Page No. :
Date :

Exercise 4(C)

1. a) Examine whether the function $f(x) = x^3(x-4)$ is increasing at $x=2$.

Solution

Given,

$f(x) = x^3(x-4)$

or, $f(x) = x^4 - 4x^3$ ——— ①

Differentiating both sides of ① with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x^4 - 4x^3)$$

$$\therefore f'(x) = 4x^3 - 12x^2$$

When $x=2$,

$$f'(2) = 4\times(2)^3 - 12\times(2)^2$$
$$= 4\times 8 - 48$$
$$= 32 - 48$$
$$\cancel{= 28 > 0 \text{ (Increasing)}}$$
$$\therefore f'(2) = -16 < 0 \text{ (Decreasing)}$$

Therefore, the function $f(x) = x^3(x-4)$ is ~~in~~decreasing at $x=2$.

b) Show that the function $f(x) = 2x^3 - 24x + 15$ is increasing at $x=3$ and decreasing at $x=\frac{3}{2}$.

Solution

Given,

$f(x) = 2x^3 - 24x + 15$ ——— ①

Page No. :
Date :

Differentiating both sides of ① with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(2x^3 - 24x + 15)$$
$$= 6x^2 - 24 + 0$$
$$\therefore f'(x) = 6x^2 - 24$$

When $x = 3$,

$$f'(3) = 6 \times (3)^2 - 24$$
$$= 54 - 24$$
$$\therefore f'(3) = 30 > 0 \text{ (Increasing)}$$

When $x = \frac{3}{2}$,

$$f'\left(\frac{3}{2}\right) = 6 \times \left(\frac{3}{2}\right)^2 - 24$$
$$= \frac{27}{2} - 24$$
$$\therefore f'\left(\frac{3}{2}\right) = -\frac{21}{2} < 0 \text{ (Decreasing)}$$

verified

c) The cost function is defined by $C(x) = 5x - 3x^2 + x^3$, where x is the quality of output. Examine whether the cost is increasing or decreasing when the output $x = 3$.

Solution

Given,

$$C(x) = 5x - 3x^2 + x^3 \quad \text{——— ①}$$

Differentiating both sides of ① with respect to x,

$$\frac{d}{dx}[C(x)] = \frac{d}{dx}(5x - 3x^2 + x^3)$$
$$\therefore C'(x) = 5 - 6x + 3x^2$$

Date :

When $x=3$,

$$C'(3) = 5-6\times3+3\times(3)^2$$
$$= 5-18+27$$
$$\therefore C'(3) = 14 > 0 \text{ (Increasing)}$$

Therefore, the cost is increasing when the output $x=3$.

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2. Examine the function $f(x) = x^3 - 10x^2 + 8$ is increasing or decreasing at the points $x=1$ and $x=2$.

Solution

Given,

$$f(x) = x^3 - 10x^2 + 8$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x^3 - 10x^2 + 8)$$
$$= 3x^2 - 20x + 0$$
$$\therefore f'(x) = 3x^2 - 20x$$

When $x=1$,

$$f'(1) = 3\times(1)^2 - 20\times1$$
$$= 3-20$$
$$\therefore f'(1) = -17 < 0 \text{ (Decreasing)}$$

When $x=2$,

$$f'(2) = 3\times(2)^2 - 20\times2$$
$$= 12-40$$
$$\therefore f'(2) = -28 < 0 \text{ (Decreasing)}$$

Therefore, the function ~~f(x) is~~ $f(x) = x^3 - 10x^2 + 8$ is decreasing at $x=1$ and $x=2$.

3. Determine whether the following functions are concave upward or concave downward.

a) $f(x) = x^3 - 3x^2 + 1$ at $x = 3$

Given,

$f(x) = x^3 - 3x^2 + 1$

Differentiating both sides with respect to x, we get,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x^3 - 3x^2 + 1)$$

$$= 3x^2 - 6x + 0$$

$$\therefore f'(x) = 3x^2 - 6x \quad \text{——} \quad (1)$$

Again, Differentiating (1) with respect to x,

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}(3x^2 - 6x)$$

$$\therefore f''(x) = 6x - 6$$

When $x = 3$,

$$f''(3) = 6 \times 3 - 6$$

$$= 18 - 6$$

$$\therefore f''(3) = 12 > 0 \text{ (Concave Upward)}$$

Therefore, the function is concave upward at $x = 3$.

b) $f(x) = 2x^3 + 4x - 8$ at $x = 4$

Given,

$$f(x) = 2x^3 + 4x - 8 \quad \text{——} \quad (1)$$

Page No. :
Date :

Differentiating both sides of ① with respect to x, we get,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(2x^3+4x-8)$$

$$= 6x^2+4-0$$

$$\therefore f'(x) = 6x^2+4 \quad \text{— (II)}$$

Again, Differentiating (II) with respect to x, we get

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}(6x^2+4)$$

$$= 12x+0$$

$$\therefore f''(x) = 12x$$

When $x=4$,

$$f''(4) = 12\times 4$$

$$\therefore f''(4) = 48 > 0 \text{ (Concave upward)}$$

Therefore, the function is concave upward at $x=4$.

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4. Find the interval in which the function $f(x)$ is increasing or decreasing.

a) $f(x) = x^2-4x+3$

Given,

$$f(x) = x^2-4x+3$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x^2-4x+3)$$

$$\therefore f'(x) = 2x-4$$

For stationary point, $f'(x) = 0$

or, $2x - 4 = 0$

or, $2x = 4$

$\therefore x = 2$

Two cases arise:

(i) when $x > 2$ and

(ii) when $x < 2$.

First Case

When $x > 2$ i.e. $x = 3$

$f'(3) = 2\times3 - 4$

$= 6 - 4$

$\therefore f'(3) = 2 > 0$ (increasing)

Therefore the above function is increasing when $x > 2$ i.e. $x \in (\cancel{-\infty}\ 2, \infty)$

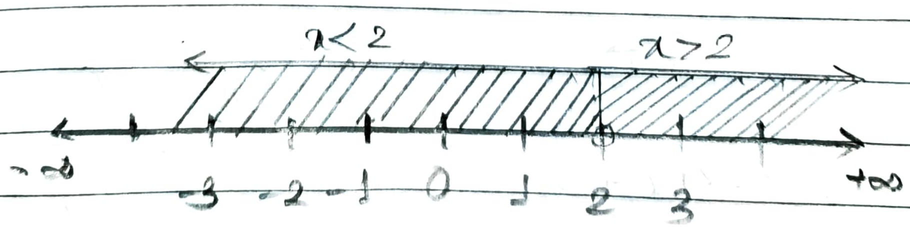
Second Case

When $x < 2$ i.e. $x = 1$

$f'(1) = 2\times1 - 4$

$\therefore f'(1) = -2 < 0$ (Decreasing)

Therefore, the above function is decreasing when $x < 2$ i.e. $x \in (-\infty, 2)$



Page No. :
Date :

b) $f(x) = 3x^2 - 6x + 5$

Given,

$f(x) = 3x^2 - 6x + 5$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(3x^2 - 6x + 5)$$

$$= 6x - 6 + 0$$

$$\therefore f'(x) = 6x - 6$$

For stationary point, $f'(x) = 0$

or, $6x - 6 = 0$

or, $6x = 6$

$\therefore x = 1$

Two cases arise:

(I) when $x > 1$ and

(II) when $x < 1$.

First case

when $x > 1$ i.e. $x = 2$

$f'(2) = 6 \times 2 - 6$

$= 12 - 6$

$\therefore f'(2) = 6 > 0$ (Increasing)

Therefore the above function is increasing when $x > 1$ i.e. $x \in (1, \infty)$

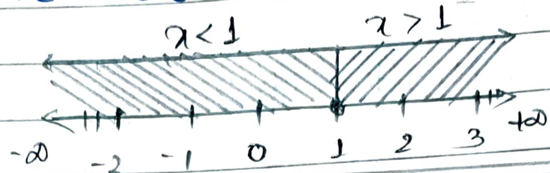
Second case

when $x < 1$ i.e. $x = 0$

$f'(0) = 6 \times 0 - 6$

$\therefore f'(0) = -6 < 0$ (Decreasing)

Therefore the function is decreasing when $x<1$ i.e. $x \in (-\infty, 1)$



c) $f(x) = x^2 - 2x + 10$

Given,

$f(x) = x^2 - 2x + 10$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x^2 - 2x + 10)$$

$$\therefore f'(x) = 2x - 2$$

For stationary point, $f'(x) = 0$

or, $2x - 2 = 0$

or, $2x = 2$

$\therefore x = 1$

Two cases arise!

(I) when $x > 1$ and

(II) when $x < 1$

First case

When $x > 1$ i.e. $x = 2$

$f'(3) = \cancel{2\times3-2}\ \cancel{2\times1-2}\ 2\times2-2$

$= \cancel{6-2}\ 4-2$

$\therefore f'(3) = 2 > 0$ (Increasing)

Therefore the above function is increasing when $x > 2$ i.e. $\cancel{x \in (4,\infty)}$ $\cancel{x \in (2,\infty)}$

$x \in (1, \infty)$

First case

When $x > 1$ i.e. $x = 2$

$$f'(2) = 2 \times 2 - 2$$
$$= 4 - 2$$
$$\therefore f'(2) = 2 > 0 \text{ (Increasing)}$$

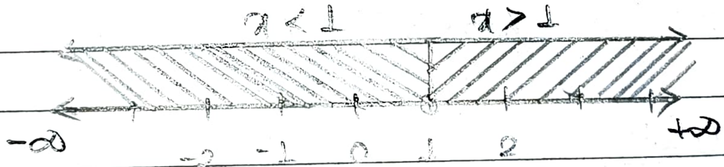
Therefore the function is increasing when $x > 1$ i.e. $x \in (1, \infty)$

Second case

When $x < 1$ i.e. $x = 0$

$$f'(0) = 2 \times 0 - 2$$
$$= 0 - 2$$
$$\therefore f'(0) = -2 < 0 \text{ (Decreasing)}$$

Therefore the function is decreasing when $x < 1$ i.e. $x \in (-\infty, 1)$



d) $f(x) = 5x^3 - 135x + 22$

Given,

$$f(x) = 5x^3 - 135x + 22$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(5x^3 - 135x + 22)$$
$$\therefore f'(x) = 15x^2 - 135$$

Date :

For stationary point, $f'(x) = 0$,

Or, $15x^2 - 135 = 0$

or, $15x^2 = 135$

or, $x^2 = 9$

$\therefore\ x = \pm 3 = -3, 3$

Three cases arise:

1. When $x > 3$
2. When $x < -3$ &
3. $-3 < x < 3$

First Case

When $x > 3$ i.e. $x = 4$

$f'(4) = 15 \times (4)^2 - 135$

$= 240 - 135$

$\therefore f'(4) = 105 > 0$ (Increasing)

Therefore, the above function is increasing when $x > 3$ i.e. $\cancel{x \in (3,\infty)}$ $\cancel{x \in (-\infty,3)}$ $x \in (3, \infty)$

Second case

when $x < -3$ i.e. $x = \cancel{2}\ -4$

$f'(-2) = 15 \times (-4)^2 - 135$

$= \cancel{60 - 135}\ 240 - 135$

$\therefore f'(2) = \cancel{-75}\ 105 > 0$ (Increasing)

The above function is increasing when $x < -3$ i.e. $\cancel{x \in (-3,\infty)}$ $x \in (-\infty, -3)$

Page No. :
Date :

Third Case

When $-3<x<3$ i.e. $x=0$

$f'(0) = 15\times(0)^2 - 135$

$\therefore f'(0) = -135 < 0$ (Decreasing)

Therefore the above function is decreasing when $-3<x<3$ i.e. $x \in (-3,3)$

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e) $f(x) = 2x^3 - 15x^2 + 36x + 1$.

Given,

$f(x) = 2x^3 - 15x^2 + 36x + 1$

Differentiating both sides with respect to x,

$\frac{d}{dx}[f(x)] = \frac{d}{dx}(2x^3 - 15x^2 + 36x + 1)$

$\therefore f'(x) = 6x^2 - 30x + 36$

At the stationary point, $f'(x) = 0$

Or, $6x^2 - 30x + 36 = 0$

~~Or, $6x^2 - 36x + 6x + 36 = 0$~~

~~Or, $6x(x-6)+$~~

Or, $6x^2 - 12x - 18x + 36 = 0$

Or, $6x(x-2) - 18(x-2) = 0$

Or, $(x-2)(6x-18) = 0$

Or, $(x-2)\times 6(x-3) = 0$

Or, $(x-2)(x-3) = 0$

$\therefore x = 2, 3$

There Three cases arise:

1. When $x<2$
2. When $x>3$
3. When $2<x<3$

Date :

First case

When $x<2$ i.e. $x=1$

$\therefore f'(1) = 6\times(1)^2 - 30\times1 + 36$

$= 6-30+36$

$\therefore f'(1) = 12 > 0$ (Increasing)

Therefore, the above function is increasing ~~at~~ when ~~$x=12$.~~ $x<2$ i.e. ~~$x\in(-\infty,2)$~~ ~~$x\in(2,\infty)$~~

$x\in(-\infty,2)$

Second case

When $x>3$ i.e. $x=4$

$\therefore f'(4) = 6\times(4)^2 - 30\times4 + 36$

$= 96 - 120 + 36$

$\therefore f'(4) = 12 > 0$ (Increasing)

Therefore the above function is increasing when $x>3$ i.e. ~~$x\in(-\infty,3)$~~ $x\in(3,\infty)$

Third case

When $2<x<3$ i.e. $x=2.5$

$f'(2.5) = 6\times(2.5)^2 - 30\times2.5 + 36$

$= 37.5 - 75 + 36$

$\therefore f'(2.5) = -1.5 < 0$ (Decreasing)

Therefore the above function is decreasing when $2<x<3$ i.e. $x\in(2,3)$

Date :

b) $f(x) = \frac{x}{x^2+1}$

Given,

$$f(x) = \frac{x}{x^2+1}$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[\frac{x}{x^2+1}\right]$$

$$= \frac{(x^2+1)\frac{d}{dx}(x) - x.\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{d}{dx}(u) - u.\frac{d}{dx}(v)}{v^2}\right]$$

$$= \frac{(x^2+1) - x.(2x+0)}{(x^2+1)^2}$$

$$\therefore f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2}$$

At stationary point, $f'(x) = 0$

or, $\frac{(x^2+1) - 2x^2}{(x^2+1)^2} = 0$

or, $x^2+1-2x^2 = 0$

or, $-x^2+1 = 0$

or, $x^2 = 1$

$\therefore x = -1, 1$

Date :

Three cases arise:
1. ~~x>~~ when $x > 1$
2. when $x < -1$ &
3. when $-1 < x < 1$

First case

When $x > 1$ i.e. $x = 2$

$$f'(2) = \frac{(2^2+1) - 2\times(2)^2}{(2^2+1)^2}$$

$$= \frac{5-8}{25}$$

$$\therefore f'(2) = -\frac{3}{25} < 0 \text{ (Decreasing)}$$

Therefore, the above function is decreasing when $x > 1$ i.e. $x \in (1, \infty)$

Second case

When $x < -1$ i.e. $x = -2$

$$f'(-2) = \frac{[(-2)^2+1] - 2\times(-2)^2}{[(-2)^2+1]^2}$$

$$= \frac{5-8}{25}$$

$$\therefore f'(-2) = -\frac{3}{25} < 0 \text{ (Decreasing)}$$

Therefore, the above function is decreasing ~~at~~ when $x < -1$ i.e. $x \in (-\infty, -1)$

Page No. :
Date :

Third case

When $-1 < x < 1$ i.e. $x = 0$

$$f'(0) = \frac{(0^2+1) - 2\times(0)^2}{(0^2+1)^2}$$

$$= \frac{1-0}{1}$$

$\therefore f'(0) = 1 > 0$ (Increasing)

Therefore the above function is decreasing when $-1 < x < 1$ i.e. $x \in (-1, 1)$

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5. Determine where the graph of the following functions are concave upwards and concave downwards.

a) $f(x) = 2x^3 - 6x^2 + 5$

Given,

$f(x) = 2x^3 - 6x^2 + 5$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(2x^3 - 6x^2 + 5)$$

$$= 6x^2 - 12x + 0$$

$$\therefore f'(x) = 6x^2 - 12x \text{ ——— } (\mathrm{I})$$

Differentiating both sides of (I) with respect to x,

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}(6x^2 - 12x)$$

$$\therefore f''(x) = 12x - 12 \text{ ——— } (\mathrm{II})$$

For the point of inflection, $f''(x) = 0$

Or, $12x - 12 = 0$

or $12x = 12$

$\therefore x = 1$

Two cases arise:

1. when $x > 1$
2. when $x < 1$

First case

When $x > 1$ i.e. $x = 2$

$f''(2) = 12 \times 2 - 12$

$= 24 - 12$

$\therefore f''(2) = 12 > 0$ (concave upward)

Therefore, the above function is concave upward when $x > 1$ i.e. $x \in (1, \infty)$

Second Case

When $x < 1$ i.e. $x = 0$

$f''(0) = 12 \times 0 - 12$

$\therefore f''(0) = -12 < 0$ (concave downward)

Therefore, the above function is concave downward when $x < 1$ i.e. $x \in (-\infty, 1)$

b) $f(x) = 2x^3 - 15x^2 + 36x$

Given,

$f(x) = 2x^3 - 15x^2 + 36x$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(2x^3 - 15x^2 + 36x)$$

$$\therefore f'(x) = 6x^2 - 30x + 36 \quad \text{———} \quad (1)$$

Differentiating both sides of (1) with respect to x,

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}(6x^2 - 30x + 36)$$

$$\therefore f''(x) = 12x - 30 \quad \text{———} \quad (II)$$

For the point of inflection, $f''(x) = 0$

or, $12x - 30 = 0$

or, $12x = 30$

$\therefore x = \frac{5}{2}$

Two cases arise:

(1) When $x > \frac{5}{2}$

(2) When $x < \frac{5}{2}$

First Case

When $x > \frac{5}{2}$ i.e. $x = 3$

$f''(3) = 12 \times 3 - 30$

$= 36 - 30$

$\therefore f''(3) = 6 > 0$ (concave upward)

Therefore, the above function is concave upward when $x \in (\frac{5}{2}, \infty)$

Second case

when $x < \frac{5}{2}$ i.e. $x = 2$

$$f''(2) = 12 \times 2 - 30$$
$$= 24 - 30$$
$$\therefore f''(2) = -6 < 0 \text{ (Concave downward)}$$

Therefore, the above function is concave downward when $x < \frac{5}{2}$ i.e. $x \in (-\infty, \frac{5}{2})$

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c) $f(x) = x^4 - 8x^3 + 18x^2 - 24$

Given,

$$f(x) = x^4 - 8x^3 + 18x^2 - 24$$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x^4 - 8x^3 + 18x^2 - 24)$$
$$= 4x^3 - 24x^2 + 36x - 0$$
$$\therefore f'(x) = 4x^3 - 24x^2 + 36x \text{ ——— } (1)$$

Differentiating both sides of (1) with respect to x,

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}(4x^3 - 24x^2 + 36x)$$
$$\cancel{\therefore f''(x) = 12x^2 - 72x^2 + 36}$$
$$\therefore f''(x) = 12x^2 - 48x + 36 \text{ ——— } (11)$$

For the point of inflection, ~~$f''(x) = 12x^2 - 48x + 36$~~.
$f''(x) = 0$

or, $12x^2 - 48x + 36 = 0$
$$\therefore x = 1, 3$$

Page No. :
Date :

Three cases arise:

1. ~~$x>3$~~ when $x>3$
2. ~~$x<1$~~ when $x<1$
3. when $1<x<3$

First case

when $x>3$ i.e. $x=4$

$$f''(4) = 12\times(4)^2 - 48\times4 + 36$$
$$= 192 - 192 + 36$$
$$\therefore f''(4) = 36 > 0 \text{ (Concave upward)}$$

Therefore, the above function is concave upward when $x>3$ i.e. $x \in (3, \infty)$

Second case

when $x<1$ i.e. $x=0$

$$f''(0) = 12\times(0)^2 - 48\times0 + 36$$
$$\therefore f''(0) = 36 > 0 \text{ (Concave upward)}$$

Therefore, the above function is concave upward when $x<1$ i.e. $(-\infty, 1)$

Third case

when $1<x<3$ i.e. $x=2$

$$f''(2) = 12\times(2)^2 - 48\times2 + 36$$
$$= 48 - 96 + 36$$
$$\therefore f''(2) = -12 < 0 \text{ (Concave downward)}$$

Therefore, the above function is concave downward when $1<x<3$ i.e. $x \in (1,3)$

d) $f(x) = x^4 - 4x^3$

Given,

$f(x) = x^4 - 4x^3$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(x^4 - 4x^3)$$

$$\therefore f'(x) = 4x^3 - 12x^2 \quad \text{--- (1)}$$

Differentiating both sides of (1) with respect to x, we get.

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}(4x^3 - 12x^2)$$

$$\therefore f''(x) = 12x^2 - 24x$$

For the point of inflection, $f''(x) = 0$

or, $12x^2 - 24x = 0$

or, $12(x^2 - 2x) = 0$

or, $x^2 - 2x = 0$

or, $x(x-2) = 0$

$\therefore x = 0, 2$

Three cases arises:

(1) when $x > 2$

(2) when $x < 0$

(3) when $0 < x < 2$

First case

when $x > 2$ i.e. $x = 3$

$f''(3) = 12 \times (3)^2 - 24 \times 3$

$= 108 - 72$

$\therefore f''(3) = 36 > 0$ (concave upward)

Page No.:
Date:

Therefore, the above function is concave upward when $x > 2$ i.e. $x \in (2, \infty)$

Second Case

When $x < 0$ i.e. $x = -1$

$f''(-1) = 12 \times (-1)^2 - 24 \times (-1)$

$= 12 + 24$

$\therefore f''(-1) = 36 > 0$ (concave upward)

Therefore, the above function is concave upward when $x < 0$ i.e. $x \in (-\infty, 0)$

Third Case

When $0 < x < 2$ i.e. $x = 1$

$f''(1) = 12 \times (1)^2 - 24 \times (1)$

$= 12 - 24$

$\therefore f''(1) = -12 < 0$ (concave downward)

Therefore, the above function is concave downward when $0 < x < 2$ i.e. $x \in (0, 2)$

---

e) $f(x) = 2x^3 - 6x^2 + 5$

Given,

$f(x) = 2x^3 - 6x^2 + 5$

Differentiating both sides with respect to x,

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}(2x^3 - 6x^2 + 5)$$

$$= 6x^2 - 12x + 0$$

$$\therefore f'(x) = 6x^2 - 12x \quad \text{——— (1)}$$

Differentiating both sides of (1) with respect to x,

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}(6x^2 - 12x)$$

$$\therefore f''(x) = 12x - 12 \quad \text{——— (11)}$$

For the point of inflection, $f''(x) = 0$

or, i.e. $12x - 12 = 0$

or, $12x = 12$

$\therefore x = 1$

Two cases arise:

① when $x > 1$

② when $x < 1$

First case

when $x > 1$ i.e. $x = 2$

$f''(2) = 12 \times 2 - 12$

$= 24 - 12$

$\therefore f''(2) = 12 > 0$ (concave upward)

Therefore, the above function is concave upward when $x > 1$ i.e. $x \in (1, \infty)$

Second case

when $x < 1$ i.e. $x = 0$

$f''(0) = 12 \times 0 - 12$

$= 0 - 12$

$\therefore f''(0) = -12 < 0$ (concave downward)

Therefore, the above function is concave downward when $x < 1$ i.e. $x \in (-\infty, 1)$