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$= 2(-5\times5 - 3\times0) - 3(2\times5 - 1\times0) + 4(2\times3 - 1\times(-5))$
$= 2(-25-0) - 3(10-0) + 4(6+5)$
$= -50 - 30 + 44$
$= -36$

Since, $|A| \neq 0$, the matrix is non-singular.

Exercise 8(C)

1. Solve the following system of equations by using Cramer's rule.

a) $3x - 5y = 14$, $2x - 3y = 10$

Solution

The given equations are:

$3x - 5y = 14$ —— (I)
$2x - 3y = 10$ —— (II)

Coeff. of x	Coeff. of y	Constant terms
3	−5	14
2	−3	10

$$D = \begin{vmatrix} 3 & -5 \\ 2 & -3 \end{vmatrix}$$

$= 3\times(-3) - 2\times(-5)$
$= -9 - (-10)$
$= -9 + 10$
$= 1 \neq 0$. So, the solution exists.

$$D_1 = \begin{vmatrix} 14 & -5 \\ 10 & -3 \end{vmatrix}$$

$= 14 \times (-3) - 10 \times (-5)$

$= -42 - (-50)$

$= -42 + 50$

$= 8$

$$D_2 = \begin{vmatrix} 3 & 14 \\ 2 & 10 \end{vmatrix}$$

$= 3 \times 10 - 2 \times 14$

$= 30 - 28$

$= 2$

Now, By using Cramer's Rule,

$$x = \frac{D_1}{D} = \frac{8}{1} = 8$$

$$y = \frac{D_2}{D} = \frac{2}{1} = 2$$

$\therefore x = 8$ & $y = 2$

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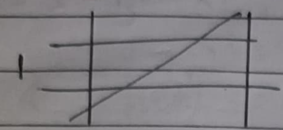
b) $x + 2y = 4, \; 3x - y = -2$

Solution

The given equations are:

$x + 2y = 4$ —— (I)

$3x - y = -2$ —— (II)



Coeff. of x	Coeff of y	Constant terms
1	2	4
3	-1	-2

$$D = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$$

$= 1 \times (-1) - 3 \times 2$

$= -1 - 6$

$= -7 \neq 0$ so the solution exists.

$$D_1 = \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix}$$

$= 4 \times (-1) - (-2) \times 2$

$= -4 + 4$

$= 0$

$$D_2 = \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix}$$

$= 1 \times (-2) - 3 \times 4$

$= -2 - 12$

$= -14$

Now, By using cramers Rule,

$$x = \frac{D_1}{D} = \frac{0}{-7} = 0$$

$$y = \frac{D_2}{D} = \frac{-14}{-7} = 2$$

$\therefore x = 0$ & $y = 2$

c) $x-y+z=-3,\ x+y+z=1,\ 3x-4y-z=1$

Solution

The given equations are:

$x-y+z=-3$ ——— (I)

$x+y+z=1$ ——— (II)

$3x-4y-z=1$ ——— (III)

~~coeff. of x~~	~~coeff. of y~~	~~constant term~~

coeff. of x	coeff. of y	coeff. of z	constant term
1	−1	1	−3
1	1	1	1
3	−4	−1	1

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 3 & -4 & -1 \end{vmatrix}$$

Expanding D using R_1

$$= 1\begin{vmatrix} 1 & 1 \\ -4 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 9 & -4 \end{vmatrix}$$

$= 1(1\times(-1) - (-4)\times 1) + 1(1\times(-1) - 3\times 1) + 1(1\times(-4) - 3\times 1)$

$= 1(-1+4) + 1(-1-3) + 1(-4-3)$

$= 3 - 4 - 7$

$= -8 \neq 0$, so, the solution exists.

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$$D_1 = \begin{vmatrix} -3 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -4 & -1 \end{vmatrix}$$

Expanding D_1 using R_1

$$= -3\begin{vmatrix} 1 & 1 \\ -4 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix}$$

$= -3(1\times(-1) - (-4)\times 1) + 1(1\times(-1) - 1\times 1) + 1(1\times(-4) - 1\times 1)$
$= -3(-1+4) + 1(-1-1) + 1(-4-1)$
$= -9 - 2 - 5$
$= -16$

$$D_2 = \begin{vmatrix} 1 & -3 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & -1 \end{vmatrix}$$

Expanding D_2 using R_1

$$= 1\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - (-3)\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}$$

$= 1(1\times(-1) - 1\times 1) + 3(1\times(-1) - 3\times 1) + 1(1\times 1 - 3\times 1)$
$= 1(-1-1) + 3(-1-3) + 1(1-3)$
$= -2 - 12 - 2$
$= -16$

$$D_3 = \begin{vmatrix} 1 & -1 & -3 \\ 1 & 1 & 1 \\ 3 & -4 & 1 \end{vmatrix}$$

Expanding D_3 using R_1

$$= 1\begin{vmatrix} 1 & 1 \\ -4 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix}$$

$$= 1(1\times1 - (-4)\times1) + 1(1\times1 - 3\times1) - 3(1\times(-4) - 3\times1)$$

$$= 1(1+4) + 1(1-3) - 3(-4-3)$$

$$= 5 - 2 + 21$$

$$= 24$$

Now, By using cramer's Rule.

$$x = \frac{D_1}{D} = \frac{-16}{-8} = 2$$

$$y = \frac{D_2}{D} = \frac{-16}{-8} = 2$$

$$z = \frac{D_3}{D} = \frac{24}{-8} = -3$$

$\therefore x = 2,\ y = 2\ \&\ z = -3$

d) $x - 2y - 3z = 1,\ 2x + y + z = 6,\ x + 3y - 2z = 13$

Solution

The given equations are:

$x - 2y - 3z = 1$ —— (i)

$2x + y + z = 6$ —— (ii)

$x + 3y - 2z = 13$ —— (iii)

Coeff. of x	Coeff of y	Coeff. of z	constant term
1	-2	-3	1
2	1	1	6
1	3	-2	13

$$D = \begin{vmatrix} 1 & -2 & -3 \\ 2 & 1 & 1 \\ 1 & 3 & -2 \end{vmatrix}$$

Expanding D by using R_1

$$= 1\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$= 1(1\times(-2) - 1\times 3) + 2(2\times(-2) - 1\times 1) - 3(2\times 3 - 1\times 1)$

$= 1(-2-3) + 2(-4-1) - 3(6-1)$

$= -5 - 10 - 15$

$= -30 \neq 0$. So, the solution exists.

$$D_1 = \begin{vmatrix} 1 & -2 & -3 \\ 6 & 1 & 1 \\ 13 & 3 & -2 \end{vmatrix}$$

Expanding D_1 by using R_1

$$= 1\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 6 & 1 \\ 13 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 6 & 1 \\ 13 & 3 \end{vmatrix}$$

$= 1(1\times(-2) - 3\times 1) + 2(6\times(-2) - 13\times 1) - 3(6\times 3 - 13\times 1)$

$= 1(-2-3) + 2(-12-13) - 3(18-13)$

$= -5 - 50 - \cancel{45}\ 15$

$= \cancel{-100}\ -70$

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$$D_2 = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 6 & 1 \\ 1 & 13 & -2 \end{vmatrix}$$

Expanding D_2 using R_1

$$= 1\begin{vmatrix} 6 & 1 \\ 13 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 6 \\ 1 & 13 \end{vmatrix}$$

$$= 1(6\times(-2) - 13\times1) - 1(2\times(-2) - 1\times1) - 3(2\times13 - 1\times6)$$
$$= 1(-12-13) - 1(-4-1) - 3(26-6)$$
$$= -25 + 5 - 60$$
$$= -80$$

$$D_3 = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & 6 \\ 1 & 3 & 13 \end{vmatrix}$$

Expanding D_3 by using R_1

$$= 1\begin{vmatrix} 1 & 6 \\ 3 & 13 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 6 \\ 1 & 13 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 1(1\times13 - 3\times6) + 2(2\times13 - 1\times6) + 1(2\times3 - 1\times1)$$
$$= 1(13-18) + 2(26-6) + 1(6-1)$$
$$= -5 + 40 + 5$$
$$= 40$$

Now, by using Cramer's rule,

$$x = \frac{D_1}{D} = \frac{-70}{-30} = \frac{7}{3}$$

$$y = \frac{D_2}{D} = \frac{-80}{-30} = \frac{8}{3}$$

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$z = \frac{D_3}{D} = \frac{40}{-30} = -\frac{4}{3}$

$\therefore x = \frac{7}{3}, \ y = \frac{8}{3} \ \& \ z = -\frac{4}{3}$

---

e) $3x + 5z = 14, \ 2x + y - 3z = 3, \ x + y + z = 4$

Solution

The given equations are:

$3x + 5z = 14$ —— (I)

$2x + y - 3z = 3$ —— (II)

$x + y + z = 4$ —— (III)

Coeff of x	Coeff of y	Coeff of z	Constant term
3	0	5	14
2	1	-3	3
1	1	1	4

$$D = \begin{vmatrix} 3 & 0 & 5 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

Expanding D by using R_1

$$= 3\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$= 3(1\times1 - 1\times(-3)) - 0 + 5(2\times1 - 1\times1)$

$= 3(1+3) - 0 + 5(2-1)$

$= 12 + 5$

$= 17 \neq 0$. So, the solution exists.

$$D_1 = \begin{vmatrix} 14 & 0 & 5 \\ 3 & 1 & -3 \\ 4 & 1 & 1 \end{vmatrix}$$

Expanding D_1 by using R_1

$$= 14\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 3 & -3 \\ 4 & 1 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix}$$

$$= 14(1\times1 - 1\times(-3)) - 0 + 5(3\times1 - 4\times1)$$

$$= 14(1+3) - 0 + 5(3-4)$$

$$= \cancel{56+5}\; 56-5$$

$$= 51$$

$$D_2 = \begin{vmatrix} 3 & 14 & 5 \\ 2 & 3 & -3 \\ 1 & 4 & 1 \end{vmatrix}$$

Expanding D_2 by using R_1

$D_3 =$

$$= 3\begin{vmatrix} 3 & -3 \\ 4 & 1 \end{vmatrix} - 14\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + 5\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= 3(3\times1 - 4\times(-3)) - 14(2\times1 - 1\times(-3)) + 5(2\times4 - 1\times3)$$

$$= 3(3+12) - 14(2+3) + 5(8-3)$$

$$= 45 - 70 + 25$$

$$= 0$$

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$$D_3 = \begin{vmatrix} 3 & 0 & 14 \\ 2 & 1 & 3 \\ 1 & 1 & 4 \end{vmatrix}$$

Expanding D_3 by using R_1

$$= 3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 14\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 3(1\times4 - 1\times3) - 0 + 14(2\times1 - 1\times1)$$
$$= 3 - 0 + 14$$
$$= 17$$

Now, By using cramer's rule,

$$x = \frac{D_1}{D} = \frac{51}{17} = 3$$

$$y = \frac{D_2}{D} = \frac{0}{17} = 0$$

$$z = \frac{D_3}{D} = \frac{17}{17} = 1$$

$\therefore x = 3,\ y = 0\ \&\ z = 1$

f) $2x - y + z = -2,\ x + y - 2z = -9,\ x + 2y + z = 9$

Solution

The given equations are:

$$2x - y + z = -2 \quad \text{——} \quad (\text{I})$$
$$x + y - 2z = -9 \quad \text{——} \quad (\text{II})$$
$$x + 2y + z = 9 \quad \text{——} \quad (\text{III})$$

Coeff. of x	coeff of y	coeff of z	Constant term
2	-1	1	-2
1	1	-2	-9
1	2	1	9

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{vmatrix}$$

Expanding D by using R_1

$$= 2\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$= 2(1\times1 - 2\times(-2)) + 1(1\times1 - 1\times(-2)) + 1(1\times2 - 1\times1)$

$= 2(1+4) + 1(1+2) + 1(2-1)$

$= 10 + 3 + 1$

$= 14 \neq 0$. So, the solution exists.

$$D_1 = \begin{vmatrix} -2 & -1 & 1 \\ -9 & 1 & -2 \\ 9 & 2 & 1 \end{vmatrix}$$

Expanding D_1 by using R_1

$$= -2\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -9 & -2 \\ 9 & 1 \end{vmatrix} + 1\begin{vmatrix} -9 & 1 \\ 9 & 2 \end{vmatrix}$$

$= -2(1\times1 - 2\times(-2)) + 1(-9\times1 - 9\times(-2)) + 1(-9\times2 - 9\times1)$

$= -2(1+4) + 1(-9+18) + 1(-18-9)$

$= -10 + 9 - 27$

$= -28$

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$$D_2 = \begin{vmatrix} 2 & -2 & 1 \\ 1 & -9 & -2 \\ 1 & 9 & 1 \end{vmatrix}$$

Expanding D_2 by using R_1

$$= 2\begin{vmatrix} -9 & -2 \\ 9 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & -9 \\ 1 & 9 \end{vmatrix}$$

$$= 2(-9\times1 - 9\times(-2)) + 2(1\times1 - 1\times(-2)) + 1(1\times9 - 1\times(-9))$$
$$= 2(-9 + 18) + 2(1+2) + 1(9+9)$$
$$= 18 + 6 + 18$$
$$= 42$$

$$D_3 = \begin{vmatrix} 2 & -1 & -2 \\ 1 & 1 & -9 \\ 1 & 2 & 9 \end{vmatrix}$$

Expanding D_3 by using R_1

$$= 2\begin{vmatrix} 1 & -9 \\ 2 & 9 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -9 \\ 1 & 9 \end{vmatrix} + (-2)\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 2(1\times9 - 2\times(-9)) + 1(1\times9 - 1\times(-9)) - 2(1\times2 - 1\times1)$$
$$= 2(9 + 18) + 1(9+9) - 2(2-1)$$
$$= 54 + 18 - 2$$
$$= 70$$

Now, by using cramer's rule,

$$x = \frac{D_1}{D} = \frac{-28}{14} = -2$$

$$y = \frac{D_2}{D} = \frac{42}{14} = 3$$

$$z = \frac{D_3}{D} = \frac{70}{14} = 5$$

$\therefore x = -2,\ y = 3\ \&\ z = 5$

Exercise 8(C)

1. Solve the following system of equations by inverse matrix method.

c) $x-y+z=-3$, $x+y+z=1$, $3x-4y-z=1$

Solution

The given equations are:

$x-y+z=-3$ ——— (i)

$x+y+z=1$ ——— (ii)

$3x-4y-z=1$ ——— (iii)

Expressing the above equations in matrix form as:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Put } A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 3 & -4 & -1 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

Then, $AX = B$

i.e. $X = A^{-1}B$ ——— (A)

For A^{-1}

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 3 & -4 & -1 \end{vmatrix}$$

Expanding $|A|$ by using R_1

$$= 1\begin{vmatrix}1 & 1\\ -4 & -1\end{vmatrix} - (-1)\begin{vmatrix}1 & 1\\ 3 & -1\end{vmatrix} + 1\begin{vmatrix}1 & 1\\ 3 & -4\end{vmatrix}$$

$= 1(1\times(-1) - (-4)\times 1) + 1(1\times(-1) - 3\times 1) + 1(1\times(-4) - 3\times 1)$

$= 1(-1+4) + 1(-1-3) + 1(-4-3)$

$= 3 - 4 - 7$

$= -8 \neq 0$. So, A^{-1} exists

Let the cofactor matrix be $\begin{bmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$

$A_{11} = +\begin{vmatrix}1 & 1\\ -4 & -1\end{vmatrix} = 1\times(-1) - (-4)\times 1 = -1+4 = 3$

$A_{12} = -\begin{vmatrix}1 & 1\\ 3 & -1\end{vmatrix} = -(1\times(-1) - 3\times 1) = -(-1-3) = 4$

$A_{13} = +\begin{vmatrix}1 & 1\\ 3 & -4\end{vmatrix} = 1\times(-4) - 3\times 1 = -4-3 = -7$

$A_{21} = -\begin{vmatrix}-1 & 1\\ -4 & -1\end{vmatrix} = -(-1\times -1 - (-4)\times 1) = -(1+4) = -5$

$A_{22} = +\begin{vmatrix}1 & 1\\ 3 & -1\end{vmatrix} = 1\times(-1) - 3\times 1 = -1-3 = -4$

$A_{23} = -\begin{vmatrix}1 & -1\\ 3 & -4\end{vmatrix} = -(1\times(-4) - 3\times(-1)) = 1$

$A_{31} = +\begin{vmatrix}-1 & 1\\ 1 & 1\end{vmatrix} = -1\times 1 - 1\times 1 = -1-1 = -2$

$A_{32} = -\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix} = -(1\times 1 - 1\times 1) = -1+1 = 0$

$A_{33} = +\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix} = 1\times 1 - 1\times(-1) = 1+1 = 2$

Now, the cofactor matrix becomes,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 4 & -7 \\ -5 & -4 & 1 \\ -2 & 0 & 2 \end{bmatrix}$$

Then,

Adjoint of A = (cofactor matrix)^T

$$= \begin{bmatrix} 3 & 4 & -7 \\ -5 & -4 & 1 \\ -2 & 0 & 2 \end{bmatrix}^T$$

$$\therefore \text{Adj. } A = \begin{bmatrix} 3 & -5 & -2 \\ 4 & -4 & 0 \\ -7 & 1 & 2 \end{bmatrix}$$

Then,

Inverse of matrix $= \dfrac{\text{Adj. } A}{|A|}$, $|A| \neq 0$

$$= \frac{\begin{bmatrix} 3 & -5 & -2 \\ 4 & -4 & 0 \\ -7 & 1 & 2 \end{bmatrix}}{-8}$$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{3}{8} & \frac{5}{8} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{7}{8} & -\frac{1}{8} & -\frac{1}{4} \end{bmatrix}$$

From (A)

$$X = A^{-1} B$$

or, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3}{8} & \frac{5}{8} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{7}{8} & -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3}{8} \times (-3) + \frac{5}{8} \times 1 + \frac{1}{4} \times 1 \\ -\frac{1}{2} \times (-3) + \frac{1}{2} \times 1 + 0 \times 1 \\ \frac{7}{8} \times (-3) - \frac{1}{8} \times 1 - \frac{1}{4} \times 1 \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$

By the definition of equality of matrices,

$\therefore x = 2,\ y = 2\ \&\ z = -3$

---

d) $x - 2y - 3z = 1,\ 2x + y + z = 6,\ x + 3y - 2z = 13$

Solution

The given equations are:

$x - 2y - 3z = 1$ —— (I)

$2x + y + z = 6$ —— (II)

$x + 3y - 2z = 13$ —— (III)

Expressing the above equations into matrix form as:

$$\begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 13 \end{bmatrix}$$

Put $A = \begin{bmatrix} 1 & -2 & -3 \\ 2 & 1 & 1 \\ 1 & 3 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 6 \\ 13 \end{bmatrix}$

Then, $AX = B$

i.e. $X = A^{-1}B$ —— (A)

<u>For A^{-1}</u>

$$|A| = \begin{vmatrix} 1 & -2 & -3 \\ 2 & 1 & 1 \\ 1 & 3 & -2 \end{vmatrix}$$

<u>Expanding $|A|$ by using R_1</u>

$$= 1\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 1(1\times(-2) - 3\times1) + 2(2\times(-2) - 1\times1) - 3(2\times3 - 1\times1)$$

$$= 1(-2-3) + 2(-4-1) - 3(6-1)$$

$$= -5 - 10 - 15$$

$= -30 \neq 0$. so, A^{-1} exists.

Let the cofactor matrix be $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$A_{11} = +\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = 1\times(-2) - 3\times1 = -2-3 = -5$$

$$A_{12} = -\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -(2\times-2 - 1\times1) = 5$$

$$A_{13} = +\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 2\times3 - 1\times1 = 6-1 = 5$$

$$A_{21} = -\begin{vmatrix} -2 & -3 \\ 3 & -2 \end{vmatrix} = -(-2\times-2 - 3\times(-3)) = -4-9 = -13$$

$$A_{22} = +\begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} = 1\times(-2) - 1\times(-3) = -2+3 = 1$$

$$A_{23} = -\begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} = -(1\times 3 - 1\times(-2)) = -(3+2) = -5$$

$$A_{31} = +\begin{vmatrix} -2 & -3 \\ 1 & 1 \end{vmatrix} = -2\times 1 - 1\times(-3) = -2+3 = 1$$

$$A_{32} = -\begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = -(1\times 1 - 2\times(-3)) = -1-6 = -7$$

$$A_{33} = +\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1\times 1 - 2\times(-2) = 1+4 = 5$$

Now, the cofactor matrix becomes,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -5 & 5 & 5 \\ -13 & 1 & -5 \\ 1 & -7 & 5 \end{bmatrix}$$

Also, Adjoint of A = (cofactor matrix)^T

$$= \begin{bmatrix} -5 & 5 & 5 \\ -13 & 1 & -5 \\ 1 & -7 & 5 \end{bmatrix}^T$$

$$\therefore \text{Adj. } A = \begin{bmatrix} -5 & -13 & 1 \\ 5 & 1 & -7 \\ 5 & -5 & 5 \end{bmatrix}$$

Then,

Inverse of matrix $= \dfrac{\text{Adj. } A}{|A|}, \; |A| \neq 0$

$$= \frac{\begin{bmatrix} -5 & -13 & 1 \\ 5 & 1 & -7 \\ 5 & -5 & 5 \end{bmatrix}}{-30}$$

$$= \begin{bmatrix} \frac{-5}{-30} & \frac{-13}{-30} & \frac{1}{-30} \\ \frac{5}{-30} & \frac{1}{-30} & \frac{-7}{-30} \\ \frac{5}{-30} & \frac{-5}{-30} & \frac{5}{-30} \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{13}{30} & -\frac{1}{30} \\ -\frac{1}{6} & -\frac{1}{30} & \frac{7}{30} \\ -\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \end{bmatrix}$$

From equation (A)

$$X = A^{-1} B$$

$$\text{or } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1/6 & 13/30 & -1/30 \\ -1/6 & -1/30 & 7/30 \\ -1/6 & 1/6 & -1/6 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 13 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \times 1 + \frac{13}{30} \times 6 - \frac{1}{30} \times 13 \\ -\frac{1}{6} \times 1 - \frac{1}{30} \times 6 + \frac{7}{30} \times 13 \\ -\frac{1}{6} \times 1 + \frac{1}{6} \times 6 - \frac{1}{6} \times 13 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{8}{3} \\ -\frac{4}{3} \end{bmatrix}$$

By the definition of equality of matrices,

$$\therefore x = \frac{7}{3}, \; y = \frac{8}{3}, \; z = -\frac{4}{3}$$

e) $3x+5z=14$, $2x+y-3z=3$, $x+y+z=4$

Solution

The given equations are:

$3x+5z=14$ ——— (1)

$2x+y-3z=3$ ——— (2)

$x+y+z=4$ ——— (3)

Expressing the above equations into matrix form as:

$$\begin{bmatrix} 3 & 0 & 5 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 3 \\ 4 \end{bmatrix}$$

Put $A = \begin{bmatrix} 3 & 0 & 5 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ 3 \\ 4 \end{bmatrix}$

Then, $AX = B$

i.e. $X = A^{-1}B$

For A^{-1}

$$|A| = \begin{vmatrix} 3 & 0 & 5 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

Expanding $|A|$ by using R_1

$$= 3\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$= 3(1\times1 - 1\times(-3)) - 0 + 5(2\times1 - 1\times1)$

$= 3(1+3) + 5(2-1)$

$= 12 + 5$

$= 17 \neq 0$. So, A^{-1} exists.

Let the cofactor matrix be $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$A_{11} = + \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 1\times1 - 1\times(-3) = 1+3 = 4$

$A_{12} = - \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -(2\times1 - (-3)\times1) = -2-3 = -5$

$A_{13} = + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2\times1 - 1\times1 = 2-1 = 1$

$A_{21} = - \begin{vmatrix} 0 & 5 \\ 1 & 1 \end{vmatrix} = -(0\times1 - 1\times5) = 5$

$A_{22} = + \begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} = 3\times1 - 1\times5 = 3-5 = -2$

$A_{23} = - \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} = -(3\times1 - 1\times0) = -3$

$A_{31} = + \begin{vmatrix} 0 & 5 \\ 1 & -3 \end{vmatrix} = 0\times(-3) - 1\times5 = -5$

$A_{32} = - \begin{vmatrix} 3 & 5 \\ 2 & -3 \end{vmatrix} = -(3\times(-3) - 2\times5) = 9+10 = 19$

$A_{33} = + \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} = 3\times1 - 2\times0 = 3-0 = 3$

Now, the cofactor matrix becomes,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 4 & -5 & 1 \\ 5 & -2 & -3 \\ -5 & 19 & 3 \end{bmatrix}$$

Adjoint of A = (cofactor matrix)^T

$$= \begin{bmatrix} 4 & -5 & 1 \\ 5 & -2 & -3 \\ -5 & 19 & 3 \end{bmatrix}^T$$

$$\therefore \text{Adj. } A = \begin{bmatrix} 4 & 5 & -5 \\ -5 & -2 & 19 \\ 1 & -3 & 3 \end{bmatrix}$$

Inverse of matrix $= \dfrac{\text{Adj. } A}{|A|}$, $|A| \neq 0$

$$= \frac{\begin{bmatrix} 4 & 5 & -5 \\ -5 & -2 & 19 \\ 1 & -3 & 3 \end{bmatrix}}{17}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{4}{17} & \frac{5}{17} & -\frac{5}{17} \\ -\frac{5}{17} & -\frac{2}{17} & \frac{19}{17} \\ \frac{1}{17} & -\frac{3}{17} & \frac{3}{17} \end{bmatrix}$$

Now, from equation (A),

$$X = A^{-1} B$$

or, $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{17} & \frac{5}{17} & -\frac{5}{17} \\ -\frac{5}{17} & -\frac{2}{17} & \frac{19}{17} \\ \frac{1}{17} & -\frac{3}{17} & \frac{3}{17} \end{bmatrix} \begin{bmatrix} 14 \\ 3 \\ 4 \end{bmatrix}$$

or, $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{17} \times 14 + \frac{5}{17} \times 3 - \frac{5}{17} \times 4 \\ -\frac{5}{17} \times 14 - \frac{2}{17} \times 3 + \frac{19}{17} \times 4 \\ \frac{1}{17} \times 14 - \frac{3}{17} \times 3 + \frac{3}{17} \times 4 \end{bmatrix}$$

or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

By the definition of equality of matrices,
$\therefore x = 3,\ y = 0\ \&\ z = 1$

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b) $2x - y + z = -2,\ x + y - 2z = -9,\ x + 2y + z = 9$

Solution

The given equations are :

$2x - y + z = -2$ ——— (i)

$x + y - 2z = -9$ ——— (ii)

$x + 2y + z = 9$ ——— (iii)

Expressing the above equations into matrix form as:

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -9 \\ 9 \end{bmatrix}$$

Put $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} -2 \\ -9 \\ 9 \end{bmatrix}$

Then, $AX = B$

i.e. $X = A^{-1}B$ ——— (A)

For A^{-1}

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{vmatrix}$$

Expanding $|A|$ by using R_1

$$= 2\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$= 2(1\times1 - 2\times(-2)) + 1(1\times1 - 1\times(-2)) + 1(1\times2 - 1\times1)$

$= 2(1+4) + 1(1+2) + 1(2-1)$

$= 10 + 3 + 1$

$= 14 \neq 0$. so A^{-1} exists.

Let the cofactor matrix be $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$A_{11} = +\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1\times1 - 2\times(-2) = 1+4 = 5$

$A_{12} = -\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = -(1\times1 - 1\times(-2)) = -1-2 = -3$

$A_{13} = +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1\times2 - 1\times1 = 2-1 = 1$

$A_{21} = -\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -(-1\times1 - 2\times1) = 1+2 = 3$

$A_{22} = +\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2\times1 - 1\times1 = 2-1 = 1$

$A_{23} = -\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = -(2\times2 - 1\times(-1)) = -4-1 = -5$

$A_{31} = +\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = -1\times-2 - 1\times1 = 2-1 = 1$

$A_{32} = -\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -(2\times -2 - 1\times 1) = 4+1 = 5$

$A_{33} = +\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2\times 1 - 1\times(-1) = 2+1 = 3$

Now, the cofactor matrix becomes,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 5 & -3 & 1 \\ 3 & 1 & -5 \\ 1 & 5 & 3 \end{bmatrix}$$

Adjoint of A = (Cofactor matrix)^T

$$= \begin{bmatrix} 5 & -3 & 1 \\ 3 & 1 & -5 \\ 1 & 5 & 3 \end{bmatrix}^T$$

$$\therefore \text{Adj. } A = \begin{bmatrix} 5 & 3 & 1 \\ -3 & 1 & 5 \\ 1 & -5 & 3 \end{bmatrix}$$

Inverse of Matrix $= \dfrac{\text{Adj. } A}{|A|}$, $|A| \neq 0$

$$= \frac{\begin{bmatrix} 5 & 3 & 1 \\ -3 & 1 & 5 \\ 1 & -5 & 3 \end{bmatrix}}{14}$$

$$\therefore A^{-1} = \begin{bmatrix} 5/14 & 3/14 & 1/14 \\ -3/14 & 1/14 & 5/14 \\ 1/14 & -5/14 & 3/14 \end{bmatrix}$$

From equation Ⓐ

$$X = A^{-1} B$$

or, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{14} & \frac{3}{14} & \frac{1}{14} \\ -\frac{3}{14} & \frac{1}{14} & \frac{5}{14} \\ \frac{1}{14} & -\frac{5}{14} & \frac{3}{14} \end{bmatrix} \begin{bmatrix} -2 \\ -9 \\ 9 \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{14} \times (-2) + \frac{3}{14} \times (-9) + \frac{1}{14} \times 9 \\ -\frac{3}{14} \times (-2) + \frac{1}{14} \times (-9) + \frac{5}{14} \times 9 \\ \frac{1}{14} \times (-2) - \frac{5}{14} \times (-9) + \frac{3}{14} \times 9 \end{bmatrix}$

or, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

By the definition of equality of matrises,

$\therefore$ $x = -2$, $y = 3$ & $z = 5$

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2. One unit of medicine A contains 10 units of chemical x and 6 units of chemical y unit of medicine B contains 15 units of chemical x and 8 units of chemical y. Total chemical requirements are 110 units of x and 62 units of y. Find out by using determinant the number of units of medicine of each type produced.

Solution

	Chemical x ~~Medicine A~~	chemical y	Total Requirement
Medicine A	10	6	110
Medicine B	15	8	62

The given information can be expressed in tabular form as:

	Medicine A	Medicine B	Total Requirement
Chemical x	10	15	110
Chemical y	6	8	62

Then the equation can be formed as:

$$10A + 15B = 110 \text{ ——— } (I)$$
$$6A + 8B = 62 \text{ ——— } (II)$$

Coeff. of A	Coeff. of B	Constant Term
10	15	110
6	8	62

$$D = \begin{vmatrix} 10 & 15 \\ 6 & 8 \end{vmatrix}$$
$$= 10\times8 - 6\times15$$
$$= 80 - 90$$
$= -10 \neq 0$, So, the solution exists.

$$D_1 = \begin{vmatrix} 110 & 15 \\ 62 & 8 \end{vmatrix}$$
$$= 110\times8 - 62\times15$$
$$= 880 - \cancel{975}\ 930$$
$$= \cancel{-95}\ -50$$

$$D_2 = \begin{vmatrix} 10 & 110 \\ 6 & 62 \end{vmatrix}$$

$= 10 \times 62 - 6 \times 110$

$= 620 - 660$

$= -40$

Now, By using determinant method,

$A = \frac{D_1}{D} = \frac{-50}{-10} = 5$

$B = \frac{D_2}{D} = \frac{-40}{-10} = 4$

Therefore 5 units of Medicine A and 4 units of medicine B is produced.

3\. 40 people are employed in a certain factory. If the daily total wage bill of the factory is Rs. 3625 and the daily wage of men and women are Rs. 100 and Rs. 75 respectively, find the number of men and women employed in the factory using cramer's rule.

Solution

Let the number of men and women employed in the factory be x and y respectively.

From the question, equation can be formed as:

$x + y = 40$ —— (I)

$100x + 75y = 3625$ —— (II)

Coeff. of x	Coeff. of y	Constant term
1	1	40
100	75	3625

$$D = \begin{vmatrix} 1 & 1 \\ 100 & 75 \end{vmatrix}$$

$= 1 \times 75 - 1 \times 100$

$= 75 - 100$

$= -25 \neq 0$. so, the solution exists.

$$D_1 = \begin{vmatrix} 40 & 1 \\ 3625 & 75 \end{vmatrix}$$

$= 40 \times 75 - 3625 \times 1$

$= 3000 - 3625$

$= -625$

$$D_2 = \begin{vmatrix} 1 & 40 \\ 100 & 3625 \end{vmatrix}$$

$= 1 \times 3625 - 100 \times 40$

$= 3625 - 4000$

$= -375$

Now, By using Cramer's Rule,

$$x = \frac{D_1}{D} = \frac{-625}{-25} = 25$$

$$y = \frac{D_2}{D} = \frac{-375}{-25} = 15$$

Therefore, 25 men and 15 women are employed in the factory.

4. An agriculture centre uses 10 litres of chemicals in a month combining two chemicals A and B that cost Rs. 150 and Rs 400 per litres respectively. The agriculture centre spends only Rs. 3250 in a month to buy those chemicals. Find the amount of each chemical. Use matrix method.

Solution

Let the amount of chemical A be x and chemical B be y respectively.

Then, From the question, the system of equations can be formed as:

$$x + y = 10 \quad \text{———} \quad \text{(i)}$$
$$150x + 400y = 3250 \quad \text{———} \quad \text{(ii)}$$

Expressing the above equations into matrix form as:

$$\cancel{\begin{bmatrix} 1 & 1 \\ 150 & 400 \end{bmatrix} = \begin{bmatrix} 10 \\ 3250 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 \\ 150 & 400 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 3250 \end{bmatrix}$$

Put $A = \begin{bmatrix} 1 & 1 \\ 150 & 400 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 10 \\ 3250 \end{bmatrix}$

Then, $AX = B$

i.e. $X = A^{-1}B$ ——— (iii)

For A^{-1}

$$|A| = \begin{vmatrix} 1 & 1 \\ 150 & 400 \end{vmatrix}$$

$$= 1 \times 400 - 150 \times 1$$

$$= 400 - 150$$

$$= 250 \neq 0. \text{ So, } A^{-1} \text{ exists}$$

Let the cofactor matrix be $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$A_{11} = +400$

$A_{12} = -150$

$A_{21} = -1$

$A_{22} = +1$

Then, the cofactor matrix becomes,

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 400 & -150 \\ -1 & 1 \end{bmatrix}$$

Now,

Adjoint of A = (cofactor matrix)^T

$$= \begin{bmatrix} 400 & -150 \\ -1 & 1 \end{bmatrix}^T$$

$$\therefore \text{Adj. } A = \begin{bmatrix} 400 & -1 \\ -150 & 1 \end{bmatrix}$$

Then,

Inverse of Matrix $= \dfrac{\text{Adj. } A}{|A|}$, $|A| \neq 0$

$$= \frac{\begin{bmatrix} 400 & -1 \\ -150 & 1 \end{bmatrix}}{250}$$

$$= \begin{bmatrix} \frac{400}{250} & -\frac{1}{250} \\ -\frac{150}{250} & \frac{1}{250} \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{8}{5} & -\frac{1}{250} \\ -\frac{3}{5} & \frac{1}{250} \end{bmatrix}$$

<u>From equation (iii)</u>

$$X = A^{-1} B$$

$$\text{or, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{8}{5} & -\frac{1}{250} \\ -\frac{3}{5} & \frac{1}{250} \end{bmatrix} \begin{bmatrix} 10 \\ 3250 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \times 10 - \frac{1}{250} \times 3250 \\ -\frac{3}{5} \times 10 + \frac{1}{250} \times 3250 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

By the definition of equality of matrices,

$\therefore x = 3$ & $y = 7$

Therefore 3 litres of chemical A and 7 litres of chemical B is used.

5. There are three brands of fertilizers X, Y and Z. 'X' contains 1 unit of nitre, 2 units of potash and 3 units of phosphate. Y contains 3 units of nitre, 1 unit of potash and 2 units of phosphate. Z contains 2 unit of nitre, 3 units of potash and 1 unit of phosphate. If 11 units of nitre, 10 units of potash and 9 units of phosphate are necessary for a field, how much each type of fertilizers required for it? Solve by Cramer's rule.

Solution

Let the number of units of ~~nitre, potash~~ fertilizers X, Y and ~~phosphate~~ Z be x, y, and z respectively.

The above information can be expressed in tabular form as:

	Fertilizer X	Fertilizer Y	Fertilizer Z	Total Requirement
Nitre	1	3	2	11
Potash	2	1	3	10
Phosphate	3	2	1	9

Then the system of equations can be formed as:

$$x + 3y + 2z = 11 \quad \text{—— (I)}$$
$$2x + y + 3z = 10 \quad \text{—— (II)}$$
$$3x + 2y + z = 9 \quad \text{—— (III)}$$

Coeff. of x	Coeff. of y	Coeff. of z	Constants
1	3	2	11
2	1	3	10
3	2	1	9

$\cancel{D}$

$$D = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

Expanding D by using R_1

$$= 1\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$= 1(1\times1 - 2\times3) - 3(2\times1 - 3\times3) + 2(2\times2 - 3\times1)$

$= 1(1-6) - 3(2-9) + 2(4-3)$

$= -5 + 21 + 2$

$= 18 \neq 0$. So, the solution exists.

$$D_1 = \begin{vmatrix} 11 & 3 & 2 \\ 10 & 1 & 3 \\ 9 & 2 & 1 \end{vmatrix}$$

Expanding D_1 by using R_1

$$= 11\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 10 & 3 \\ 9 & 1 \end{vmatrix} + 2\begin{vmatrix} 10 & 1 \\ 9 & 2 \end{vmatrix}$$

$= 11(1\times1 - 2\times3) - 3(10\times1 - 9\times3) + 2(10\times2 - 9\times1)$

$= 11(1-6) - 3(10-27) + 2(20-9)$

$= -55 + 51 + 22$

$= 18$

$$D_2 = \begin{vmatrix} 1 & 11 & 2 \\ 2 & 10 & 3 \\ 3 & 9 & 1 \end{vmatrix}$$

Expanding D_2 by using R_1,

$$= 1\begin{vmatrix} 10 & 3 \\ 9 & 1 \end{vmatrix} - 11\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 10 \\ 3 & 9 \end{vmatrix}$$

$= 1(10\times1 - 9\times3) - 11(2\times1 - 3\times3) + 2(2\times9 - 3\times10)$
$= 1(10-27) - 11(2-9) + 2(18-30)$
$= -17 + 77 - 24$
$= 36$

$$D_3 = \begin{vmatrix} 1 & 3 & 11 \\ 2 & 1 & 10 \\ 3 & 2 & 9 \end{vmatrix}$$

Expanding D_3 by using R_1

$$= 1\begin{vmatrix} 1 & 10 \\ 2 & 9 \end{vmatrix} - 3\begin{vmatrix} 2 & 10 \\ 3 & 9 \end{vmatrix} + 11\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$= 1(1\times9 - 2\times10) - 3(2\times9 - 3\times10) + 11(2\times2 - 3\times1)$
$= 1(9-20) - 3(18-30) + 11(4-3)$
$= -11 + 36 + 11$
$= 36$

Now, By using Cramer's Rule,

$x = \frac{D_1}{D} = \frac{18}{18} = 1$

$y = \frac{D_2}{D} = \frac{36}{18} = 2$

$z = \frac{D_3}{D} = \frac{36}{18} = 2$

$\therefore x=1,\ y=2\ \&\ z=2$

Therefore, 1 unit of fertilizer X, 2 units of fertilizer Y and 2 units of fertilizer Z are required.

6. There are the three commodities X, Y and Z. A purchases 4 units of Z and sells 3 units of X and 5 units of Y. B purchases 3 units of Y and sells 2 units of X and 1 unit of Z. C purchases 1 unit of X and sells 4 units of Y and 6 units of Z. In the process, A, B and C get Rs. 600, 500 and 1300 respectively. Find the price per unit of them by using matrix method.

Solution

Let x, y and z be the per unit price of commodities x, y and z respectively.

The above information can be expressed in tabular form.

	X	Y	Z	
A	3	5	-4	600
B	2	-3	1	500
C	-1	4	6	1300

The system of equations can be formed as:

$3x+5y-4z=600$ ——— (I)

$2x-3y+1z=500$ ——— (II)

$-x+4y+6z=1300$ ——— (III)

Expressing the above equations into matrix form as:

$$\begin{bmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 600 \\ 500 \\ 1300 \end{bmatrix}$$

Put $A = \begin{bmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & 4 & 6 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 600 \\ 500 \\ 1300 \end{bmatrix}$

Then, $AX = B$

i.e. $X = A^{-1}B$ —— (A)

For A^{-1}

$$|A| = \begin{vmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & 4 & 6 \end{vmatrix}$$

Expanding $|A|$ by using R_1

$$= 3\begin{vmatrix} -3 & 1 \\ 4 & 6 \end{vmatrix} - 5\begin{vmatrix} 2 & 1 \\ -1 & 6 \end{vmatrix} + (-4)\begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix}$$

$= 3(-3\times6 - 4\times1) - 5(2\times6 - (-1)\times1) - 4(2\times4 - (-1)\times(-3))$

$= 3(-18-4) - 5(12+1) - 4(8-3)$

$= -66 - 65 - 20$

$= -151 \neq 0$, so, A^{-1} exists.

Let the cofactor matrix be $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$A_{11} = +\begin{vmatrix} -3 & 1 \\ 4 & 6 \end{vmatrix} = -3\times6 - 4\times1 = -18-4 = -22$$

$$A_{12} = -\begin{vmatrix} 2 & 1 \\ -1 & 6 \end{vmatrix} = -(2\times6 - (-1)\times1) = -12-1 = -13$$

$$A_{13} = +\begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix} = 2\times4 - (-3)\times(-1) = 8-3 = 5$$

$$A_{21} = -\begin{vmatrix} 5 & -4 \\ 4 & 6 \end{vmatrix} = -(5\times6 - 4\times(-4)) = -30-16 = -46$$

$$A_{22} = +\begin{vmatrix} 3 & -4 \\ -1 & 6 \end{vmatrix} = 3\times6 - (-1)\times(-4) = 18-4 = 14$$

$$A_{23} = -\begin{vmatrix} 3 & 5 \\ -1 & 4 \end{vmatrix} = -(3\times4 - (-1)\times5) = -12-5 = -17$$

$$A_{31} = +\begin{vmatrix} 5 & -4 \\ -3 & 1 \end{vmatrix} = 5\times1 - (-3)\times(-4) = 5-12 = -7$$

$$A_{32} = -\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} = -(3\times1 - 2\times(-4)) = -3-8 = -11$$

$$A_{33} = +\begin{vmatrix} 3 & 5 \\ 2 & -3 \end{vmatrix} = 3\times(-3) - 2\times5 = -9-10 = -19$$

Now, the cofactor matrix becomes,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -22 & -13 & 5 \\ -46 & 14 & -17 \\ -7 & -11 & -19 \end{bmatrix}$$

Adjoint of A = (Cofactor matrix)^T

$$= \begin{bmatrix} -22 & -13 & 5 \\ -46 & 14 & -17 \\ -7 & -11 & -19 \end{bmatrix}^T$$

$$\therefore \text{Adj. } A = \begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}$$

Now,

Inverse of Matrix $= \dfrac{\text{Adj. } A}{|A|}$, $|A| \neq 0$

$$= \frac{\begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}}{-151}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{22}{151} & \frac{46}{151} & \frac{7}{151} \\ \frac{13}{151} & -\frac{14}{151} & \frac{11}{151} \\ -\frac{5}{151} & \frac{17}{151} & \frac{19}{151} \end{bmatrix}$$

Now,

From equation (A)

$$X = A^{-1} B$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{22}{151} & \frac{46}{151} & \frac{7}{151} \\ \frac{13}{151} & -\frac{14}{151} & \frac{11}{151} \\ -\frac{5}{151} & \frac{17}{151} & \frac{19}{151} \end{bmatrix} \begin{bmatrix} 600 \\ 500 \\ 1300 \end{bmatrix}$$

$$\text{or} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{22}{151} \times 600 + \frac{46}{151} \times 500 + \frac{7}{151} \times 1300 \\ \frac{13}{151} \times 600 - \frac{14}{151} \times 500 + \frac{11}{151} \times 1300 \\ -\frac{5}{151} \times 600 + \frac{17}{151} \times 500 + \frac{19}{151} \times 1300 \end{bmatrix}$$

$$\text{or} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 300 \\ 100 \\ 200 \end{bmatrix}$$

By the definition of matrices,

$\therefore x = 300, y = 100$ & $z = 200$

The price per unit of commodity x, y and z are 300, 100 and 200 respectively.

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7. A chemist produces 100 units of special drug. To produce the drug, three different chemical should be mixed up in a specific proportion i.e., the quantity of chemical A should be one-third of the difference between the chemicals B and C used, further quantity of chemical B should be twice that of C. The cost per unit of chemical A, B and C are respectively Rs 20, Rs 8 and Rs 16. If the total cost of the component chemicals is Rs 1160, determine by using matrix method the units of each chemicals used.

Solution

Let the quantity of chemical A, chemical B and chemical C be x, y & z respectively.

From the question, system of equations can be formed as:

$$x + y + z = 100 \quad \text{—— (I)}$$

$$x = \frac{1}{3}(y - z)$$

$$\text{or } x = \frac{1}{3}y - \frac{1}{3}z$$

$$\therefore \cancel{\text{or}},\ x - \frac{1}{3}y + \frac{1}{3}z = 0 \quad \text{—— (II)}$$

$$20x + 8y + 16z = 1160 \quad \text{—— (III)}$$

$$y = 2z$$
$$\text{or } y - 2z = 0$$

Expressing the above equations into matrix form as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{3} & \frac{1}{3} \\ 20 & 8 & 16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 1160 \end{bmatrix}$$

$$\text{Put } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{3} & \frac{1}{3} \\ 20 & 8 & 16 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 100 \\ 0 \\ 1160 \end{bmatrix}$$

Then, $AX = B$

i.e. $X = A^{-1}B$ —— (A)

For A^{-1}

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{3} & \frac{1}{3} \\ 20 & 8 & 16 \end{vmatrix}$$

Expanding $|A|$ by using R_1

$$= 1\begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ 8 & 16 \end{vmatrix} - 1\begin{vmatrix} 1 & \frac{1}{3} \\ 20 & 16 \end{vmatrix} + 1\begin{vmatrix} 1 & -\frac{1}{3} \\ 20 & 8 \end{vmatrix}$$

$$= 1(-\frac{1}{3}\times 16 - 8\times\frac{1}{3}) - 1(16\times 1 - 20\times\frac{1}{3}) + 1(1\times 8 - 20\times(-\frac{1}{3}))$$

$= \frac{-8}{3} \neq 0$ so, A^{-1} exists

Let the cofactor matrix be $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$A_{11} = +\begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ 8 & 16 \end{vmatrix} = -\frac{1}{3}\times 16 - 8\times\frac{1}{3} = -8$$

$$A_{12} = -\begin{vmatrix} 1 & \frac{1}{3} \\ 20 & 16 \end{vmatrix} = -(1\times 16 - 20\times\frac{1}{3}) = -\frac{28}{3}$$

$$A_{13} = +\begin{vmatrix} 1 & -\frac{1}{3} \\ 20 & 8 \end{vmatrix} = 1\times 8 - 20\times(-\frac{1}{3}) = \frac{44}{3}$$

$$A_{21} = -\begin{vmatrix} 1 & 1 \\ 8 & 16 \end{vmatrix} = -(1\times 16 - 8\times 1) = -16+8 = -8$$

$$A_{22} = +\begin{vmatrix} 1 & 1 \\ 20 & 16 \end{vmatrix} = 1\times 16 - 20\times 1 = -4$$

$$A_{23} = -\begin{vmatrix} 1 & 1 \\ 20 & 8 \end{vmatrix} = -(1\times 8 - 20\times 1) = 12$$

$$A_{31} = +\begin{vmatrix} 1 & 1 \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = 1\times\frac{1}{3} - (-\frac{1}{3})\times 1 = \frac{2}{3}$$

$$A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & \frac{1}{3} \end{vmatrix} = -\left(1\times\frac{1}{3} - 1\times1\right) = \frac{2}{3}$$

$$A_{33} = +\begin{vmatrix} 1 & 1 \\ 1 & -\frac{1}{3} \end{vmatrix} = 1\times\left(-\frac{1}{3}\right) - 1\times1 = -\frac{4}{3}$$

Now, the cofactor matrix becomes,

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} -8 & -\frac{28}{3} & \frac{44}{3} \\ -8 & -4 & 12 \\ \frac{2}{3} & \frac{2}{3} & -\frac{4}{3} \end{bmatrix}$$

Adjoint of A = (cofactor matrix)^T

$$= \begin{bmatrix} -8 & -\frac{28}{3} & \frac{44}{3} \\ -8 & -4 & 12 \\ \frac{2}{3} & \frac{2}{3} & -\frac{4}{3} \end{bmatrix}^T$$

$$\therefore \text{Adj. } A = \begin{bmatrix} -8 & -8 & \frac{2}{3} \\ -\frac{28}{3} & -4 & \frac{2}{3} \\ \frac{44}{3} & 12 & -\frac{4}{3} \end{bmatrix}$$

Then,

$$\text{Inverse Matrix} = \frac{\text{Adj. } A}{|A|}$$

$$= \frac{\begin{bmatrix} -8 & -8 & \frac{2}{3} \\ -\frac{28}{3} & -4 & \frac{2}{3} \\ \frac{44}{3} & 12 & -\frac{4}{3} \end{bmatrix}}{-\frac{8}{3}}$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & 3 & -\frac{1}{4} \\ \frac{7}{2} & \frac{3}{2} & -\frac{1}{4} \\ -\frac{11}{2} & -\frac{9}{2} & \frac{1}{2} \end{bmatrix}$$

From equation (A)

$$X = A^{-1} B$$

$$\text{or, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 3 & -\frac{1}{4} \\ \frac{7}{2} & \frac{3}{2} & -\frac{1}{4} \\ -\frac{11}{2} & -\frac{9}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 100 \\ 0 \\ 1160 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3\times100 + 3\times0 - \frac{1}{4}\times1160 \\ \frac{7}{2}\times100 + \frac{3}{2}\times0 - \frac{1}{4}\times1160 \\ -\frac{11}{2}\times100 - \frac{9}{2}\times0 + \frac{1}{2}\times1160 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 60 \\ 30 \end{bmatrix}$$

By definition of equality of matrices,

$\therefore x = 10, y = 60 \;\&\; z = 30$

Therefore 10 units of chemical A, 60 units of chemical B and 30 units of chemical C is used.