= 72 - x^3|_0^2 - 4x^2|_0^2 - 8x|_0^2$

$= 72 - (2^3 - 0^3) - 4(2^2 - 0^2) - 8(2-0)$

$= 72 - 8 - 16 - 16$

$= 32$

$\therefore PS = 32$

Now,

Total surplus = Consumer's surplus + Producers surplus

$$= \frac{16}{3} + 32$$

$$\therefore TS = \frac{112}{3}$$

21) There are three brands of fertilizers x, y and z. 'x' contains 1 unit of nitre, 2 units of potash and 3 units of phosphate. y contains 3 units of nitre, 1 unit of potash and 2 units of phosphate. z contains 2 units of nitre, 3 units of potash and 1 units of phosphate. If 11 units of nitre, 10 units of potash and 9 units of phosphate are necessary for a field, how much each type of fertilizers required for it? Solve by cramers rule or inverse matrix method.

Solution

The given information can be represented in table as:

Fertilizers	Nitre	Potash	Phosphate
x	1	2	3
y	3	1	2
z	2	3	1
Total	11	10	9

Let the amount of ~~chemical~~ fertilizer X be 'x', ~~chemical~~ fertilizer Y be 'y' and ~~chemical~~ fertilizer Z be 'z' respectively.

Then the system of equations can be formed as:

$$x + 3y + 2z = 11 \quad \text{—— (i)}$$
$$2x + y + 3z = 10 \quad \text{—— (ii)}$$
$$3x + 2y + z = 9 \quad \text{—— (iii)}$$

Coeff. of x	Coeff. of y	Coeff. of z	Constant term
1	3	2	11
2	1	3	10
3	2	1	9

$$D = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

Expanding D by using R_1

$$= 1\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 1(1\times1 - 2\times3) - 3(2\times1 - 3\times3) + 2(2\times2 - 3\times1)$$

$$= -5 + 21 + 2$$

$= 18 \neq 0$; so, the solution exists.

$$D_1 = \begin{vmatrix} 11 & 3 & 2 \\ 10 & 1 & 3 \\ 9 & 2 & 1 \end{vmatrix}$$

Expanding D_1 by using R_1

$$= 11\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 10 & 3 \\ 9 & 1 \end{vmatrix} + 2\begin{vmatrix} 10 & 1 \\ 9 & 2 \end{vmatrix}$$

$$= 11(1\times1 - 2\times3) - 3(10\times1 - 9\times3) + 2(10\times2 - 9\times1)$$

$$= -55 + 51 + 22$$

$$= 18$$

$$D_2 = \begin{vmatrix} 1 & 11 & 2 \\ 2 & 10 & 3 \\ 3 & 9 & 1 \end{vmatrix}$$

Expanding D_2 by using R_1

$$= 1\begin{vmatrix} 10 & 3 \\ 9 & 1 \end{vmatrix} - 11\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 10 \\ 3 & 9 \end{vmatrix}$$

$$= 1(10\times1 - 9\times3) - 11(2\times1 - 3\times3) + 2(2\times9 - 3\times10)$$

$$= -17 + 77 - 24$$

$$= 36$$

$$D_3 = \begin{vmatrix} 1 & 3 & 11 \\ 2 & 1 & 10 \\ 3 & 2 & 9 \end{vmatrix}$$

Expanding D_3 by using R_1

$$= 1\begin{vmatrix} 1 & 10 \\ 2 & 9 \end{vmatrix} - 3\begin{vmatrix} 2 & 10 \\ 3 & 9 \end{vmatrix} + 11\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 1(1\times9 - 2\times10) - 3(2\times9 - 3\times10) + 11(2\times2 - 3\times1)$$

$$= -11 + 36 + 11$$

$$= 36$$

Now,

By using Cramer's Rule

$$x = \frac{D_1}{D} = \frac{18}{18} = 1$$

$$y = \frac{D_2}{D} = \frac{36}{18} = 2$$

$$z = \frac{D_3}{D} = \frac{36}{18} = 2$$

Hence, 1 unit of ~~chemical x~~ fertilizer x, 2 units of fertilizer y and 2 units of fertilizer z is required.

Group-D [1x20 = 20]

22) The demand ~~function~~ and the cost function of a firm are given by $p = 100 - x$ and $c = 100 + 5x + 4x^2$ respectively.

a) Find the maximum revenue

b) Find the maximum profit

c) Find the break-even points.

d) Find the price at which the revenue is maximum.

Solution

Given,

$P(x) = 100 - x$

$C(x) = 100 + 5x + 4x^2$

a) We know,

$$R(x) = P \times x$$
$$= (100 - x) \times x$$
$$\therefore R(x) = 100x - x^2 \quad \text{——— (I)}$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[R(x)] = \frac{d}{dx}(100x - x^2)$$
$$\therefore R'(x) = 100 - 2x \quad \text{——— (II)}$$

At the critical point, $R'(x) = 0$,

i.e. $100 - 2x = 0$

Or, $2x = 100$

$\therefore x = \cancel{100}\ 50.$

Differentiating both sides of (II) with respect to x,

$$\frac{d}{dx}[R'(x)] = \frac{d}{dx}(100 - 2x)$$
$$\therefore R''(x) = -2$$

when $x = 50$,

$R''(50) = -2 < 0$ (maximum)

Hence, revenue is maximum at $x = 50$,

when $x = 50$,

$$R(50)max = 100 \times 50 - (50)^2$$
$$= 5000 - 2500$$
$$\therefore R(50)max = 2500$$

Therefore the maximum revenue is Rs 2500

(b) We know,
Profit Function = Revenue Function - Cost function

i.e. $\pi(x) = R(x) - C(x)$
$= 100x - x^2 - 100 - 5x - 4x^2$
$\therefore \pi(x) = 95x - 5x^2 - 100$ —— (ii)

Differentiating both sides with respect to x,

$$\frac{d}{dx}[\pi(x)] = \frac{d}{dx}(95x - 5x^2 - 100)$$

$\therefore \pi'(x) = 95 - 10x$ —— (iii)

At the critical point $\pi'(x) = 0$,

or $95 - 10x = 0$
or $10x = 95$
$\therefore x = 9.5$

Differentiating both sides of (iii) with respect to x,

$$\frac{d}{dx}[\pi'(x)] = \frac{d}{dx}(95 - 10x)$$

$\therefore \pi''(x) = -10$
when $x = 9.5$,
$\pi''(9.5) = -10 < 0$ (maximum)
Hence, profit is maximum when
$x = 9.5$

When $x = 9.5$,

$\pi(9.5)_{max} = 95 \times 9.5 - 5 \times (9.5)^2 - 100$

$= 902.5 - 451.25 - 100$

$\therefore \pi(9.5)_{max} = 351.25$

Therefore, the maximum profit is 351.25.

(c) For break-even points,

$\pi(x) = 0$

or $95x - 5x^2 - 100 = 0$

or $+5x^2 + 95x - 100 = 0$

~~or 5x~~

$\therefore x = 1.11, 17.88$

Therefore, the breakeven is attained at either $x = 1.11$ or $x = 17.88$

(d) We know that revenue is maximum at $x = 50$,

when $x = 50$,

$P = 100 - 50$

$\therefore P = 50$

Therefore, the price is Rs 50 ~~at~~ at which the revenue is maximum.