Exercise 7(c)

1. If $\vec{a} = 5\vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{b} = -3\vec{i} + \vec{j} + \vec{k}$. Show that $\vec{a} \times \vec{b}$ is perpendicular (orthogonal) to $\vec{a}$ & $\vec{b}$.

Solution
Given,

$$\vec{a} = 5\vec{i} - 2\vec{j} + 3\vec{k} = (5, -2, 3)$$

$$\vec{b} = -3\vec{i} + \vec{j} + \vec{k} = (-3, 1, 1)$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -2 & 3 \\ -3 & 1 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \vec{i}\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 5 & 3 \\ -3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 5 & -2 \\ -3 & 1 \end{vmatrix}$$

$$= \vec{i}(-2-3) - \vec{j}(5+9) + \vec{k}(5-6)$$

$$= -5\vec{i} - 14\vec{j} - \vec{k}$$

$$\therefore \vec{a} \times \vec{b} = (-5, -14, -1)$$

~~Now,~~ Now,
To show that $\vec{a} \times \vec{b}$ is perpendicular (orthogonal) to $\vec{a}$,
we should show that $\vec{a} \times \vec{b} \cdot \vec{a} = 0$

$$LHS = \vec{a} \times \vec{b} \cdot \vec{a}$$
$$= (-5, +14, -1) \cdot (5, -2, 3)$$
$$= -5 \times 5 + -14 \times (-2) + (-1) \times 3$$
$$= -25 + 28 - 3$$
$$= 0 \text{ RHS.}$$

Hence, proved.

To show that $\vec{a} \times \vec{b}$ is perpendicular (orthogonal) to $\vec{b}$,
we should show that $\vec{a} \times \vec{b} \cdot \vec{b} = 0$

$$LHS = \vec{a} \times \vec{b} \cdot \vec{b}$$
$$= (-5, -14, -1) \cdot (-3, 1, 1)$$
$$= -5 \times (-3) + (-14) \times 1 + (-1) \times 1$$
$$= 15 - 14 - 1$$
$$= 0 \text{ RHS.}$$

Hence, proved.

2. If $\vec{a} = (2, -1, 3)$, $\vec{b} = (1, 2, -3)$ and $\vec{c} = (2, 3, 5)$.
verify that,

a) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Solution

Given,

$\cancel{\vec{a} \times \vec{b}}$ $\vec{a} = (2, -1, 3)$

$\vec{b} = (1, 2, -3)$

$\cancel{\vec{c} = (2, 3, 5)}$

Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= \vec{i}(3-6) - \vec{j}(-6-3) + \vec{k}(4+1)$$

$$= -3\vec{i} + 9\vec{j} + 5\vec{k}$$

$$\therefore \vec{a} \times \vec{b} = (-3, 9, 5)$$

Also,

$$\vec{b} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= \vec{i}(6-3) - \vec{j}(3+6) + \vec{k}(-1-4)$$

$$= 3\vec{i} - 9\vec{j} - 5\vec{k}$$

$$\therefore \vec{b} \times \vec{a} = (3, -9, -5)$$

$$\therefore -\vec{b} \times \vec{a} = (-3, 9, 5)$$

$$\therefore \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

Proved

b) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

We have,

$\vec{a} \times \vec{b} = -3\vec{i} + 9\vec{j} + 5\vec{k}$

Now,

$\vec{b} + \vec{c} = (1, 2, -3) + (2, 3, 5)$

$= (1+2, 2+3, -3+5)$

$\therefore \vec{b} + \vec{c} = (3, 5, 2)$

Then,

$$LHS = \vec{a} \times (\vec{b} + \vec{c})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 3 & 5 & 2 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -1 & 3 \\ 5 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$$

$$= \vec{i}(-2-15) - \vec{j}(4-9) + \vec{k}(10+3)$$

$$= -17\vec{i} + 5\vec{j} + 13\vec{k}$$

$$\vec{a} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -1 & 3 \\ 3 & 5 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= \vec{i}(-5-9) - \vec{j}(10-6) + \vec{k}(6+2)$$

$$= -14\vec{i} - 4\vec{j} + 8\vec{k}$$

RHS $= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$= (-3, 9, 5) + (-14, -4, 8)$

$= -17\vec{i} + 5\vec{j} + 13\vec{k}$

$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

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c) $\vec{a} \times \vec{b}$ is perpendicular (orthogonal) to $\vec{a}$ & $\vec{b}$

Solution

To show $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.
we need to show that $\vec{a} \times \vec{b} \cdot \vec{a} = 0$

LHS $= \vec{a} \times \vec{b} \cdot \vec{a}$

$= (-3, 9, 5) \cdot (2, -1, 3)$

$= (-3) \times 2 + 9 \times (-1) + 5 \times 3$

$= -6 - 9 + 15$

$= 0$

To show $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$
we need to show that $\vec{a} \times \vec{b} \cdot \vec{b} = 0$

LHS $= \vec{a} \times \vec{b} \cdot \vec{b}$

$= (-3, 9, 5) \cdot (1, 2, -3)$

$= -3 \times 1 + 9 \times 2 + 5 \times (-3)$

$= -3 + 18 - 15$

$= 0$

Proved

3. Find the area of parallelogram determined by the following vectors.

a) $2\vec{i} + 4\vec{j} + \vec{k}$ and $-2\vec{i} + \vec{j} + \vec{k}$

Solution

Let $\vec{a} = 2\vec{i} + 4\vec{j} + \vec{k}$

$\vec{b} = -2\vec{i} + \vec{j} + \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & 1 \\ -2 & 1 & 1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 4 \\ -2 & 1 \end{vmatrix}$$

$$= \vec{i}(4-1) - \vec{j}(2+2) + \vec{k}(2+8)$$

$$= 3\vec{i} - 4\vec{j} + 10\vec{k}$$

Then,

$$|\vec{a} \times \vec{b}| = \sqrt{(3)^2 + (-4)^2 + (10)^2} = 5\sqrt{5}$$

$\therefore$ Area of parallelogram $= |\vec{a} \times \vec{b}|$ sq. units

$= 5\sqrt{5}$ sq. units

b) $\vec{i} - 2\vec{j}$ and $3\vec{i} - 2\vec{j} + \vec{k}$

Solution

Let $\vec{a} = \vec{i} - 2\vec{j} = (1, -2, 0)$

$\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k} = (3, -2, 1)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 0 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -2 & 0 \\ -2 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -2 \\ 3 & -2 \end{vmatrix}$$

$= \vec{i}(-2+0) - \vec{j}(1-0) + \vec{k}(-2+6)$

$= -2\vec{i} - \vec{j} + 4\vec{k}$

Then,

$|\vec{a} \times \vec{b}| = \sqrt{(-2)^2 + (-1)^2 + (4)^2} = \sqrt{21}$

We know that,

Area of parallelogram $= |\vec{a} \times \vec{b}|$ sq units

$= \sqrt{21}$ sq. unit

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c) $2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{i} + 2\vec{j} + 5\vec{k}$

Solution

Let $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k} = (2,3,-1)$

$\vec{b} = \vec{i} + 2\vec{j} + 5\vec{k} = (1,2,5)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & 2 & 5 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

$= \vec{i}(15+2) - \vec{j}(10+1) + \vec{k}(4-3)$

$= 17\vec{i} - 11\vec{j} + \vec{k}$

Then,

$$|\vec{a} \times \vec{b}| = \sqrt{(17)^2 + (-11)^2 + (1)^2} = \sqrt{411}$$

We know that,

Area of parallelogram $= |\vec{a} \times \vec{b}|$ sq. units

$= \sqrt{411}$ sq. units

d) $\vec{i} - 2\vec{k}$ and $4\vec{i} + 3\vec{j} + \vec{k}$

Solution

Let $\vec{a} = \vec{i} - 2\vec{k} = (1, 0, -2)$

$\vec{b} = 4\vec{i} + 3\vec{j} + \vec{k} = (4, 3, 1)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 0 \\ 4 & 3 \end{vmatrix}$$

$$= \vec{i}(0+6) - \vec{j}(1+8) + \vec{k}(3-0)$$

$$= 6\vec{i} - 9\vec{j} + 3\vec{k}$$

Then,

$$|\vec{a} \times \vec{b}| = \sqrt{(6)^2 + (-9)^2 + (3)^2} = 3\sqrt{14}$$

We know,

~~Area of p~~

Area of parallelogram $= |\vec{a} \times \vec{b}|$ sq. units

$= 3\sqrt{14}$ sq. units

4. Find the area formed by triangle determined by:

a) $\vec{a} = (3,2,-1)$ and $\vec{b} = (1,-2,4)$

Solution

Given,

$$\vec{a} = (3,2,-1) = 3\vec{i} + 2\vec{j} - \vec{k}$$
$$\vec{b} = (1,-2,4) = \vec{i} - 2\vec{j} + 4\vec{k}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 2 & -1 \\ -2 & 4 \end{vmatrix} - \vec{j} \begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= \vec{i}(8-2) - \vec{j}(12+1) + \vec{k}(-6-2)$$
$$= 6\vec{i} - 13\vec{j} - 8\vec{k}$$

Then,

$$|\vec{a} \times \vec{b}| = \sqrt{(6)^2 + (-13)^2 + (-8)^2} = \sqrt{269}$$

We know that,

$$\text{Area of triangle} = \frac{|\vec{a} \times \vec{b}|}{2} \text{ sq. units}$$
$$= \frac{\sqrt{269}}{2} \text{ sq. units.}$$

b) $\vec{u} = (2, 5, 0)$ and $\vec{v} = (3, -2, 1)$

Solution

Given,

$$\vec{u} = (2, 5, 0)$$
$$\vec{v} = (3, -2, 1)$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 5 & 0 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 5 & 0 \\ -2 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 5 \\ 3 & -2 \end{vmatrix}$$

$$= \vec{i}(5-0) - \vec{j}(2-0) + \vec{k}(-4-15)$$
$$= 5\vec{i} - 2\vec{j} - 19\vec{k}$$

Then,

$$|\vec{u} \times \vec{v}| = \sqrt{(5)^2 + (-2)^2 + (-19)^2} = \sqrt{390}$$

we know that,

$$\text{Area of triangle} = \frac{|\vec{a} \times \vec{b}|}{2} \text{ sq. units}$$

$$= \frac{\sqrt{390}}{2} \text{ sq. units}$$

5. Find the sine of the angle between the vectors.

a) $2\vec{i} - \vec{j} + \vec{k}$ and $2\vec{i} + 3\vec{j} - 2\vec{k}$

Solution:

Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k} = (2, -1, 1)$

$\vec{b} = 2\vec{i} + 3\vec{j} - 2\vec{k} = (2, 3, -2)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 2 & 3 & -2 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= \vec{i}(2 - 3) - \vec{j}(-4 - 2) + \vec{k}(6 + 2)$$

$$= -\vec{i} + 6\vec{j} + 8\vec{k}$$

Then,

$$|\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + (6)^2 + (8)^2} = \sqrt{101}$$

$$|\vec{a}| = \sqrt{(2)^2 + (-1)^2 + (-1)^2} = \sqrt{6}$$

$$|\vec{B}| = \sqrt{(2)^2 + (3)^2 + (-2)^2} = \sqrt{17}$$

We know,

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{B}|}$$

$$= \frac{\sqrt{101}}{\sqrt{6}\,\sqrt{17}}$$

$$= \frac{\sqrt{101}}{\sqrt{102}}$$

$$\therefore \sin\theta = \frac{\sqrt{101}}{\sqrt{102}}$$

b) $2\vec{j} + 3\vec{k}$ and $-3\vec{i} + 3\vec{j} - 2\vec{k}$.

Solution

Let $\vec{a} = 2\vec{j} + 3\vec{k} = (0, 2, 3)$

$\vec{b} = -3\vec{i} + 3\vec{j} - 2\vec{k} = (-3, 3, -2)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 3 \\ -3 & 3 & -2 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & 3 \\ -3 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & 2 \\ -3 & 3 \end{vmatrix}$$

$$= \vec{i}(-4-9) - \vec{j}(0+9) + \vec{k}(0+6)$$

$$= -13\vec{i} - 9\vec{j} + 6\vec{k}$$

Then,

$$|\vec{a} \times \vec{b}| = \sqrt{(-13)^2 + (-9)^2 + (6)^2} = \sqrt{286}$$

$$|\vec{a}| = \sqrt{(0)^2 + (2)^2 + (3)^2} = \sqrt{13}$$

$$|\vec{b}| = \sqrt{(-3)^2 + (3)^2 + (-2)^2} = \sqrt{22}$$

We know,

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{\sqrt{286}}{\sqrt{13}\sqrt{22}}$$

$$= \frac{\sqrt{286}}{\sqrt{286}}$$

$$= 1$$

$\therefore \sin\theta = 1$

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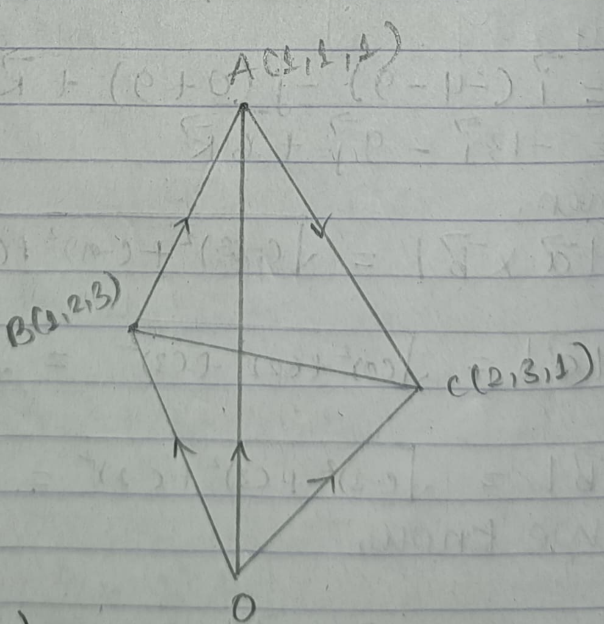
6. Find the area of triangle formed by the points:

a) (1,1,1), (1,2,3) and (2,3,1)

Solution

Let ABC be a triangle whose vertices are A(1,1,1), B(1,2,3) and C(2,3,1) respectively.

Let O be the origin. Join OA, OB & OC.



$\vec{OA} = (1,1,1)$

$\vec{OB} = (1,2,3)$

$\vec{OC} = (2,3,1)$

In Δ OBA

$\vec{OB} + \vec{BA} = \vec{OA}$ [using triangle law of vector addition]

or $\vec{BA} = \vec{OA} - \vec{OB}$

or $\vec{BA} = (1,1,1) - (1,2,3)$

or, $\vec{BA} = (1-1, 1-2, 1-3)$

$\therefore \vec{BA} = (0, -1, -2)$

In Δ OAC

$\vec{OA} + \vec{AC} = \vec{OC}$ [using triangle law of vector addition]

or $\vec{AC} = \vec{OC} - \vec{OA}$

or, $\vec{AC} = (2,3,1) - (1,1,1)$

or $\vec{AC} = (2-1, 3-1, 1-1)$

$\therefore \vec{AC} = (1,2,0)$

Now,

$$\vec{BA} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & -2 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & -2 \\ 2 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= \vec{i}(0+4) - \vec{j}(0+2) + \vec{k}(0+1)$$

$$= 4\vec{i} - 2\vec{j} + \vec{k}$$

Now,
we know that,

Area of triangle $= |\vec{BA} \times \vec{AC}| / 2$ sq units

$= \sqrt{(-4)^2 + (-2)^2 + (1)^2} / 2$

$= \sqrt{16+4+1} / 2$

$= \dfrac{\sqrt{21}}{2}$ sq units

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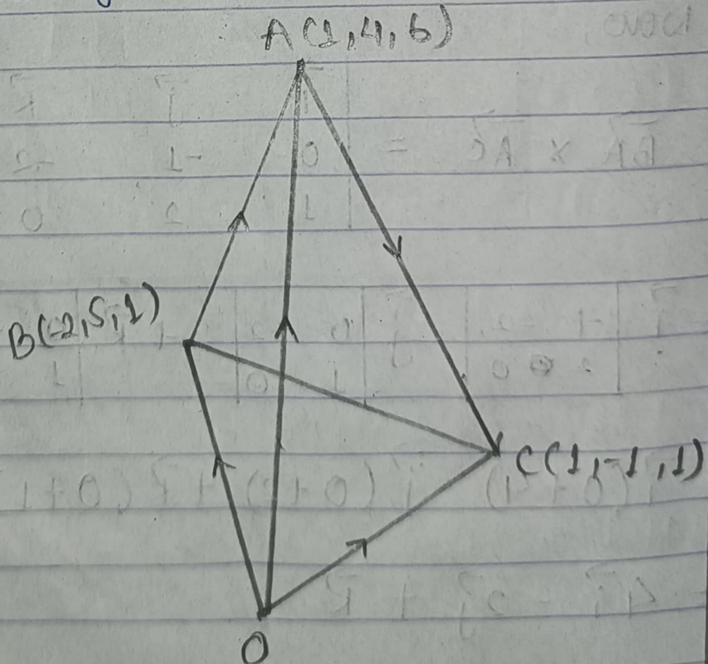
b) (1,4,6), (-2,5,1) and (1,-1,1)

Solution

Let ABC be a triangle whose vertices are A(1,4,6), B(-2,5,1) and C(1,-1,1) respectively.

Let O be the origin. Join OA, OB & OC.



$\vec{OA} = (1,4,6)$

$\vec{OB} = (-2,5,1)$

$\vec{OC} = (1,-1,1)$

In ΔOBA

$\vec{OB} + \vec{BA} = \vec{OA}$ [Using triangle law of vector addition]

Or, $\vec{BA} = \vec{OA} - \vec{OB}$

Or, $\vec{BA} = (1,4,6) - (-2,5,1)$

$\therefore \vec{BA} = (3,-1,5)$

In ΔOAC,

~~OB~~ $\vec{OA} + \vec{AC} = \vec{OC}$ [Using triangle law of vector addition]

or, $\vec{AC} = \vec{OC} - \vec{OA}$

or, $\vec{AC} = (1,-1,1) - (1,4,6)$

or, $\vec{AC} = (1-1, -1-4, 1-6)$

$\therefore \vec{AC} = (0,-5,-5)$

Now,

$$\vec{BA} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 5 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & 5 \\ -5 & -5 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & 5 \\ 0 & -5 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & -1 \\ 0 & -5 \end{vmatrix}$$

$$= \vec{i}(5+25) - \vec{j}(-15-0) + \vec{k}(-15-0)$$

$$= 30\vec{i} + 15\vec{j} - 15\vec{k}$$

Now,
we know that,

$$\text{Area of triangle} = \frac{|\vec{BA} \times \vec{AC}|}{2} \text{ square units}$$

$$= \frac{\sqrt{(30)^2 + (15)^2 + (-15)^2}}{2} \text{ square units}$$

$$= \frac{\sqrt{1350}}{2} \text{ square units}$$

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7. A pair of vectors is given below in each of the following cases:

i) $\vec{a} = (2, 3, -1)$ and $\vec{b} = (3, -1, 2)$

ii) $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j} - \vec{k}$

iii) $\vec{a} = -2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{b} = \vec{i} - 4\vec{j} - 2\vec{k}$

For each of the above pair ~~ve~~ of vectors:

a) find $\vec{a} \times \vec{b}$

b) find the unit vector perpendicular to $\vec{a}$ and $\vec{b}$.

c) find the sine of the angle between $\vec{a}$ and $\vec{b}$.

Solution.

i)

a)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}$$

$$= \vec{i}(6-1) - \vec{j}(4+3) + \vec{k}(-2-9)$$
$$= 5\vec{i} - 7\vec{j} - 11\vec{k}$$

$$\therefore \vec{a} \times \vec{b} = (5, -7, -11)$$

b)

$$|\vec{a} \times \vec{b}| = \sqrt{(5)^2 + (-7)^2 + (-11)^2} = \sqrt{195}$$

Now,

The unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is

$$= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$= \frac{(5, -7, -11)}{\sqrt{195}}$$

$$= \frac{5}{\sqrt{195}}\vec{i} - \frac{7}{\sqrt{195}}\vec{j} - \frac{11}{\sqrt{195}}\vec{k}$$

c)

$$|\vec{a}| = \sqrt{2^2+3^2+(-1)^2} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{3^2+(-1)^2+2^2} = \sqrt{14}$$

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{\sqrt{195}}{\sqrt{14}\,\sqrt{14}}$$

$$\therefore \sin\theta = \frac{\sqrt{195}}{14}$$

ii)

a)

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 1 & 3 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix}$$

$$= \vec{i}(2-9) - \vec{j}(-1-3) + \vec{k}(3+2)$$

$$= -7\vec{i} + 4\vec{j} + 5\vec{k}$$

$$\therefore \vec{a}\times\vec{b} = (-7, 4, 5)$$

b)

$$|\vec{a} \times \vec{b}| = \sqrt{(-7)^2 + (4)^2 + (5)^2} = 3\sqrt{10}$$

Now,

The unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is

$$= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$$= \frac{(-7, 4, 5)}{3\sqrt{10}}$$

$$= -\frac{7}{3\sqrt{10}}\vec{i} + \frac{4}{3\sqrt{10}}\vec{j} + \frac{5}{3\sqrt{10}}\vec{k}$$

c)

$$|\vec{a}| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{(1)^2 + (3)^2 + (1)^2} = \sqrt{11}$$

we know,

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

$$= \frac{3\sqrt{10}}{\sqrt{14}\sqrt{11}}$$

$$\therefore \sin\theta = \frac{3\sqrt{385}}{77}$$

iii)

a)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & -1 \\ 1 & -4 & -2 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 3 & -1 \\ -4 & -2 \end{vmatrix} - \vec{j} \begin{vmatrix} -2 & -1 \\ 1 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} -2 & 3 \\ 1 & -4 \end{vmatrix}$$

$$= \vec{i}(-6-4) - \vec{j}(4+1) + \vec{k}(8-3)$$

$$= -10\vec{i} - 5\vec{j} + 5\vec{k}$$

$$\therefore \vec{a} \times \vec{b} = (-10, -5, 5)$$

b)

$$|\vec{a} \times \vec{b}| = \sqrt{(-10)^2 + (-5)^2 + (5)^2} = 5\sqrt{6}$$

Now,

The unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is $= \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$$= \frac{(-10, -5, 5)}{5\sqrt{6}}$$

$$= -\frac{2}{\sqrt{6}}\vec{i} - \frac{1}{\sqrt{6}}\vec{j} + \frac{1}{\sqrt{6}}\vec{k}$$

c)

$$|\vec{a}| = \sqrt{(-2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{(1)^2 + (-4)^2 + (-2)^2} = \sqrt{21}$$

We know,

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{5\sqrt{6}}{\sqrt{14}\,\sqrt{21}}$$

$$= \frac{5\sqrt{6}}{7\sqrt{6}}$$

$$\therefore \sin\theta = \frac{5}{7}$$

9. Prove that a unit vector perpendicular to each of the vectors $2\vec{i} - \vec{j} + \vec{k}$ and $3\vec{i} + 4\vec{j} - \vec{k}$ is $\frac{1}{\sqrt{155}}(-3\vec{i} + 5\vec{j} + 11\vec{k})$ and the sine of an angle between them is

$$\sqrt{\frac{155}{156}}.$$

Solution

Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$

$\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} -1 & 1 \\ 4 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

$= \vec{i}(1-4) - \vec{j}(-2-3) + \vec{k}(8+3)$

$= -3\vec{i} + 5\vec{j} + 11\vec{k}$

$\therefore \vec{a} \times \vec{b} = (-3, 5, 11)$

$|\vec{a} \times \vec{b}| = \sqrt{(-3)^2 + (5)^2 + (11)^2} = \sqrt{155}$

the unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$= \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$= \dfrac{(-3, 5, 11)}{\sqrt{155}}$

$= \dfrac{1}{\sqrt{155}}(-3\vec{i} + 5\vec{j} + 11\vec{k})$

proved

Also,

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + (1)^2} = \sqrt{6}$$

$$|\vec{b}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$$

We know,

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{\sqrt{155}}{\sqrt{6}\,\sqrt{26}}$$

$$= \frac{\sqrt{155}}{\sqrt{156}}$$

$$= \sqrt{\frac{155}{156}}$$

proved