Exercise 3(c)

1. Evaluate the following limits:

a) $\lim_{x \to 1} (x^2 - 10x + 3)$

$= (1)^2 - 10 \times 1 + 3$

$= 4 - 10$

$= -6$

b) $\lim_{x \to 3} \left[\dfrac{x+3}{x^2+5x+6} \right]$

Solution

$= \lim_{x \to 3} \left[\dfrac{x+3}{x^2+3x+2x+6} \right]$

$\neq \lim_{x \to 3} \left[\dfrac{(x+3)}{x(x+3)+2(x+3)} \right] \qquad = \dfrac{3+3}{3^2+5 \times 3+6}$

$\neq \lim_{x \to 3} \left[\dfrac{(x+3)}{(x+3)(x+2)} \right] \qquad = \dfrac{6}{9+15+6}$

$= \lim_{x \to 3} \left[\dfrac{1}{(x+2)} \right] \qquad = \dfrac{6}{30}$

$= \dfrac{1}{3+2}$

$= \dfrac{1}{5}$

$\therefore \lim_{x \to 3} \left[\dfrac{x+3}{x^2+5x+6} \right] = \dfrac{1}{5}$

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c) $\lim_{x \to 7} \left[\frac{x^2-49}{x-7} \right]$, $\left(\frac{0}{0}\right)$

Solution

$= \lim_{x \to 7} \left[\frac{(x)^2 - (7)^2}{x-7} \right]$

$= \lim_{x \to 7} \left[\frac{(x+7)(x-7)}{(x-7)} \right]$

$= \lim_{x \to 7} (x+7)$

$= 7+7$

$= 14$

$\therefore \lim_{x \to 7} \left[\frac{x^2-49}{x-7} \right] = 14$

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d) $\lim_{x \to 3} \left[\frac{1}{x-3} - \frac{9}{x^3-3x^2} \right]$, $(\infty - \infty)$

Solution

$= \lim_{x \to 3} \left[\frac{1}{x-3} - \frac{9}{x^2(x-3)} \right]$

$= \lim_{x \to 3} \left[\frac{x^2-9}{x^2(x-3)} \right]$

$= \lim_{x \to 3} \left[\frac{(x+3)(x-3)}{x^2(x-3)} \right]$

$= \lim_{x \to 3} \frac{(x+3)}{x^2}$

$= \frac{3+3}{3^2}$

$= \frac{6}{9}$

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$= \frac{2}{3}$

$\therefore \lim_{x \to 3} \left[\frac{1}{x-3} - \frac{9}{x^3 - 3x^2} \right] = \cancel{6}\frac{2}{3}$

2. Evaluate:

a) $\lim_{x \to \infty} \left[\frac{4x^2 + 2x + 5}{x^2 + x - 3} \right] \quad \left(\frac{\infty}{\infty}\right)$

Solution

Dividing numerator and denominator by x^2,

$= \lim_{x \to \infty} \left[\frac{\frac{4x^2 + 2x + 5}{x^2}}{\frac{x^2 + x - 3}{x^2}} \right]$

$= \lim_{x \to \infty} \left[\frac{\frac{4x^2}{x^2} + \frac{2x}{x^2} + \frac{5}{x^2}}{\frac{x^2}{x^2} + \frac{x}{x^2} - \frac{3}{x^2}} \right]$

$= \lim_{x \to \infty} \left[\frac{4 + \frac{2}{x} + \frac{5}{x^2}}{1 + \frac{1}{x} - \frac{3}{x^2}} \right]$

$= \frac{4 + \frac{2}{\infty} + \frac{5}{(\infty)^2}}{1 + \frac{1}{\infty} - \frac{3}{(\infty)^2}}$

$= \frac{4 + \frac{2}{\infty} + \frac{5}{\infty}}{1 + \frac{1}{\infty} - \frac{3}{\infty}} \quad [\because \infty^2 = \infty]$

$$= \frac{4+0+0}{1+0-0}$$

$$= 4$$

$$\therefore \lim_{x\to\infty}\left[\frac{4x^2+2x+5}{x^2+x-3}\right] = 4$$

b) $\lim_{x\to\infty}\left[\frac{x^2}{4x^2+2x-1}\right]$, $\left(\frac{\infty}{\infty}\right)$

Solution

Dividing nomerator and denominator by x^2,

$$= \lim_{x\to\infty}\left[\frac{\frac{x^2}{x^2}}{\frac{4x^2}{x^2}+\frac{2x}{x^2}-\frac{1}{x^2}}\right]$$

$$= \lim_{x\to\infty}\left[\frac{1}{4+\frac{2}{x}-\frac{1}{x^2}}\right]$$

$$= \frac{1}{4+\frac{2}{\infty}-\frac{1}{(\infty)^2}}$$

$$= \frac{1}{4+\frac{2}{\infty}-\frac{1}{\infty}}$$

$$= \frac{1}{4+0-0}$$

$$= \frac{1}{4}$$

$$\therefore \lim_{x\to\infty}\left[\frac{x^2}{4x^2+2x-1}\right] = \frac{1}{4}$$

4. Evaluate the following limits.

a) $\lim_{x \to \infty} \left[\frac{\sqrt{1+2x} - \sqrt{1-2x}}{x} \right]$, $\left(\frac{\infty}{\infty}\right)$

Solution

$= \lim_{x \to \infty} \left[\frac{\sqrt{1+2x} - \sqrt{1-2x}}{x} \times \frac{\sqrt{1+2x} + \sqrt{1-2x}}{\sqrt{1+2x} + \sqrt{1-2x}} \right]$

$= \lim_{x \to \infty} \left[\frac{(\sqrt{1+2x})^2 - (\sqrt{1-2x})^2}{x(\sqrt{1+2x} + \sqrt{1-2x})} \right]$

$= \lim_{x \to \infty} \left[\frac{(1+2x) - (1-2x)}{x(\sqrt{1+2x} + \sqrt{1-2x})} \right]$

$= \lim_{x \to \infty} \left[\frac{1+2x-1+2x}{x(\sqrt{1+2x} + \sqrt{1-2x})} \right]$

$= \lim_{x \to \infty} \left[\frac{4x}{x(\sqrt{1+2x} + \sqrt{1-2x})} \right]$

$= \lim_{x \to \infty} \left[\frac{4}{(\sqrt{1+2x} + \sqrt{1-2x})} \right]$

$= \frac{4}{\sqrt{1+\infty} + \sqrt{1-\infty}}$

$= \frac{4}{\infty}$

$= 0$

$\therefore \lim_{x \to \infty} \left[\frac{\sqrt{1+2x} - \sqrt{1-2x}}{x} \right] = 0$

b) $\lim_{x \to \infty} \left[\sqrt{1+x} - \sqrt{x} \right]$, $(\infty - \infty)$

Solution

$\lim_{x \to \infty} \left[\sqrt{1+x} - \sqrt{x} \right]$

$= \lim_{x \to \infty} \left[(\sqrt{1+x} - \sqrt{x}) \times \frac{(\sqrt{1+x} + \sqrt{x})}{(\sqrt{1+x} + \sqrt{x})} \right]$

$= \lim_{x \to \infty} \left[\frac{(\sqrt{1+x})^2 - (\sqrt{x})^2}{(\sqrt{1+x} + \sqrt{x})} \right]$

$= \lim_{x \to \infty} \left[\frac{1+x-x}{(\sqrt{1+x} + \sqrt{x})} \right]$

$= \lim_{x \to \infty} \left(\frac{1}{\sqrt{1+x} + \sqrt{x}} \right)$

$= \frac{1}{\sqrt{1+\infty} + \sqrt{\infty}}$

$= \frac{1}{\infty + \infty}$

$= \frac{1}{\infty}$

$= 0$

$\therefore \lim_{x \to \infty} \left[\sqrt{1+x} - \sqrt{x} \right] = 0$

c) $\lim_{x \to 0} \left[\frac{\sqrt{1-x^2} - \sqrt{1+x^2}}{x^2} \right] \quad \left(\frac{0}{0}\right)$

Solution

$$= \lim_{x \to 0} \left[\frac{\sqrt{1-x^2} - \sqrt{1+x^2}}{x^2} \times \frac{\sqrt{1-x^2} + \sqrt{1+x^2}}{\sqrt{1-x^2} + \sqrt{1+x^2}} \right]$$

$$= \lim_{x \to 0} \left[\frac{(\sqrt{1-x^2})^2 - (\sqrt{1+x^2})^2}{x^2(\sqrt{1-x^2} + \sqrt{1+x^2})} \right]$$

$$= \lim_{x \to 0} \left[\frac{1-x^2-1-x^2}{x^2(\sqrt{1-x^2} + \sqrt{1+x^2})} \right]$$

$$= \lim_{x \to 0} \left[\frac{-2x^2}{x^2(\sqrt{1-x^2} + \sqrt{1+x^2})} \right]$$

$$= \lim_{x \to 0} \left[\frac{-2}{\sqrt{1-x^2} + \sqrt{1+x^2}} \right]$$

$$= \frac{-2}{\sqrt{1-0^2} + \sqrt{1+0^2}}$$

$$= \frac{-2}{1+1}$$

$$= -1$$

$$\therefore \lim_{x \to 0} \left[\frac{\sqrt{1-x^2} - \sqrt{1+x^2}}{x^2} \right] = -1$$

d) $\lim_{x \to 0} \left[\dfrac{2x}{(\sqrt{a+x} - \sqrt{a-x})} \right] , \left(\dfrac{0}{0}\right)$

Solution

$$\lim_{x \to 0} \left[\frac{2x}{(\sqrt{a+x} - (\sqrt{a-x})} \times \frac{(\sqrt{a+x} + \sqrt{a-x})}{(\sqrt{a+x} + \sqrt{a+x})} \right]$$

$$= \lim_{x \to 0} \left[\frac{2x(\sqrt{a+x} + \sqrt{a-x})}{(\sqrt{a+x})^2 - (\sqrt{a-x})^2} \right]$$

$$= \lim_{x \to 0} \left[\frac{2x(\sqrt{a+x} + \sqrt{a-x})}{a+x-a+x} \right]$$

$$= \lim_{x \to 0} \left[\frac{2x(\sqrt{a+x} + \sqrt{a-x})}{2x} \right]$$

$$= \lim_{x \to 0} (\sqrt{a+x} + \sqrt{a-x})$$

$$= (\sqrt{a+0} + \sqrt{a-0})$$

$$= \sqrt{a} + \sqrt{a}$$

$$= 2\sqrt{a}$$

$$\therefore \lim_{x \to 0} \left[\frac{2x}{(\sqrt{a+x} - \sqrt{a-x})} \right] = 2\sqrt{a}$$

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e) $\lim_{x \to a} \left[\dfrac{\sqrt{3a-x} - \sqrt{x+a}}{4(x-a)} \right] , \left(\dfrac{0}{0}\right)$

Solution

$$= \lim_{x \to a} \left[\frac{\sqrt{3a-x} - \sqrt{x+a}}{4(x-a)} \times \frac{\sqrt{3a-x} + \sqrt{x+a}}{\sqrt{3a-x} + \sqrt{x+a}} \right]$$

$$= \lim_{x \to a} \left[\frac{(\sqrt{3a-x})^2 - (\sqrt{x+a})^2}{4(x-a)(\sqrt{3a-x} + \sqrt{x+a})} \right]$$

$$= \lim_{x \to a} \frac{3a - x - x - a}{4(x-a)(\sqrt{3a-x} + \sqrt{x+a})}$$

$$= \lim_{x \to a} \frac{2a - 2x}{4(x-a)(\sqrt{3a-x} + \sqrt{x+a})}$$

$$= \lim_{x \to a} \frac{-2(x-a)}{4(x-a)(\sqrt{3a-x} + \sqrt{x+a})}$$

$$= \lim_{x \to a} \frac{-1}{2(\sqrt{3a-x} + \sqrt{x+a})}$$

$$= \frac{-1}{2(\sqrt{3a-a} + \sqrt{a+a})}$$

$$= - \frac{1}{2(\sqrt{2a} + \sqrt{2a})}$$

$$= - \frac{1}{2 \times 2\sqrt{2a}}$$

$$= - \frac{1}{4\sqrt{2a}}$$

3\. Evaluate:

a) $\lim_{x \to 3} \left[\frac{x^4 - 81}{x-3} \right]$

Solution

$$= \lim_{x \to 3} \left[\frac{(x)^4 - (3)^4}{x-3} \right]$$

$$= 4 \times 3^{4-1} \qquad \left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x-a} \right) = na^{n-1} \right]$$

$$= 4 \times 3^3$$

$$= 4 \times 27$$

$$= 108$$

$$\therefore \lim_{x \to 3} \left[\frac{x^4 - 81}{x-3} \right] = 108$$

b) $\lim_{x \to a} \left[\frac{x^5 - a^5}{x^3 - a^3} \right]$

Solution

$$\lim_{x \to a} \left[\frac{x^5 - a^5}{x^3 - a^3} \right]$$

Dividing numerator & denominator by $x-a$,

$$= \lim_{x \to a} \left[\frac{\frac{x^5 - a^5}{x-a}}{\frac{x^3 - a^3}{x-a}} \right]$$

$$= \frac{\lim_{x \to a} \frac{x^5 - a^5}{x-a}}{\lim_{x \to a} \frac{x^3 - a^3}{x-a}}$$

$$= \frac{5a^{5-1}}{3a^{3-1}} \qquad \left[\because \lim_{x \to a} \left[\frac{(x^n - a^n)}{x-a} \right] = na^{n-1} \right]$$

$$= \frac{5a^4}{3a^2}$$

$$= \frac{5a^2}{3}$$

$$\therefore \lim_{x \to a} \left[\frac{x^5 - a^5}{x^3 - a^3} \right] = \frac{5a^2}{3}$$

c) $\lim_{x \to 16} \left[\frac{x^{1/4} - 2}{x^{1/2} - 4} \right]$

Solution

$$= \lim_{x \to 16} \left[\frac{(x)^{1/4} - (16)^{1/4}}{(x)^{1/2} - (16)^{1/2}} \right]$$

Dividing numerator and denominator by $x-16$,

$$= \frac{\lim_{x \to 16} \frac{(x)^{1/4} - (16)^{1/4}}{x-16}}{\lim_{x \to 16} \frac{(x)^{1/2} - (16)^{1/2}}{x-16}}$$

$$= \frac{\frac{1}{4} \times 16^{1/4 - 1}}{\frac{1}{2} \times 16^{1/2 - 1}} \qquad \left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1} \right]$$

$$= \frac{\frac{1}{4} \times \frac{1}{8}}{\frac{1}{2} \times \frac{1}{4}}$$

$$= \frac{\frac{1}{32}}{\frac{1}{8}}$$

$$= \frac{1}{4}$$

$$\therefore \lim_{x \to 16} \left[\frac{x^{1/4} - 2}{x^{1/2} - 4} \right] = \frac{1}{4}$$

5. If $\lim_{x \to a} \left[\frac{x^9 - a^9}{x - a} \right] = 9$, find the possible values of a.

Solution

Given, $\lim_{x \to a} \left[\frac{x^9 - a^9}{x - a} \right] = 9$

or, $9a^{9-1} = 9$ $\left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1} \right]$

or, $9a^8 = 9$

or, $a^8 = 1$

$\therefore a = \pm 1$

6. If $\lim_{x \to 2} \left[\frac{x^n - 2^n}{x - 2} \right] = 80$ and $n \in N$, find n.

Solution

Given, $\lim_{x \to 2} \left[\frac{x^n - 2^n}{x - 2} \right] = 80$

or, $n \times 2^{n-1} = 80$ $\left[\because \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1} \right]$

or, $n \times 2^{n-1} = 5 \times 16$

or, $n \times 2^{n-1} = 5 \times 2^4$

or, $n \times 2^{n-1} = 5 \times 2^{5-1}$

By using hit and trial method,

$\therefore n = 5$

7. Find 'K' when, if $\lim_{x\to 1}\left[\frac{x^4-1}{x-1}\right] = \lim_{x\to K}\left[\frac{x^3-K^3}{x-K}\right]$

Given,

$$\lim_{x\to 1}\left[\frac{x^4-1^4}{x-1}\right] = \lim_{x\to K}\left[\frac{x^3-K^3}{x-K}\right]$$

or, $4.1^{4-1} = 3.K^{3-1}$ $\left[\because \lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = na^{n-1}\right]$

or, $4 = 3K^2$

or, $K^2 = \frac{4}{3}$

$\therefore K = \pm \frac{2}{\sqrt{3}}$

8. If $f(x) = \frac{ax+b}{x-5}$, $\lim_{x\to 0} f(x) = -1$ and $\lim_{x\to\infty} f(x) = 3$, find $f(6)$.

Solution

Given,

$\lim_{x\to 0} f(x) = -1$

or, $\lim_{x\to 0}\left[\frac{ax+b}{x-5}\right] = -1$ $\left[\because f(x) = \frac{ax+b}{x-5}\right]$

or, $\frac{a\times 0+b}{0-5} = -1$

or, $\frac{b}{-5} = -1$

$\therefore b = 5$

Again,

$$\lim_{x \to \infty} f(x) = 3$$

or, $\lim_{x \to \infty} \left[\frac{ax+b}{x-5} \right] = 3 \quad \left[\because f(x) = \frac{ax+b}{x-5} \right]$

or, $\lim_{x \to \infty} \left[\frac{ax+5}{x-5} \right] = 3 \quad [\because b = 5]$

Dividing numerator and denominator by x,

or, $\lim_{x \to \infty} \left[\dfrac{\frac{ax}{x} + \frac{5}{x}}{\frac{x}{x} - \frac{5}{x}} \right] = 3$

or, $\lim_{x \to \infty} \left[\dfrac{a + \frac{5}{x}}{1 - \frac{5}{x}} \right] = 3$

or, $\cancel{\lim_{x \to \infty}} \left[\dfrac{a + \frac{5}{\infty}}{1 - \frac{5}{\infty}} \right] = 3$

or, $\frac{a+0}{1-0} = 3$

$\therefore a = 3$

$\therefore a = 3$ & $b = 5$

Finally,

$$f(x) = \frac{ax+b}{x-5} = \frac{3x+5}{x-5}$$

~~when $x=6$~~

~~wt~~

when $x=6$,

$$f(6) = \frac{3\times6+5}{6-5}$$

$$= 23$$

9. If $f(x) = \frac{2ax+b}{x-1}$, $\lim_{x\to0} f(x) = -3$, $\lim_{x\to\infty} f(x) = 4$, prove that $f(2) = 11$.

Solution

Given,

$$\lim_{x\to0} f(x) = -3$$

or, $\lim_{x\to0}\left[\frac{2ax+b}{x-1}\right] = -3 \quad \left[\because f(x) = \frac{2ax+b}{x-1}\right]$

or, $\frac{2a\times0+b}{0-1} = -3$

or, $\frac{b}{-1} = -3$

$\therefore b = 3$

Again,

$$\lim_{x\to\infty} f(x) = 4$$

or, $\lim_{x\to\infty}\left[\frac{2ax+b}{x-1}\right] = 4 \quad \left[\because f(x) = \frac{2ax+b}{x-1}\right]$

or, $\lim_{x\to\infty}\left[\frac{2ax+3}{x-1}\right] = 4 \quad [\because b=3]$

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Dividing numerator and denominator by x,

or $\lim_{x\to\infty}\boxed{\dfrac{\frac{2ax}{x}+\frac{3}{x}}{\frac{x}{x}-\frac{1}{x}}}=4$

or, $\lim_{x\to\infty}\boxed{\dfrac{2a+\frac{3}{x}}{1-\frac{1}{x}}}=4$

or, $\dfrac{2a+\frac{3}{\infty}}{1-\frac{1}{\infty}}=4$

or, $\dfrac{2a+0}{1-0}=4$

or, $2a=4$

$\therefore a=2$

$\therefore a=2$ & $b=3$

Finally,

$f(x)=\dfrac{2ax+b}{x-1}$

$=\dfrac{2\times 2\times x+3}{x-1}$

$=\dfrac{4x+3}{x-1}$

$\therefore f(x)=\dfrac{4x+3}{x-1}$

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When $x = 2$,

$$f(2) = \frac{4\times2+3}{2-1}$$

$$= \frac{8+3}{1}$$

$$= 11$$

$\therefore f(2) = 11$ proved.

2. Evaluate:
prove that:

c) $\lim_{x\to3}\left[\frac{1}{x-3} - \frac{3}{x(x^2-5x+6)}\right] = \frac{4}{3}$

Solution

$$LHS = \lim_{x\to3}\left[\frac{1}{x-3} - \frac{3}{x(x^2-5x+6)}\right]$$

$$= \lim_{x\to3}\left[\frac{1}{x-3} - \frac{3}{x(x^2-3x-2x+6)}\right]$$

$$= \lim_{x\to3}\left[\frac{1}{x-3} - \frac{3}{x(x-3)(x-2)}\right]$$

$$= \lim_{x\to3}\left[\frac{x(x-2)-3}{x(x-2)(x-3)}\right]$$

$$= \lim_{x\to3}\left[\frac{x^2-2x-3}{x(x-2)(x-3)}\right]$$

$$= \lim_{x\to3}\left[\frac{x^2-3x+x-3}{x(x-2)(x-3)}\right]$$

$$= \lim_{x\to3}\left[\frac{x(x-3)+1(x-3)}{x(x-2)(x-3)}\right]$$

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$$= \lim_{x \to 3} \left[\frac{(x-3)(x+1)}{x(x-2)(x-3)} \right]$$

$$= \lim_{x \to 3} \left[\frac{x+1}{x(x-2)} \right]$$

$$= \frac{3+1}{3(3-2)}$$

$$= \frac{4}{3} \text{ proved}$$

4. Evaluate the following limits:

Prove that:

i) $\lim_{x \to 1} \left[\frac{x - \sqrt{2-x^2}}{2x - \sqrt{2+2x^2}} \right] = 2$

Solution

$$\text{LHS} = \lim_{x \to 1} \left[\frac{x - \sqrt{2-x^2}}{2x - \sqrt{2+2x^2}} \right]$$

$$= \lim_{x \to 1} \left[\frac{x - \sqrt{2-x^2}}{2x - \sqrt{2+2x^2}} \times \frac{x+\sqrt{2-x^2}}{x+\sqrt{2-x^2}} \times \frac{2x+\sqrt{2+2x^2}}{2x+\sqrt{2+2x^2}} \right]$$

$$= \lim_{x \to 1} \left[\frac{(x)^2 - (\sqrt{2-x^2})^2}{(2x)^2 - (\sqrt{2+2x^2})^2} \times \frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-x^2}} \right]$$

$$= \lim_{x \to 1} \left[\frac{x^2 - 2 + x^2}{4x^2 - 2 - 2x^2} \times \frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-x^2}} \right]$$

$$= \lim_{x \to 1} \left[\frac{2x^2 - 2}{2x^2 - 2} \times \frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-x^2}} \right]$$

$$= \lim_{x \to 1} \left[\frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-x^2}} \right]$$

$$= \frac{2 \times 1 + \sqrt{2 + 2 \times (1)^2}}{1 + \sqrt{2 - (1)^2}}$$

$$= \frac{2 + \sqrt{4}}{1 + \sqrt{1}}$$

$$= \frac{2+2}{1+1}$$

$$= \frac{4}{2}$$

$= 2$ proved.