Kriti's

Solution Manual of

Principles of MATHEMATICS

Grade XII

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CHAPTER 1

PERMUTATION AND COMBINATION

EXERCISE 1.1

1. There are 5 air flights and 15 buses per day to travel from Bhairahawa to Kathmandu. In how many ways can a man travel from Bhairahawa to Kathmandu per day?

Solution:

Total no. of air flights (n_1) = 5
Total no. of buses (n_2) = 15
As from the addition rule
The no. of ways to travel from Bhairahawa to Kathmandu = 5 + 15 = 20
Hence, there are 20 ways to travel.

3. There are 5 routes from station A to station B and 4 routes from station B to C. Answer the questions

a. How many different routes are possible from station A to station C?
b. In how many ways can a person travel from A to C and come back from C to A?
c. In how many ways can a person travel from A to C and return back to A by using different routes?

Solution:

If there are 5 routes from station A to station B and 4 routes from station B to C then, the no. of possible routes from A to B is 5 and from B to C is 4.

a. Here,
∴ The no. of possible routes from A to C is 5×4 = 20 routes.

b. Here,
The no. of possible routes from A to C is 20 and so as to return from C to A there is also 20 routes (i.e. 4×5).
Hence, the no. of required ways = 20×20 = 400 ways

c. Here, the no. of routes to travel from A to C is 20. If the same route is not used more than once, the–n the no. of ways to travel and return back is 20×12 = 240 ways

4. How many numbers of three different digits can be made by using the digits 1, 2, 3, 4, 5, 6? How many of them are even?

Solution:

The no. of digits = 6
So, hundred place can be arranged in 6 ways
Tens place can be arranged in 5 ways
Units place can be arranged in 4 ways
∴ Required numbers = 6×5×4 = 120
Next, If these numbers formed must be even, the digit in the units place can be arranged in 3 ways
Ten's place can be arranged in 5 ways
Hundred place can be arranged in 4 ways
∴ Required number's place = 3×5×4 = 60 ways

5. How many 4 digit numbers are possible by using the digits 0, 1, 2, 3, 4, 5? If (a) the repetition of the digits is not allowed? (b) if the repetition of the digits is allowed?

Solution:

The no. of digits = 6

a. As we know, units place can never be filled by zero, so units place can be filled by 5 ways

Tens place can be filled by 5 ways

Hundred place can be filled by 4 ways

Thousand place can be filled by 3 ways

$\therefore$ The required no.s of 4 digit when repetition is not allowed

$= 5\times5\times4\times3 = 300$ ways

b. If the repetition is allowed, the unit place can be arranged/filled by 5 ways and then after all remaining places can be filled by 6 ways.

$\therefore$ The required no.s of 4 digit when repetition is allow $= 5\times6\times6\times6 = 1080$ ways

6. How many positive numbers less than 100 are possible by using the digits 0, 1, 2, 3? If the repetition of the digits is not allowed, how many such numbers are possible?

Solution:

The given digits are 0, 1, 2, 3

If the digits may repeat: then

For 1 digits: For the units place, number of way = 4

For the ten's place, number of ways = 3

$\therefore$ Number of ways $= 4 \times 3 = 12$

For 1 digit: The number of ways = 3

So, total number of ways = 12 + 3 = 15 ways

If the digits may not repeat:

For 1 digit: Number of ways = 3

For two digits = Number of ways in tens place = 3

Number of ways in ones place = 3

$\therefore$ Number of ways $= 3\times3 = 9$

So, total number of ways = 3 + 9 = 12 ways

7. How many even numbers are possible between 2000 and 3000 by using the digits 0, 1, 2, 3, 4, only once?

Solution:

The no. of digits = 5

The number must lies between 2000 and 3000 and so each no. should be started with 2. As the formed no. should be even each no. must be ended with 0,or 2 but here digits can be used only once.

So, units place can be filled by 1 ways

Tens place can be filled by 4 ways

Hundred place can be filled by 3 ways

Thousand place can be filled by 1 ways

$\therefore$ Required no. of digits $= 1\times4\times3\times1 = 12$ ways

8. How many numbers of three digits can be formed from the integres 2, 3 ,4 5, 6 ? How many of them will be divisible by 5 ?

Solution:

Here, the number of given digits = 5

And, the number of digits to be selected = 3

The hundred digit of a number can be choosen in 5 ways

The ten digits of a number can be choosen in (5 – 1) ways = 4 ways

The unit of digit of a number can be choosen in (5 – 2) ways = 3 ways

$\therefore$ By basic principle of counting, the total no. of ways $= 5 \times4\times3 = 60$ ways

Again, for the numbers divisible by 5, we fix the digit 5 in the units place. So, there is only one choice for filling up the unit place. There are 4 ways of filling up the ten's place and 3 ways of filling up the hundred's place. So, the number of digits that are divisible by 5 = 1×4×3 = 12 ways.

EXERCISE 1.2

1. Solve for n, if $\frac{(n+1)!}{(n-1)!} = 12$ where, n is whole number.

Solution

We have, $\frac{(n+1)!}{(n-1)!} = 12$

or, $\frac{(n+1)\, n\, (n-1)!}{(n-1)!} = 12$

or, $n(n+1) = 12$

or, $n^2 + n - 12 = 0$

or, $n^2 + 4n - 3n - 12 = 0$

or, $(n+4)(n-3) = 0$

either $n = -4$

or, $n = 3$

Since, $n \neq -4$, so $n = 3$

2. If P (5, r) = 5 find the value of r.

Solution:

We have, P(5, r) = 5

r = 1 [$\because$ if P(n, r) = n, then r = 1)

3. How many numbers of 4 digits can be formed from the digits 0, 1, 2, 3, 4, 5

a. If the repetition of the digits is allowed.

b. If the repetition of the digits is not allowed.

Solution:

The no. of digits (n) = 6

a. Units place can be filled only by 5 digits but the remaining 3 places can be filled 6 digits as the repetition is allowed.

$\therefore$ The required no. of 4 digits = 5×6×6×6 = 1080

b. Unit first place can be filled by 5 digit as the repletion not allowed.

2^{nd} first place can be filled by 5 digit

3^{rd} first place can be filled by 4 digit

4^{th} first place can be filled by 3 digit

$\therefore$ The required no. of 4 digit = 5×5×4×3 = 300 ways

4. In how many ways can 4 boys and 3 girls be seated in a row containing 7 seats if they sit anywhere? If they seat alternately how many such arrangements are possible?

Solution:

There are 4 boys and 3 girls be seated in a row containing 7 seats.

$\therefore$ Required arrangement is $p(7, 7) = \frac{7!}{(7-7)!} = \frac{7!}{0!} = 5040$

Again,

If they seat alternatively, then 4 boys can set in 4! ways and 3 girls can seat in 3! ways.

$\therefore$ Required arrangement is = 4! ×3! = 24×6 = 144 ways

5. How many 6 digits numbers can be formed by using the digits 0 to 9 only once? How many such arrangements are divisible by 10?

Solution:

The total no. of digits = 10

The first digit can be chosen from only 1 to 9 so there is only 9 choices for first digit. The remaining 5 digits can be chosen from remaining 9 digits in p(9, 5) ways

i.e. $\frac{9!}{(9-5)!} = \frac{15}{20}$ ways

$\therefore$ The total numbers of 6 digits is 9×15120 ways = 136080 ways

Next: For the divisible by 10. Last digit must be zero, so the last digit can be chosen from 0, so there is 1 choice for last digit. The remaining 5 digits can be chosen from 9 digits in p(95) way

i.e. $\frac{9!}{(9-5)!}$ = 15120 ways

6. How many even numbers of at most 2 digits can be formed by using the digits 1, 2, 3, 4, 5?

Solution:

The numbers given in the question is 1, 2, 3, 4, 5

For one digit: No. of ways for even = 2

For two digits: No. of ways for ones place = 2

Number of ways for ten's place = 4

$\therefore$ Total no. of ways $2 + 2 \times 4 = 10$

7. In how many ways can 9 different colour beads be set in a bracelet?

Solution:

In a bracelet, beads are arrangement in circular form and the anticlockwise and clockwise arrangements are not different.

Here the total number of beds n = 9

They can be arranged in $\frac{(n-1)!}{2}$ ways = $\frac{1}{2} \times 8!$ ways = 20160

8. How many numbers lying between 100 and 1000 can be formed with the digits 0, 1, 2, 3, 4, 5 if the repetition of the digits is not allowed?

Solution:

The no. lying between 100 and 1000 is of 3 digit. In which at unit place can be chosen only from 5 digit and hundred place can only be chosen from 5 digit where as remaining tens place can be chosen from remaining 4 digit.

$\therefore$ The no. formed between 100 and 1000 = $\frac{5!}{(5-1)!} \times \frac{4!}{(4-1)!} \times \frac{5!}{(5-1)!}$

$$= \frac{5 \times 4!}{4!} \times \frac{4 \times 3!}{3!} \times \frac{5 \times 4!}{4!} = 5 \times 4 \times 5 = 100 \text{ ways}$$

9. In how many ways can a man post 5 post cards if 4 post boxes are available?

Solution:

Each post cards can be posted in 4 ways

Hence, required number of ways = $n^r = 5^5 = 1024$

10. In how many ways can letters of following words be arranged without any restriction?

a. PERMUTATION b. INTERMEDIATE

c. EXAMINATION

Solution:

a. Here,
PERMUTATION
Total no. of letters (n) = 11
No. of letter 'T' (p) = 2
$\therefore$ Total number of way of arrangement = $\frac{n!}{p!} = \frac{11!}{2!}$

b. INTERMEDIATE
Here, the total number of letters (n) = 12
No. of letter 'I' (p) = 2
No. of letter 'T' (q) = 2
No. of letter 'E' (r) = 3
$\therefore$ The total no. of arrangement = $\frac{n!}{p!\ q!\ r!} = \frac{12!}{2!\ 2!\ 3!}$

c. EXAMINATION
Here, the total number of letter (n) = 11
No. of letters 'A' (p) = 2
No. of letters 'I' (q) = 2
No. of letters 'N' (r) = 2
$\therefore$ Total no. of arrangement = $\frac{n!}{p!\ q!\ r!} = \frac{11!}{2!\ 2!\ 2!}$

11. In how many ways can the letters of the word 'ARRANGE' be arranged? How many such arrangements are having two R's together? How many of them are not having two R's together?

Solution:

In 'ARRANGE'
Total no. of letters (n) = 7
No. of letter 'A' (p) = 2
No. off letter 'R' (q) = 2
$\therefore$ Total no. of ways of arrangement = $\frac{n!}{p!\ q!} = \frac{7!}{2!\ 2!}$ = 1260 ways
If we suppose (RR) as one letter, then the no. of letters will be 6
$\therefore$ The no. of ways of arrangement when R comes together = $\frac{n!}{p!} = \frac{6!}{2!}$ = 360 ways
Thus, the required no. of ways of arrangement when two R not comes together = 1260 – 360 = 900 ways

12. In how many ways can the letters of the word "SUNDAY" be arranged? How many of these arrangements donot begin with S? How many begin with S and donot end with Y?

Solution:

Total number of letters in the word 'SUNDAY' = 6.
Since each of the letters is distinct, they can be arranged in 6! ways. i.e. 720 ways.
If the arrangements do not begin with S, the beginning letter can be chosen in 5 ways.
The remaining successive letters can be arranged in 5! ways.
So, the total number of ways = 5 × 5! = 600.
If it is to be begun with S the first letter can be chosen in 1 ways. If it doesn't end with Y, the last letter can be chosen in 4 ways and the remaining middle 4 letters can be chosen in 4! ways.
So, total number of ways = 1 × 4 × 4! = 96 ways.

13. In how many ways can the letters of the word 'UNIVERSITY' be arranged? How many such arrangements begin with 'U'? How many of these begin with U but do not end with Y?

Solution:

In UNIVERSITY'
The no. of letters (n) = 10
No. of letter 'I' (p) = 2

$\therefore$ Total no. of arrangement = $\frac{n!}{p!} = \frac{10!}{2!}$ = 1814400 ways

Since the arrangement begin with U there is only. Nine letters to arrange. So, the nine letters can be arranged in = $\frac{n!}{p!} = \frac{9!}{2!}$ = 181440

$\therefore$ Required no. of arrangement = 1×181440 = 181440 ways

Next: The total no. of ways in which the arrangement begin with U but do not end with

'Y' = $4 \times p(8, 8) = 4 \times \frac{8!}{0!}$ = 161280 ways

14. If 8 secretary level representatives of SAARC countries sit in round table conference, how many arrangements are possible? If Nepali and Indian secretary level sit always together, how many such arrangements will be there?

Solution:

Total no. of countries (n) = 8
If they sit in round table then they form a circle, so its arrangement is = $(n - 1)! = 7!$ = 5040 ways
If Nepali and Indian always sit together, then we take it as one. Then the total no. will be 6. So, the arrangement $(n–r+1)! = (7–2+1)! = 6! = 720$
If they sit together, then they also can interchange there seat between themselves in 2 ways.
Hence, the required no. of arrangement = 21×720 = 1440 ways

15. How many arrangements are possible using the letters of the word 'EQUATION' only once? If all vowels come together how many arrangements are possible?

Solution:

In 'EQUATION'
The no. of total letters (n) = 8

$\therefore$ The total no. of arrangement = 8! = 40320

Next, The no. of vowels = 5
When we take all vowels as one then there will be total letters left = 4. Also the vowel letters be arranged themselves in 5 ways.

$\therefore$ The required no. of arrangement = $4! \times 5!$ = 2880

EXERCISE 1.3

1. Find the value of n and r if possible
   a. If C(n, 10) = C(n, 12) find 'n' and hence C(n, 6)
   b. If C(n, 8) = C(n, 6), find C(n, 2).
   c. If C(n, 30) = C(n, 4) find the value of C(n, 30) + C(n, 4)
   d. If C(9, 2r) = C(9, 3r – 1), find the value of r.

Solution:

a. Here, c(n, 10) = (ln, 12)

$$\Rightarrow \frac{n!}{(n-10)!\,10!} = \frac{n!}{(n-12)!\,2!}$$

$$\Rightarrow \frac{(n-12)!\,12!}{(n-10)!\,10!} = \frac{n!}{n!}$$

$$\Rightarrow \frac{(n-12)!\,12\times11\times10!}{(n-10)\,(n-11)\,(n-12)!\,10!} = 1$$

$$\Rightarrow \frac{12\times11}{(n-10)\,(n-11)} = 1$$

$\Rightarrow$ $132 = n^2 - 11n - 10n + 110$

$\Rightarrow$ $n^2 - 21n - 22 = 0$

$\Rightarrow$ $n^2 - 22n + n - 22 = 0$

$\Rightarrow$ $n(n - 22)\ 11(n - 22) = 0$

$\Rightarrow$ $(n - 22)\,(n + 1) = 0$

either n = 22

or, n = –1 (This is not possible, so rejected)

$\therefore$ n = 22

Next $C(n, 6) = C(22, 6) = \frac{22!}{(22-6)!\,6!} = \frac{22!}{16!\,16!}$

b. Here, C(n, 8) = C(n, 6)

Then, n = 8 + 6 = 14

Now, $C(14, 2) = \frac{14!}{12!\,2!} = 91$ ways

c. Given, C(n, 30) = C(n, 4)

$\Rightarrow$ $C(n, r) = C(n, r^1)$

$\Rightarrow$ $r + r^1 = n$

Then, 30 + 4 = n

$\therefore$ n = 34

Now, $C(n, 30) + C(n, 4) = \frac{34!}{(34-4)!\,14!} + \frac{34!}{20!\,14!}$

$C(n, 30) + C(n, 4) = \frac{34!}{4!30!} + \frac{34!}{30!\,4!} = 46376 + 46376 = 92752$ ways

d. Given, c(9, 2r) = c(9, 3r – 1)

$\Rightarrow$ $2r + 3r - 1 = 9$ [$\because C(n, r) = C(n, r^1) = r + r^1 = n$]

$\Rightarrow$ $5r = 10$

$\Rightarrow$ $r = 2$ or, $2r = 3r - 1$

$\Rightarrow$ $r = 1$

$\therefore$ r = 1 or 2

2. In an advertisement of 3 workers in a factory, if 10 applicants apply, how many ways can the selection be made?

Solution:

The no. of workers required in 3 where total applicant is 10

$\therefore$ The selection be made as $c(10, 3) = \frac{10!}{7!\,3!}$ $\left[\because c(n, r) = \frac{n!}{(n-n!\,r!)}\right]$

= 120 Ans.

3. In a class of 25 girls and 20 boys, a boy and a girl are to be chosen for debate competition. In how many ways can the selection be made?

Solution:

The no. of girls and boys are 25 and 20 respectively. If a boy and a girl are to be chosen for debate competition, then
The no. of ways of selection would be 25×20 = 500 ways.

4. A person has 12 friends of whom 8 are relatives. In how many ways can be invite 7 guests such that 5 of may be relatives?

Solution:

To invite 7 guests out of 12 friends

Relatives (8)	Non-relative (4)	Selection
5	2	c(8, 5) × c(4, 2)

$\therefore$ The required selection is 6(8, 9) × c(4, 2) = $\frac{8!}{3!\,5!} \times \frac{4!}{2!\,2!}$ = 56 × 6 = 336 Ans.

5. There are 10 questions in group A of which 6 are to be solved. In group B there are only 6 questions of which 4 are to be solved. In how many ways an examinee can make up his choices?

Solution:

Here,

Group A (10)	Group B(6)	Selection
6	4	c(10, 6) × (6, 4)

$\therefore$ The required selection is c(10, 6) × c(6, 4) = $\frac{10!}{4!\,6!} \times \frac{6!}{2!\,4!}$ = 210 × 15 = 3150

6. A bag contains 5 red balls. In how many ways can be selected at most 3 balls ?

Solution:

Total number of balls = 5 and maximum number of balls to select = 3.
Thus, we can select in C(5, 0) + C(5, 1) + C(5, 2) + C(5, 3) = 1 + 5 + 10 + 10 = 26 ways.

7. Bidur has 4 different coins of 1 rupee, 2 rupee, 5 rupee and 10 rupee each. How many different sums can be made by using these coins?

Solution:

From the 4 coins, the sum can be made in the following ways:

$$C(4, 1) + C(4, 2) + C(4, 3) + C(4, 4) = \frac{4!}{3!\,1!} + \frac{4!}{2!\,2!} + \frac{4!}{1!\,3!} + \frac{4!}{0!\,4!}$$
$$= 4 + 6 + 4 + 1 = 15$$

8. There are 15 cricket players in a class. In how many ways can a team of 11 be made if
   a. Two particular persons are always included
   b. Two persons are always excluded.

Solution:

The no. of player in class = 15
The no. of players taken in team (r) = 11

$\therefore$ Required no. of ways of selection = C(15, 11) = $\frac{15!}{4!\,11!}$ = 1365

a. Here, if 2 particular persons are always included then there will be 13 players in class and 9 players required to be selected.

$\therefore$ Required selection is C(13, 9) = $\frac{13!}{4!\,9!}$ = 715

b. Here, If 2 persons are always excluded then there will be 13 players in class and 11 players to be selected.

$\therefore$ Required selection = C(13, 11) = $\frac{13!}{2!\ 11!}$ = 78

9. Out of 5 members belonging to party A and 6 members belonging to party B, in how many ways can a committee of 8 members be formed so that party B has always majority in the committee?

Solution:

Party A(5)	Party B(6)	Selection
3	5	(15, 3) × C(6, 5)
2	6	C(5, 2) × C(6, 6)

$\therefore$ Required selection = C(5, 3) × C(6, 5) + C(5, 2) × C(6, 6)

$$= \frac{5!}{2!\ 3!} \times \frac{6!}{1!\ 5!} = \frac{5!}{3!\ 2!} \times \frac{6!}{0!\ 6!} = 10\times6 + 10\times1 = 70 \text{ Ans.}$$

10. In a paper of 2 groups of 5 questions each, in how many ways can 6 questions be answered if at least 2 questions from each group are to be attempted?

Solution:

Group A(5)	Group B(5)	Selection
2	4	C(5, 2) × c(5, 4)
3	3	C(5, 3) × c(5, 3)
4	2	C(5, 4) × c(5, 2)

$\therefore$ The required selection = C(5, 2)×C(5,4) + C(5, 3)×C(5, 3) + C(5, 4)×C(5, 2)

$$= \frac{5!}{3!\ 2!} \times \frac{5!}{1!\ 4!} + \frac{5!}{2!\ 3!} \times \frac{5!}{2!\ 3!} \times \frac{5!}{1!\ 4!} \times \frac{5!}{3!\ 2!}$$

$$= 10 \times 5 + 10 \times 10 + 5 \times 10 = 50 + 100 + 50 = 200.$$

11. Out of 6 ladies and 8 gentle men a committee of 11 is to be formed. In how many ways can this be done if the committee contains (a) exactly 4 ladies (b) at least 4 ladies?

Solution:

a.

Ladies (6)	Gentle (8)	Selection
4	7	c(6, 4) × c(8, 7)

$\therefore$ Required selection is C(6, 4) × C(8, 7) = $\frac{6!}{2!\ 4!} \times \frac{8!}{1!\ 7!}$ = 15 × 8 = 120

b.

Ladies (6)	Gentle (8)	Selection
4	7	C(6, 4) × C(8, 7)
5	6	C(6, 5) × C(8, 6)
6	5	C(6, 6) × C(8, 5)

$\therefore$ Required selection = C(6, 4) × C(8, 7) + C(6, 5) × C(8, 6) + C(6, 6) × C(8, 5)

$$= \frac{6!}{2!\ 4!} \times \frac{8!}{1!\ 7!} + \frac{6!}{1!\ 5!} \times \frac{8!}{2!\ 6!} + \frac{6!}{6!\ 1!} \times \frac{8!}{3!\ 5!}$$

$$= 15 \times 8 + 6 \times 28 + 1 \times 56 = 120 + 168 + 56 = 344$$

c.

Ladies (6)	Gentle (8)	Selection
4	7	C(6, 4) × C(8, 7)
3	8	C(6, 3) × C(8, 8)

$\therefore$ Required selection = C(6, 4) × C(8, 7) + C(6, 3) × C(8, 8)

$$= \frac{6!}{2!\,4!} \times \frac{8!}{1!\,7!} + \frac{6!}{3!\,3!} \times \frac{8!}{0!\,8!}$$
$$= 15 \times 8 + 20 \times 1 = 120 + 20 = 140$$

12. There are 6 questions in a question paper. In how many ways can an examinee solve one or more questions?

Solution:

From the 6 question, a examinee can pass/solve one or more of questions in following ways.

A examinee can solve 1 or 2 or 3 or 4 or 5 or all

Thus, total no. of ways to solve = C(6, 1) + (6, 2) + C(6, 3) + C(6, 4) + C(6, 5) + C(6, 6)

$$= \frac{6!}{5!\,1!} + \frac{6!}{4!\,2!} + \frac{6!}{3!\,3!} + \frac{6!}{2!\,4!} + \frac{6!}{1!\,5!} + \frac{6!}{0!\,6!} = 6 + 15 + 20 + 6 + 1 = 63 \text{ Ans.}$$

13. A candidate has to pass each of the five subjects to get through. In how many ways can the candidate fail?

Solution:

A candidate fails in an examination if he cannot pass either in 1 or 2 or 3 or 4 or 5 subjects

$\therefore$ Total no. of ways by which he falls = C(5, 1)+ C(5, 2)+C(5, 3)+C(5, 4) + C(5, 5)

$$= \frac{5!}{4!\,1!} + \frac{5!}{3!\,2!} + \frac{5!}{2!\,3!} + \frac{5!}{1!\,4!} + \frac{5!}{0!\,5!} = 5 + 10 + 10 + 5 + 1 = 31 \text{ ways}$$

■

CHAPTER 2

BINOMIAL THEOREM

EXERCISE 2.1

1. Use binomial theorem to expand

a. $(3x+2y)^5$ b. $(2x-3y)^6$ c. $\left(x+\dfrac{1}{x}\right)^6$

d. $\left(x-\dfrac{1}{x}\right)^7$ e. $\left(\dfrac{2x}{3}-\dfrac{3}{2x}\right)^6$

Solution:

a. We know that,

$(a + x)^n = c(n, 0)\, a^n + c(n, 1)\, a^{n-1} x + c(n, 2)\, a^{n-2} x^2 + .. + c(n, r)\, a^{n-r} x^r + ... + c(n, n)\, x^n$.

$\therefore$ $(3x + 2y)^5 = c(5, 0)\,(3x)^5 + c(5, 1)\,(3x)^4\,(2y) + c(5, 2)\,(3x)^3\,(2y)^2 + c(5, 3)\,(3x)^2\,(2y)^3 + c(5, 4)\,(3x)^1\,(2y)^4 + c(5, 5)\,(2y)^5$

$= 243x^5 + 810x^4y + 1080x^3y^2 + 270x^2y^3 + 240xy^4 + 32y^5$

b. $(2x - 3y)^6 = {}^6c_0(2x)^6 + {}^6c_1(2x)^5\,(-3y) + {}^6c_2(2x)^4(-3y)^2 + {}^6c_3(2x)^3(-3y)^3 + {}^6c_4(2x)^2(-3y)^4 + {}^6c_5(2x)\,(-3y)^5 + {}^6c_6(-3y)^6$

$= 64x^6 - 576x^5y + 2160x^4y^2 - 4320x^3y^3 + 4860x^2y^4 - 2916xy^5 + 729y^6$.

c. $\left(x+\dfrac{1}{x}\right)^6 = 6c_0\, x^6 + 6c_1\, x^5 \dfrac{1}{x} + 6c_2\, x^4 . \dfrac{1}{x^2} + 6c_3\, x^3 \dfrac{1}{x^3} + 6c_4\, x^2 \dfrac{1}{x^4} + 6c_5\, x \dfrac{1}{x^5} + 6c_6 \dfrac{1}{x^6}$

$= x^6 + 6x^4 + 15x^2 + 20 + \dfrac{15}{x^2} + \dfrac{6}{x^4} + \dfrac{1}{x^6}$

d. $\left(x-\dfrac{1}{x}\right)^7 = x^7 + {}^7c_1\, x^6 \left(-\dfrac{1}{x}\right) + {}^7c_2\, x^5 \left(-\dfrac{1}{x}\right)^2 + {}^7c_3\, x^4 \left(-\dfrac{1}{x}\right)^3 + {}^7c_4\, x^3 \left(-\dfrac{1}{x}\right)^4 + {}^7c_5\, x^2 \left(-\dfrac{1}{x}\right)^5 + {}^7c_6\, x \left(-\dfrac{1}{x}\right)^6 + \left(-\dfrac{1}{x}\right)^7$

$= x^7 - 7x^5 + 21x^3 - 35x + 35. \dfrac{1}{x} - 21. \dfrac{1}{x^3} + 7. \dfrac{1}{x^5} - \dfrac{1}{x^7}$

e. $\left(\dfrac{2x}{3}-\dfrac{3}{2x}\right)^6 = \left(\dfrac{2x}{3}\right)^6 + {}^6c_1\left(\dfrac{2x}{3}\right)^5 . \left(\dfrac{-3}{2x}\right) + {}^6c_2 \left(\dfrac{2x}{3}\right)^4 {}^6c_3\left(\dfrac{2x}{3}\right)^3 . \left(\dfrac{-3}{2x}\right)^4 + {}^6c_4 \left(\dfrac{2x}{3}\right)^2 . \left(\dfrac{-3}{2x}\right)^4 + {}^6c_5 \left(\dfrac{2x}{3}\right) . \left(\dfrac{-3}{2x}\right)^5 + \left(\dfrac{-3}{2x}\right)^6$

$= \dfrac{64x^6}{729} - \dfrac{96}{81} x^4 + \dfrac{20}{3} x^2 - 21 + \dfrac{135}{4} . \dfrac{1}{x^2} - \dfrac{243}{8} . \dfrac{1}{x^4} + \dfrac{729}{64} . \dfrac{1}{x^6}$

2. Find

a. the 7th term in the expansion of $\left(\dfrac{2x}{3}+\dfrac{3}{2x}\right)^6$

b. the 10th term in the expansion of $\left(\dfrac{x}{y}-\dfrac{2y}{x^2}\right)^6$

c. the fifth term in the expansion of $(2x+y)^{12}$

d. the fifth term in the expansion of $\left(2x^2 + \frac{1}{x}\right)^8$

e. the 6^{th} tern in the expansion of $\left(x - \frac{1}{x}\right)^7$

Solution:

a. We know that the general term t_{r+1} of expansion of $(a + x)^n$ is given by $t_{r+1} = n_{C_r}\, a^{n-r}\, x^r$

Here,

$(a + x)^n \Rightarrow \left(\frac{2x}{3} + \frac{3}{2x}\right)^6$

$\therefore\ a \Rightarrow \frac{2x}{3}$, $x \Rightarrow \frac{3}{2x}$ and $n \Rightarrow 6$

For 7^{th} term, put r = 6

$t_{6+1} = {}^6C_6 \left(\frac{2x}{3}\right)^{6-6} \left(\frac{3}{2x}\right)^6$

$\therefore\ t_7 = 1.\ 1.\ \frac{729}{64x^6} = \frac{729}{64x^6}$

b. The total number of terms of the expansion of $\left(\frac{x}{y} - \frac{2y}{x^2}\right)^6$ is 7.

So, there is no 10^{th} term.

c. For 5^{th} term, put r = 4.

$t_{r+1} = t_{4+1} = 12_{C_4}\, (2x)^{2-4}\, y^4 = 495 \times 2^8x^8y^4 = 126720\, x^8y^4$

d. Given,

$\left(2x^2 + \frac{1}{x}\right)^8$

Which is in the form of $(a + b)^n$; where $a = 2x^2$, $b = \frac{1}{x}$, $n = 8$

We know that, $t_{r+1} = n_{C_r}\, a^{n-r}\, x^r$

$t_5 = t_{4+1} = 8_{C_4}\, (2x^2)^4 \left(\frac{1}{x}\right)^4 = 1120x^4$

e. $\left(x - \frac{1}{x}\right)^7$

$t_6 = t_{5+1} = {}^7C_5\, x^{7-5} \left(-\frac{1}{x}\right)^5 = 21x^2 \left(-\frac{1}{x^5}\right) = -\frac{21}{x^3}$

3. Find the general term in the expansion of

a. $(x^2\text{-}y)^6$ b. $\left(x^2 - \frac{1}{x}\right)^{12}$ c. $\left(\frac{x}{b} - \frac{b}{x}\right)^{10}$ d. $\left(x - \frac{1}{x}\right)^{12}$

Solution:

a. $(x^2 - y)^6$

Here, n = 6

The general term $(t_{r+1}) = {}^6C_r\, (x^2)^{6-r}\, (-y)^r = (-1)^r\, {}^6C_r\, x^{12-2r}\, y^r$

b. Given, $\left(x^2 - \frac{1}{x}\right)^{12}$

Here, n = 12

The general term $(t_{r+1}) = {}^{12}C_r\, (x^2) \left(-\frac{1}{x}\right)^r = (-1)^r\, {}^{12}C_r\, x^{24-3r}$

c. Here, n = 10

The general term $(t_{r+1}) = {}^{10}C_r \left(\frac{x}{b}\right)^{10-r} \left(-\frac{b}{x}\right)^r = (-1)^r\, {}^{10}C_r \left(\frac{x}{b}\right)^{10-2r}$

d. Given, $\left(x - \frac{1}{x}\right)^{12}$

The general term $(t_{r+1}) = {}^{12}C_r\,(x)^{12-r} \left(-\frac{1}{x}\right)^r = (-1)^r\, {}^{12}C_r\, x^{12-2r}$

4. Find the coefficient of

a. x^7 in the expansion of $\left(x^2 + \frac{1}{x}\right)^{11}$ b. x^5 in the expansion of $\left(x + \frac{1}{2x}\right)^7$

c. x^6 in the expansion of $\left(3x^2 - \frac{1}{3x}\right)^9$ d. x^{12} in the expansion of $(ax^4 - bx)^9$

e. x^{-6} i.e. $\frac{1}{x^6}$ in the expansion of $\left(2x - \frac{1}{3x^2}\right)^9$

Solution:

a. The general term $(t_{r+1}) = {}^{11}C_r\,(x^2)^{11-r} \left(-\frac{1}{x}\right)^r = (-1)^r\, {}^{12}C_r\, x^{12-2r}$

For x^7, $22 - 3r = 7$

$15 = 3r$

$\therefore$ $r = 5$

$\therefore$ the coeff. of x^7 is $11c_r$ i.e. ${}^{11}C_5 = 462$

b. The general term $(t_{r+1}) = {}^7C_r\,(x)^{7-r} \left(\frac{1}{2x}\right)^r = {}^7C_r \cdot \frac{1}{2^r}\, x^{7-2r}$

For x^5, we must have

$7 - 2r = 5$

$r = 1$

$\therefore$ Coeff. $x^5 = 7c_1 . \frac{1}{2} = \frac{7}{2}$

c. We have,

$t_{r+1} = 9c_r\,(3x^2)^{9-r} \left(\frac{-1}{3x}\right)^r = 9c_r\,.\,3^{9-2r}\,.\,x^{18-2r-r}.\,(-1)^r = (-1)^r\,3^{9-2r}\,.\,9c_r\,x^{18-3r}$

Here, $18 - 3r = 6$

$\therefore$ $r = 4$

$\therefore$ The coeff. of x^6 is $(-1)^4\,3^{9-8}\,.\,9c_4 = 3 \times 9c_4 = 378$

d. The general term $(t_{r+1}) = {}^9C_r\,(ax^4)^{9-r}\,(-bx)^r = {}^9C_r\,a^{9-r}\,(-b)^r\,x^{36-3r}$]

For x^{12}, we must have

$36 - 3r = 12$

$\therefore$ $r = 8$

The required coeff. of x^{12} is ${}^9C_8\,a^{9-8}\,(-b)^8 = 9ab^8$

e. We have,

$t_{r+1} = {}^9C_r\,(2x)^{9-r} \left(-\frac{1}{3x^2}\right)^r = (-1)^r\,{}^9C_r\,\frac{2^{9-r}}{3^r}\,x^{9-3}$

For x^{-6}, $9 - 3r = -6$, $9 + 6 = 3r$

$\therefore$ $r = 5$

Coeff. of $x^{-6} = (-1)^5\,{}^9C_5\,\frac{2^{9-5}}{3^5} = -\frac{2016}{243} = -\frac{224}{27}$

5. Find the term free (independent) of x in the expansion of

a. $\left(2x - \frac{1}{3x^2}\right)^8$ b. $\left(x + \frac{1}{x}\right)^{10}$ c. $\left(x^2 - \frac{1}{x^3}\right)^{15}$

d. $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{10}$ e. $\left(x^2 - \frac{1}{x^2}\right)^{14}$

Solution:

a. The general term $(t_{r+1}) = {}^8C_r\,(2x)^{8-r}\left(-\frac{1}{3x^2}\right)^r = {}^8C_r\left(-\frac{1}{3}\right)^r 2^{8-r}\,x^{8-3r}$

For the independent of x,
we must have $8 - 3r = 0$

$\therefore$ $r = \frac{8}{3}$ (not possible)

$\therefore$ There is no term which is free from x.

b. $\left(x + \frac{1}{x}\right)^{10}$

Here, $t_{r+1} = 10c_r\,x^{10-r}\left(\frac{1}{x}\right)^r = 10c_r\,x^{10-2r}$

For free from x, $10-2r = 0$

$\therefore$ $r = 5$

$\therefore$ $t_{r+1} = t_{5+1} = t_6$ is the required term.

c. $t_{r+1} = 15c_r\,(x^2)^{15-r}\left(-\frac{1}{x^3}\right)^r = 15c_r\,(-1)^r\,x^{30-4r}$

$\therefore$ $30 - 4r = 0$

$r = \frac{15}{2}$ (not possible)

$\therefore$ no term has free from x

d. $t_{r+1} = 10c_r\left(\frac{3x^2}{2}\right)^{10-r}\left(-\frac{1}{3x}\right)^r = (-1)r\,10c_r\,\frac{3^{10-2r}}{2^{10-r}}\,x^{20-3r}$

For x^0, $20-3r = 0$

$r = \frac{20}{3}$ (not possible)

$\therefore$ No term is free from x.

e. The general term $(t_{r+1}) = 14c_r\,(x^2)^{14-r}\left(-\frac{1}{x^2}\right)^r = (-1)^r\,14c_r\,x^{24-4r}$

$\therefore$ For free of x, we have $24 - 4r = 0$

$\therefore$ 7^{th} term is required term.

6. Find the middle term in the expansion of

a. $(3+x)^6$ b. $\left(x - \frac{1}{2y}\right)^{10}$ c. $\left(1 - \frac{x^2}{2}\right)^{14}$

d. $\left(x^2 - \frac{2}{x}\right)^{10}$ e. $\left(ax - \frac{1}{ax}\right)^{2n}$ f. $\left(2x^2 + \frac{1}{x}\right)^8$

Solution:

a. $(3 + x)^6$

Here $n = 6$, there is a single middle term.

$\therefore$ Middle term is

$t_{\frac{n}{2}+1} + t_{3+1} = t_4$

Using $t_{r+1} = n_{C_r}\, a^{n-r}\, n^r$

$t_{3+1} = 6_{C_3}\, 3^{6-3}\, x^3 = 6_{C_3}\, 3^3\, x^3 = 540x^3$

b. $\left(x - \frac{1}{2y}\right)^{10}$

Since n = 10 (even), there is a single middle term

$\therefore$ Middle term is

$t_{\frac{n}{2}+1} = t_{5+1} = 10_{C_5}\,(x)^{10-5}\left(-\frac{1}{2y}\right)^5 = 10_{C_5}\, x^5 \left(-\frac{1}{2}\right)^5 \frac{1}{y^5} = -\frac{3}{8}\left(\frac{x}{y}\right)^5 = \frac{-3}{8}\cdot\frac{x^5}{y^5}$

c. $\left(1 - \frac{x^2}{2}\right)^{14}$

Here, n = 14 (even), there is a single middle term

$t_{\frac{n}{2}+1} = t_{7+1}$

$\therefore\quad t_{7+1} = 14_{C_7}\,(1)^{14-7}\left(-\frac{x^2}{2}\right)^7 \qquad (\because t_{r+1} = n_{C_r}\, a^{n-r}\, x^2)$

$= -14_{C_7}\,\frac{x^{14}}{2^7} = -\frac{429}{16}\, x^{14}$

d. Since n = 10, there is a middle term.

$t_{5+1} = 10_{C_5}\,(x^2)^5\left(-\frac{2}{x}\right)^5 = -2^5 \,.\, 10_{C_3}\, x^5 = -8064x^5$

e. Since 2n is even, there is single middle term

$t_{\frac{2n}{2}+1}$ i.e. t_{n+1}

$= 2n_{C_n}\,(an)^n\left(-\frac{1}{an}\right)^n = 2n_{C_n}\,(-1)^n = (-1)^n\,\frac{(2n)\,!}{n!\;n!})$

$= (-1)^n\,\frac{1.2.3....\,(2n-2)\,(2n-1).\,2n}{n!\;n!}$

$= \frac{(-1)^n\, 2^n(1.2.3....\,n)\,(1.3.5....\,(2n-1))}{n!\;n!} = \frac{(-2)^n\,\{1.3.5....\,(2n-1)}{n!}$

f. There is a single middle term

$t_{\frac{n}{2}+1} = t_{4+1} = 8_{C_4}\,(2x^2)^4\left(\frac{1}{x}\right)^4 = 8_{C_4}\, 2^4 \,.\, x^4 = 1120x^4$

7. Find the middle terms in the following expansion

a. $(x^2+a^2)^5$ b. $\left(x^4 - \frac{1}{x^3}\right)^{11}$ c. $\left(x - \frac{1}{x}\right)^{13}$

d. $\left(2x + \frac{1}{x}\right)^{17}$ e. $\left(x + \frac{1}{x}\right)^{2n+1}$

Solution:

a. Here, n = 5 (odd), there are two middle terms.

i.e. $t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$

i.e. t_{2+1} and t_{3+1}

$t_{2+1} = 5_{C_2}\,(x^2)^3\,(a^2)^2 = 10x^6a^4$

$t_{3+1} = 5_{C_3}\,(x^2)^2\,(a^2)^3 = 10x^4a^6$

b. Here, n = 11 (odd), there are two middle terms.

i.e. $t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$

i.e t_{5+1} and t_{6+1}

Now, $t_{5+1} = 11c_5\,(x^4)^{11-5}\left(-\frac{1}{x^3}\right)^5 = (-1)^5\,11c_5\,x^9 = -462x^9$

$t_{6+1} = 11c_6\,(x^4)^{11-6}\left(-\frac{1}{x^3}\right)^6 = 11c_6\,x^2 = 462x^2$

c. Here, n = 13 (odd), there are two middle terms.

$t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$ i.e. t_{6+1} and t_{7+1}

Now, $t_{6+1} = 13c_6\,(x)^7\left(-\frac{1}{x}\right)^6 = 13c_6\,x = 1716x$

$t_{7+1} = 13c_7\,x^6\left(-\frac{1}{x}\right)^7 = -\frac{1716}{x}$

d. Given, $\left(2x+\frac{1}{x}\right)^{17}$

Since n = 17, there are two middle terms.

$t_{\frac{n-1}{2}+1}$ and $t_{\frac{n+1}{2}+1}$

i.e. t_{8+1} and t_{9+1}

$t_{8+1} = 17c_8\,(2x)^9\left(\frac{1}{x}\right)^8 = 2^9.c(17,8)x = 144446720x$

$t_{9+1} = 17c_9\,(2x)^8\left(\frac{1}{x}\right)^9 = 2^8\,c(17, 9)\,.\,\frac{1}{x} = \frac{6223360}{x}$

e. Here, $\left(x+\frac{1}{x}\right)^{2n+1}$

Since (2n+1) is odd, there are two middle terms.

$t_{\frac{2n+1-1}{2}+1}$ and $t_{\frac{2n+1+1}{2}+1}$

i.e. t_{n+1} and $t_{(n+1)+1)}$

Now, $t_{n+1} = 2n+1c_n\,(x)^{2n+1-n}\left(\frac{1}{x}\right)^n = 2n+1c_n\,.\,x$

$t_{(n+1)+1} = 2n+1c_{n+1}\,x^{2n+1-n-1}\left(\frac{1}{x}\right)^{n+1} = 2n+1c_{n+1}\,.\,x^{-1}$

8. Find the middle term or terms in the expansion of

a. $(1+x)^{2n}$ b. $\left(x-\frac{1}{x}\right)^{2n}$ c. $\left(\frac{x}{y}-\frac{y}{x}\right)^{2n+1}$

Solution:

a. $(1+x)^{2n}$

Since 2n is even for any n, there is a single middle term.

$t_{\frac{n}{2}+1}$ i.e. $t_{\frac{2n}{2}+1}$ t_{n+1}

$\therefore$ $t_{n+1} = {}^{2n}c_n\,x^n = c(2n, n)\,x^n$.

b. Since (2n+1) is an odd number, there is only one middle term given by

$t_{\frac{2n}{2}+1}$ i.e. t_{n+1}

We know $t_{r+1} = {}^nc_r\, a^{n-r}\, n^r$ where $a \Rightarrow x\ x - \frac{1}{x}$

$n \Rightarrow 2n$

$\therefore$ Using r = n is above formula, we get

$$t_{n+1} = 2nc_n\, a^{2n-n}\left(-\frac{1}{x}\right)^n = \frac{2n!}{n!\,n!}(-1)^n$$

$$= \frac{1\times2\times3\times4\times5\times6\times\ldots\times(2n-3)\times(2n-2)\times(2n-1)\times 2n}{n!\,n!}(-1)^n$$

$$= \frac{\{1\times3\times5\times\ldots\times(2n-1)\}\,[2\times4\times6\times\ldots\times(2n-2)\times 2n]}{n!\times n!}(-1)^n$$

$$= \frac{\{1\times3\times5\times\ldots\times(2n-1)\}\times 2^n\{1\times2\times3\times\ldots(n-1)\times n\}}{n!\,n!}(-1)^n$$

$$= \frac{1.3.5.\ \ldots\ (2n-1)}{n!}(-2)^n$$

c. $\left(\frac{x}{y}-\frac{y}{x}\right)^{2n+1}$

Since (2n+1) is odd for any n, the number of terms of the expansion is (2n+2), which is even, so there are two middle terms, given by

$t_{\frac{2n+1-1}{2}+1}$ and $t_{\frac{2n+1+1}{2}+1}$

i.e. t_{n+1} and $t_{(n+1)+1}$

Now, $t_{n+1} = 2n+1c_n\left(\frac{x}{y}\right)^{2n+1-n}\left(-\frac{-y}{x}\right) = 2n+1c_n\left(\frac{x}{y}\right) = c(2n+1, n)\,\frac{x}{y}$

$t_{(n+1)+1} = c(2n+1, n+1)\,\frac{y}{x}$

9. Prove that the middle term in the expansion of $(1 + x)^{2n}$ is $\frac{1.3.5\ \ldots\ (2n-1)}{n!}2^n.\,x^n$.

Solution:

The number of terms in the expansion of $(1 + x)^{2n}$ is (2n + 1) which is odd. So, there is only one mid–term, given by $t\left(\frac{2n+1+1}{2}\right)$ i.e. $t_{(n+1)}$ term is mid–term.

Now, $t_{n+1} = {}^{2n}c_n\,.\,(x)^{2n-n} = \frac{2n!}{n!\,n!}\,.\,x^n$

$$= \frac{1.2.3.4.5.6.\ ..\ (2n-1)\,.\,2n}{n!\,n!}x^n = \frac{[2.4.6\ \ldots\ 2n]\,[1.3.5\ldots\,[2n-1]}{n!\,n!}.x^n$$

$$= \frac{n!\,.\,2^n.\,[1.3.5-(2\,n-1)]x^n}{n!\,n!} = \frac{[1.3.5..\ (2n-1)\ 2^n.x^n}{n!}$$

10. In the expansion of $(1+x)^n$, the three successive (consecutive) coefficients are 462, 330 and 165, respectively. Find the values of n and r.

Solution:

Let ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ be the three consecutive coefficients in the expansion of $(1+x)^n$.

Then,

${}^nc_{r-1} = 165$ .... (i)

${}^nc_r = 330$ .... (ii)

${}^nc_{r+1}$ 462 .... (iii)

Dividing (i) by (ii), we get,

$$\frac{nc_{r-1}}{nc_r} = \frac{165}{330}$$

$$\Rightarrow \frac{n!}{(n-r+1)!\,(r-1)!} \times \frac{(n-r)!\,r!}{n!} = \frac{1}{2}$$

or, $\dfrac{(n-r)!\,r!}{(r-1)!\,(n-r+1)!} = \dfrac{1}{2}$

or, $\dfrac{(n-r)!\,(r-1)!\,r}{(r-1)!\,(n-r+1)\,(n-r)!} = \dfrac{1}{2}$

$\therefore$ $\dfrac{r}{n-r+1} = \dfrac{1}{2}$

or, $2r = n-r+1$

$3r = n+1$ .... (iv)

Again, dividing (ii) by (iii) we get

$\dfrac{nc_r}{nc_{r+1}} = \dfrac{330}{462} \Rightarrow \dfrac{n!}{(n-r)!\,r!} \times \dfrac{(n-r-1)!\,(r+1)!}{n!} = \dfrac{5}{7}$

or, $\dfrac{(n-r-1)!\,(r+1)!}{(n-r)!\,r!} = \dfrac{5}{7}$

or, $\dfrac{(n-r-1)!\,r!\,(r+1)}{(n-r)\,(n-r-1)!\,r!} = \dfrac{5}{7}$

or, $\dfrac{r+1}{n-r} = \dfrac{5}{7}$

or, $7r + 7 = 5n - 5r$

$12r = 5n - 7$ .... (v)

from (iv) and (v), we get

$4(n+1) = 5n - 7$ $\qquad\qquad$ $4n + 4 = 5n - 7$

$\therefore$ $n = 11$ and $r = 4$

11. The coefficients of three successive terms in the expansion of $(1+x)^n$ are in the ratio 1:7:42. Find the value of n.

Solution:

Let nc_{r-1}, nc_r and nc_{r+1} be three consecutive coefficients of the expansion of $(1+n)^n$.

Acc^r to question $nc_{r-1} : nc_r : nc_{r+1} = 1 : 7 : 42$

Let nc_{r-1} k .... (i) $nc_r = 7k$ .... (ii) and $nc_{r+1} = 42k$ .... (iii)

Dividing (ii) by (i), $\dfrac{nc_r}{nc_{r-1}} = \dfrac{7k}{k}$

$$\frac{n!}{(n-r)!\,r!} \times \frac{(n-r+1)\,!\,(r-1)!}{n!} = 7$$

or, $\dfrac{(n-r+1)\,(n-r)!}{(n-r)!\,(r-1)!} = 7$

or, $n-r+1 = 7r$

$n+1 = 8r$ .... (iv)

Again, dividing (iii) by (ii)

$\dfrac{nc_{r+1}}{nc_r} = \dfrac{42k}{7k}$

$\dfrac{n!}{(n-r-1)!\,(r+1)!} \times \dfrac{(n-r)!\,r!}{n!} = 6$

or, $\dfrac{(n-r)\,(n-r-1)!\,r!}{(r+1)\,r!\,(n-r-1)!} = 6$

or, $n-r = 6r + 6$

$n = 7r+6$ .... (v)

From (iv) and (v)

$7r + 6 + 1 = 8r$

$7 = r$

$\therefore$ $r = 7$

$\therefore$ $n = 55$

12. In the expansion of $(1+x)^{2x+1}$, the coefficient of x^r and x^{r+1} are equal. Find the value of r.

Solution:

Let us suppose that x^r and x^{r+1} occurs in the $(r+1)^{th}$ and $(r+2)^{th}$ terms in the expansion of $(1+x)^{2n+1}$

Then,

$t_{r+1} = nc_r\, a^{n-r}\, x^r$ and $t_{r+2} = nc_{r+1}\, a^{n-r-1}\, x^{r+1}$

where $a \Rightarrow 1$, $x \Rightarrow x$ $n \Rightarrow 2n+1$

$\therefore$ $t_{r+1} = {}^{2n+1}c_r\, 1^{2n+1-r}\, x^r$ and $t_{r+2} = {}^{2n+1}c_{r+1}\, (1)^{2n+1-r-1}\, x^{r+1}$

$\Rightarrow$ $t_{r+1} = {}^{2n+1}c_r\, x^r$ and $t_{r+2} = {}^{2n+1}c_{r+1}\, x^{r+1}$ .... (i)

Now, by question, coefficient x^r = coefficient of x^{r+1}

$\Rightarrow$ ${}^{2n+1}c_r = {}^{2n+1}c_{r+1}$

or, $\dfrac{(2n+1)!}{r!\,(2n+1-r)!} = \dfrac{(2n+1)!}{(r+1)!\,(2n+1-r-1)!}$

or, $r!(2n-r+1)! = (r+1)!\,(2n-r)!$

or, $r!\,(2n-r)!\,(2n-r+1) = r!(r+1)\,(2n-r)!$

or, $2n-r+1 = r+1$

$2n = 2r$

$\therefore$ $r = n$

13. Prove that the coefficient of the middle term of the expansion of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of the expansion of $(1+x)^{2n-1}$.

Solution:

Since the number of terms in the expansion of $(1+x)^{2n}$ is 2n+1, odd number. So there is only one middle term given by $t_{\frac{2n}{2}+1}$ i.e. t_{n+1}.

Now, coefficient of $(n+1)^{th}$ term = ${}^{2n}c_3$ m

Again, the number of terms in the expansion of $(1+x)^{2n-1}$ is $2n-1+1 = 2n$, even number. So, there are two middle terms given by $t_{\frac{2n-1+1}{2}}$, $t_{\frac{2n-1+1}{2}+1}$ i.e. t_n, t_{n+1}

Now, the coefficients of two middle terms are ${}^{2n-1}c_{n-1}$ and ${}^{2n-1}c_n$

$\therefore$ ${}^{2n-1}c_{n-1} + {}^{2n-1}c_n = \dfrac{(2n-1)!}{(n-1)!\,n!} + \dfrac{(2n-1)!}{n!(n-1)!} = \dfrac{2(2n-1)!}{n!(n-1)!} = \dfrac{2n(2n-1)!}{n!\,n(n-1)!} = \dfrac{(2n)!}{n!n!} = 2nc_n$

Hence proved

14. If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_rx^r + \ldots + C_nx^n$, prove that
   a. $C_1 - 2C_2 + 3C_3 - \ldots + n(-1)^{n-1} . C_n = 0$
   b. $C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n.2^{n-1}$
   c. $C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n = (n+2)2^{n-1}$
   d. $C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \ldots + C_nC_0 = \dfrac{(2n)!}{n!.n!}$.
   e. $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \dfrac{(2n)!}{n!.n!}$

f. $C_0C_1 + C_1C_2 + ... + C_{n-2}C_{n-1} + C_{n-1}C_n = \dfrac{(2n)!}{(n-1)!\,.\,(n+1)!}$

g. $C_0C_2 + C_1C_3 + C_2C_4 + ... + C_{n-2}C_n = \dfrac{(2n)!}{(n-2)!\,.\,(n+2)!}$

h. $C_0C_r + C_1C_{r+1} + ... + C_{n-r}\,C_n = \dfrac{(2n)!}{(n-r)!\,.\,(n+r)!}$

Solution:

Since $(1+x)^n = nc_0 + nc_1\,x + nc_2\,x^2 + ... + nc_n\,x^n$

Using x = 1 and –1, we get

$(1+1)^n = nc_0 + nc_1 + nc_2 + ... + nc_n$ .... (*)

$(1–n)^n = nc_0 – nc_1 + nc_2 – .... + nc_n$ .... (**)

Again, $(1+x)^{n-1} = {}^{n-1}c_0 + {}^{n-1}c_1\,x^1 + {}^{n-1}c_2\,x^2 + ... + {}^{n-1}c_{n-1}\,x^{n-1}$

Using x=1 and –1, we get

$(1+1)^{n-1} = {}^{n-1}c_0 + {}^{n-1}c_1 + {}^{n-1}c_2 + ... + {}^{n-1}c_{n-1}$ .... (***)

$(1-1)^{n-1} = {}^{n-1}c_0 - {}^{n-1}c_1 + {}^{n-1}c_2 + ... + (-1)^{n-1}\,{}^{n-1}c_{n-1}$ .... (****)

a. $C_1 - 2.C_2 + 3.C_3 - ..... + n(-1)^{n-1}\,C_n$

$= n - 2\,\dfrac{n(n-1)}{2!} + 3.\,\dfrac{n(n-1)\,(n-2)}{3!} - ..... + n.\,(-1)^{n-1}.\,1$

$= n\left[1 - \dfrac{(n-1)}{1!} + \dfrac{(n-1)\,(n-2)}{2!} - ...... + (-1)n-1\right] =$

$n[{}^{n-1}c_0 - {}^{n-1}c_1 + {}^{n-1}c_2 - ...... + (-1)^{n-1}\,{}^{n-1}c_{n-1}]$

$= n(1-1)^{n-1}$ (By using formula (****) above) $= n\times0 = 0$ Hence, proved

b. $C_1 + 2.C_2 + 3.C_3 + ... + n.\,C_n$

$n + 2.\dfrac{n(n-1)}{2!} + 3.\dfrac{n(n-1)\,(n-2)}{3!} + ... + n.1 = n[\,{}^{n-1}c_0 + {}^{n-1}c_1 + {}^{n-1}c_2 + ... + {}^{n-1}c_{n-1}]$

$= n(1+1)^{n-1}$ (Using formula (***) above) $= n.2^{n-1}$ Hence proved

c. $C_0 + 2.C_1 + 3.C_2 + ... + (n+1)\,.\,C_n$

$= (C_0 + C_1 + C_2 + ... + C_n) + (C_1 + 2C_2 + 3C_3 + ... + n.\,C_n)$

$= (1+1)^n + \left[n + \dfrac{n(n-1)}{1!} + \dfrac{n(n-1)\,(n-2)}{2!} + ... + n\right] = 2^n +$

$n\left[1 + \dfrac{(n-1)}{1!} + \dfrac{(n-1)(n-2)}{2!} + ... + 1\right]$

$= 2^n + n[\,{}^{n-1}c_0 + {}^{n-1}c_1 + {}^{n-1}c_2 + ... + {}^{n-1}c_{n-1}]$

$= 2^n + n.\,(1+1)^{n-1}$ (By using formula *** above)

$= 2^n + n\,.\,2^{n-1} = 2^{n-1}.\,2 + n.2^{n-1} = (n+2)2^{n-1}$ Hence proved

d. Since $(1+x)^n = C_0 + C_1\,x + C_2\,x^2 + ... + C_n\,x^n$

$\therefore$ $(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + ... C_nx^n)\,(C_0 + C_1x + C^2x^2 + ... + C_nx^n)$

$\therefore$ ${}^{2n}c_0 + {}^{2n}c_1\,x + {}^{2n}c_2\,x^2 + ... + {}^{2n}c_n\,x^n + ... + {}^{2n}c_{2n}\,x^{2n} = (\,"\,)\,(\,"\,)$

Equating the coefficient of x^n in both sides.

Coefficient of x^n in LHS $= {}^{2n}c_n = \dfrac{(2n)!}{n!\,n!}$ .... (i)

Coefficient of x^n in RHS $= C_0C_n + C_1C_{n-1} + C_2C_{n-2} + ... + ... + C_nC_0$ ...... (ii)

Equating (i) and (ii), we get

$C_0C_n + C_1C_{n-1} + ... + C_nC_0 = \dfrac{(2n)!}{n!\,n!}$ Hence proved

e. Since $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ .... (i)

$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n$ .... (ii)

Multiplying (i) and (ii), we get

$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + ... + C_nx^n)\,(C_0x^n + C_1x^{n-1} + ... + C_n)$

Equating the coefficient of x^n both sides, we get

Coefficient of x^n in LHS $= {}^{2n}c_n = \dfrac{(2n)!}{n!\,n!}$ .... (iii)

Coefficient of x^n in RHS $= C^2 + C_1^{\,2} + C_2^{\,2} + \ldots\ldots + C_n^{\,2}$ .... (iv)

Equating (iii) and (iv), we get,

$C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + \ldots\ldots + C_n^{\,2} = \dfrac{(2n)!}{n!\,n!}$ Hence proved

f. Since $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots.. + C_nx^n$ .... (i)

$(x+1)^n = C_0x^n = C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n$ ..... (ii)

Multiplying (i) and (ii), we get

$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + \ldots.. + C_nx^n)\,(C_0x^n + C_1x^{n-1} + \ldots\ldots. + C_n)$

Equating the coefficient of x^{n-1} both sides,

Coeff. of x^{n-1} in LHS $= {}^{2n}c_{n-1} = \dfrac{(2n)!}{(2n-n+1)!\,(n-1)!} = \dfrac{(2n)!}{(n+1)!\,(n-1)!}$ ...... (iii)

Coefficient of x^{n-1} in RHS $= C_0C_1 + C_1C_2 + \ldots. + C_{n-2}\,C_{n-1} + C_{n-1}\,C_n$ .... (iv)

Equating (iii) and (iv), we get

$C_0C_1 + C_1C_2 + \ldots.. + C_{n-2}\,C_{n-1} + C_{n-1}\,C_n = \dfrac{(2n)!}{(n+1)!\,(n-1)!}$ Proved.

g. Since, $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ ...... (i)

$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n$ .... (ii)

Multiplying (i) and (ii)

$(1+x)^{2n} = (C_0+C_1x + C_2x^2 + \ldots\ldots + C_nx^n)\,(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots.. + C_n)$

This is identify, so coeff. of any power of x in LHS and coeff. of same power of x in RHS must be equal.

Coeff. of x^{n-2} in LHS $= {}^{2n}c_{n-2} = \dfrac{(2n)!}{(n+2)!\,(n-2)!}$ .... (iii)

Coeff. of x^{n-2} in RHS $= C_0C_2 + C_1C_3 + \ldots\ldots + C_{n-2}\,C_n$ .... (iv)

Equating (iii) and (iv) we get

$C_0C_2 + C_1C_3 + \ldots.. + C_{n-2}\,C_n = \dfrac{(2n)!}{(n-2)!\,(n+2)!}$ Hence proved

h. Equating the coeff. of x^{n-r} in both sides as in above question g.

Coeff. of x^{n-r} in LHS $= {}^{2n}c_{n-r} = \dfrac{(2n)!}{(n+r)!\,(n-r)!}$ .... (iii)

Coeff. of x^{n-r} in RHS $= C_0C_r + C_1C_{r+1} + \ldots\ldots. + C_{n-r}\,C_n$ ...... (iv)

Equating (iii) and (iv), we get

$C_0C_r + C_1C_{r+1} + \ldots\ldots + C_{n-r}\,C_n = \dfrac{(2n)!}{(n-r)!\,(n+r)!}$ Proved.

EXERCISE 2.2

1. State the condition of validity of expansion and expand the following up to four terms.

a. $(2 - 3x)^{-3}$ b. $(2 + 3x)^{\frac{5}{2}}$ c. $\dfrac{1}{\sqrt{5 + 4x}}$ d. $\dfrac{1}{(3 - 2x^2)^{\frac{2}{3}}}$

Solution:

a. $(2 - 3x)^{-3} = 2^{-3}\left(1 - \dfrac{3}{2}x\right)^{-3}$

The expansion is valid when $\left|\frac{3}{2}x\right| < 1$ i.e. $|x| < \frac{2}{3}$

Now, $(2 - 3x)^{-3} = 2^{-3}\left(1 - \frac{3}{2}x\right)^{-3}$

$$= \frac{1}{2^3}\left[1 + (-3)\left(-\frac{3}{2}x\right) + \frac{(-3)(-3-1)}{2!}\left(-\frac{3}{2}x\right)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}\left(-\frac{3}{2}x\right)^3 + \ldots \text{ to } \infty\right]$$

$$= \frac{1}{2^3}\left[1 + \frac{9}{2}x + \frac{27}{2}x^2 + \frac{135}{4}x^3 + \ldots \text{ to } \infty\right]$$

$$= \frac{1}{8}\left(1 + \frac{9x}{2} + \frac{27}{2}x^2 + \frac{135}{4}x^3 + \ldots \text{ to } \infty\right)$$

b. Here $(2 + 3x)^{5/2}$

$2^{5/2}\left(1 + \frac{3}{2}x\right)^{5/2}$

The expansion is valid when $\left|\frac{3x}{2}\right| < 1$ i.e. $|x| < \frac{2}{3}$

We know that,

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}n^2 + \frac{n(n-1)(n-2)}{3!}n^3 + \ldots$

Now, $2^{5/2}\left(1 + \frac{3}{2}x\right)^{5/2}$

$$= 2^{5/2}\left[1 + \left(\frac{5}{2}\right)\left(\frac{3}{2}x\right) + \frac{\frac{5}{2}\left(\frac{5}{2} - 1\right)}{2!}\left(\frac{3x}{2}\right)^2 + \frac{\frac{5}{2}\left(\frac{5}{2} - 1\right)\left(\frac{5}{2} - 2\right)}{3!}\left(\frac{3}{2}x\right)^3 + \ldots\right]$$

$$= 2^{5/2}\left[1 + \frac{15}{4}x + \frac{135x^2}{64} + \frac{135x^3}{128} + \ldots \text{ to } \infty\right]$$

c. $(5 + 4x)^{-1/2}$

$5^{-1/2}\left[1 + \frac{4}{5}x\right]^{-1/2}$

This expansion is valid when $\left|\frac{4x}{5}\right| < 1$ i.e. $|x| < \frac{5}{4}$

$$5^{-1/2}\left[1 + \left(-\frac{1}{2}\right)\frac{4}{5}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)}{2!}\left(\frac{4x}{5}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)\left(\frac{-1}{2} - 2\right)}{3!}\left(\frac{4x}{5}\right)^3 + \ldots\right]$$

$$= \frac{1}{\sqrt{5}}\left[1 - \frac{2x}{5} + \frac{6x^2}{25} - \frac{4x^3}{25} + \ldots \text{ to } \infty\right]$$

d. $(3 - 2x^2)^{-2/3}$

$3^{-2/3}\left(1 - \frac{2}{3}x^2\right)^{-2/3}$

The expansion is valid when $\left|\frac{2x^2}{3}\right| < 1$ i.e. $|x^2| < \frac{3}{2}$

Now, $3^{-2/3}\left[1 - \frac{2}{3}x^2\right]^{-2/3}$

$$= 3^{-2/3}\left[1 + \left(\frac{-2}{3}\right)\left(-\frac{2}{3}x^2\right)\right] + \frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3} - 1\right)}{2!}\left(\frac{-2}{3}x^2\right)^2 + \ldots \text{ to } \infty)$$

$$= 3^{-2/3}\left[1 + \frac{4x^2}{9} + \frac{20x^4}{81} + \ldots \text{ to } \infty\right]$$

2. Expand the following up to four terms.

a. $\sqrt{1+x}$ b. $\dfrac{1}{\sqrt{1+x^2}}$ c. $\sqrt[4]{1+x}$ d. $\dfrac{1}{\sqrt[3]{1-x^2}}$

Solution:

a. $(1+x)^{1/2}$

$$1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)x^2}{2!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \ldots = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16}x^3 \ldots \text{ to } \infty$$

b. $(1+x^2)^{-1/2}$

$$1 + \left(\frac{-1}{2}\right)x^2 + \frac{\left(\frac{-1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}(x^2)^3 + \ldots$$

$$= 1 - \frac{x^2}{2} + \frac{3x^4}{8} - \frac{5x^6}{16} + \ldots \text{ to } \infty$$

c. $(1+x)^{1/4}$

$$1 + \left(+\frac{1}{4}\right)x + \frac{\left(+\frac{1}{4}\right)\left(+\frac{1}{4}-1\right)}{2!}x^2 + \frac{\left(\frac{+1}{4}\right)\left(+\frac{1}{4}-2\right)\left(\frac{1}{4}-2\right)}{3!}x^3 + \ldots$$

$$= 1 + \frac{x}{4} - \frac{3x^2}{32} + \frac{7x^3}{128} - \ldots \text{ to } \infty$$

d. $(1-x^2)^{-1/3}$

$$1 + \left(-\frac{1}{3}\right)(-x^2) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}(-x^2)^2 + \ldots \text{ to } \infty = 1 + \frac{1}{3}x^2 + \frac{2x^4}{9} + \ldots \text{ to } \infty$$

3. Calculate each of the following correct to three places of decimal places.

a. $(1.03)^{-5}$ b. $(0.01)^{\frac{1}{2}}$ c. $\sqrt[3]{(28)}$

d. $\sqrt{17}$ e. $\sqrt[3]{\dfrac{96}{101}}$

Solution:

a. $(1.03)^{-5}$

$(1 + 0.03)^{-5}$

$$= 1 + (-5)(0.03) + \frac{(-5)(-5-1)(0.03)^2}{2!} + \frac{(-5)(-5-1)(-5-2)}{3!}(0.03)^3 + \ldots$$

$= 0.915$

b. $(0.01)^{1/2}$

$$(1-0.99)^{1/2} = 1 + \frac{1}{2}(-0.99) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-0.99)^2 + \ldots = 0.1$$

c. $(28)^{1/3}$

$$(27 + 1)^{1/3} = 3\left(1 + \frac{1}{27}\right)^{1/3} = 3\left[1 + \frac{1}{3}\cdot\frac{1}{27} + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}\left(\frac{1}{27}\right)^2 + \ldots\right] = 3.037$$

d. $$\sqrt{17} = (16 + 1)^{1/2} = 4\left(1 + \frac{1}{16}\right)^{1/2} = 4\left[1 + \frac{1}{2}\,\frac{1}{16} + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(\frac{1}{16}\right)^2 + \ldots\right] = 4.123$$

e. $$\left(\frac{96}{101}\right)^{1/3} = \left(1 - \frac{5}{101}\right)^{1/3} = 1 + \frac{1}{3}\left(\frac{-5}{101}\right) + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}\left(\frac{-5}{101}\right)^2 + \ldots = 0.983$$

4. Prove that

a. $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots \text{ to } \infty = \sqrt{2}$ b. $1 + \frac{1.2}{2.3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \ldots \text{ to } \infty = \sqrt{3}$

c. $1 - \frac{1}{6} + \frac{1.3}{6.12} - \frac{1.3.5}{6.12.18} + \ldots \text{ to } \infty = \frac{\sqrt{3}}{2}$ d. $1 + \frac{1}{4} + \frac{1.4}{4.8} + \frac{1.4.7}{4.8.12} + \ldots \text{ to } \infty = (2)^{\frac{2}{3}}$

e. $1 + \frac{1}{4} - \frac{1}{4}\cdot\frac{1}{8} + \frac{1.1.3}{4.8.12} - \frac{1.1.3.5}{4.8.12.16} + \ldots \text{ to } \infty = \sqrt{\frac{3}{2}}$

Solution:

a. Let $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.2} + \ldots \text{ to } \infty = (1 + x)^n$

Then, $1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots \text{ to } \infty = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$

Equating corresponding term, we get,

$nx = \frac{1}{4}$

$\therefore\ x = \frac{1}{4n}$ ... ... (i) and $\frac{n(n-1)}{2!}x^2 = \frac{1.3}{4.8}$

or, $\frac{n(n-1)}{2}\frac{1}{(4n)^2} = \frac{1.3}{4.8}$

$\frac{n(n-1)}{2.16\,n^2} = \frac{3}{32}$

$n - 1 = 3n$

$-2n = 1 \therefore n = -\frac{1}{2}$

from (i) $x = \frac{1}{4(-1/2)} = -\frac{1}{2}$

Hence, $(1 + x)^n = \left(1 - \frac{1}{2}\right)^{-1/2} = \left(\frac{1}{2}\right)^{-1/2} = 2^{1/2} = \sqrt{2}$

$\therefore\ 1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \ldots \text{ to } \infty = \sqrt{2}$

b. Let $(1+x)^n$ be equal to $1 + \frac{1.2}{2.3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \ldots \text{ to } \infty$

i.e. $1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots \text{ to } \infty = 1 + \frac{1.2}{2.3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \ldots \text{ to } \infty$

Equating corresponding term, we get

$nx = \frac{1.2}{2.3} \Rightarrow x = \frac{1}{3n}$ ... ... (i) and $\frac{n(n-1)}{2}x^2 = \frac{1.3}{3.6}$

$\frac{n(n-1)}{2}\frac{1}{n^2} = \frac{1}{6}$

$n - 1 = 3n$

$-2n = 1 \therefore n = -\frac{1}{2}$

from (i) $x = \frac{1}{3\left(\frac{-1}{2}\right)} = -\frac{2}{3}$

$\therefore \quad (1+x)^n = \left(1 - \frac{2}{3}\right)^{-1/2} = \left(\frac{1}{3}\right)^{-1/2} = 3^{1/2} = \sqrt{3}$

c. Let $1 - \frac{1}{6} + \frac{1.3}{6.12} + \frac{1.3.5}{6.12.18} + \ldots$ to $\infty = (1+x)^n$

$1 - \frac{1}{6} + \frac{1.3}{6.12} + \frac{1.3.5}{6.12.18} + \ldots$ to $\infty = (1+x)^n$

Equating corresponding term

$nx = -\frac{1}{6} \;\; \therefore x = -\frac{1}{6n}$ ... ... (i)

$\frac{n(n-1)}{2}x^2 = \frac{1.3}{6.12}$

or, $\frac{n(n-1)}{2}\left(\frac{-1}{6n}\right)^2 = \frac{1}{24}$

$\frac{n(n-1)}{2}\frac{1}{36.n^2} = \frac{1}{24}$

or, $n - 1 = 3n, n = -\frac{1}{2}$

from (i) $x = -\frac{1}{6\left(-\frac{1}{2}\right)} = \frac{1}{3}$

$\therefore \quad (1+x)^n = \left(1 + \frac{1}{3}\right)^{-1/2} = \left(\frac{4}{3}\right)^{-1/2} = \left(\frac{3}{4}\right)^{1/2} = \frac{\sqrt{3}}{2}$

d. Let $1 + \frac{1}{4} + \frac{1.4}{4.8} + \ldots$ to $\infty = (1+x)^n$

$1 + \frac{1}{4} + \frac{1.4}{4.8} + \ldots$ to $\infty = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots$

$\therefore \quad nx = \frac{1}{4}$ and $\frac{n(n-1)}{2}x^2 = \frac{1.4}{4.8}$

$x = \frac{1}{4n}$ ... ... (i) $\qquad$ or, $\frac{n(n-1)}{2} \cdot \frac{1}{16n^2} = \frac{1}{8}$

$\therefore x = -\frac{3}{4}$ $\qquad$ or, $n - 1 = 4n$

$3n = -1 \therefore n = -\frac{1}{3}$

$\therefore \quad (1+x)^n = \left(1 - \frac{3}{4}\right)^{-1/3} = \left(\frac{1}{4}\right)^{-1/3} = 4^{1/3} = 2^{2/3}$ proved.

e. Let $1 + \frac{1}{4} - \frac{1.1}{4.8} + \frac{1.1.3}{4.8.12} - \ldots = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots$

Equating, $nx = \frac{1}{4}$ $\qquad$ $\frac{n(n-1)}{2}x^2 = -\frac{1}{32}$

$\therefore \quad x = \frac{1}{4n}$ ... ... (i) $\qquad$ $\frac{n(n-1)}{2} \cdot \frac{1}{16n^2} = -\frac{1}{32}$

$\therefore \quad x = \frac{1}{2}$

$n - 1 = -n$

$2n = 1 \therefore n = \frac{1}{2}$

$$\therefore \quad (1 + n)^n = \left(1 + \frac{1}{2}\right)^{1/2} = \left(\frac{3}{2}\right)^{1/2} = \sqrt{\frac{3}{2}}$$

Hence, $1 + \frac{1}{4} - \frac{1.1}{4.8} + \frac{1.1.3}{4.8.12} - \frac{1.1.3.5}{4.8.12.16} + \ldots = \sqrt{\frac{3}{2}}$

EXERCISE 2.3

1. Find the values of

a. $e + \frac{1}{e}$ b. $e - \frac{1}{e}$

Solution:

a. We know

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} \ldots\ldots.$$

Putting x = 1, we get,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} \ldots\ldots. \text{ (i)}$$

Again, putting x = –1, we get,

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \ldots\ldots\text{(ii)}$$

Adding (i) and (ii) we get,

$$e + \frac{1}{e} = 2 + \frac{2}{2!} + \frac{2}{4!} + \ldots\ldots\ldots$$

$$\therefore e + \frac{1}{e} = 2\left[1 + \frac{1}{2!} + \frac{1}{4!} \ldots\ldots\right]$$

b. To prove $e - \frac{1}{e}$, Subtracting (ii) from (i)

$$e - \frac{1}{e} = 2 + \frac{2}{3!} + \frac{2}{5!} + \ldots\ldots\ldots = 2\left[1 + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots\right]$$

2. Expand in ascending power of x

a. $\left(\frac{e^{5x} + e^x}{e^{3x}}\right)$ b. $\frac{e^{7x} + e^x}{2e^{4x}}$

Solution:

a. $\frac{e^{5x} + e^x}{e^{3x}} = e^{2x} + e^{-2x}$

We know that, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ to ∞

$$\therefore e^{2x} + e^{-2x} = \left(1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots \text{ to } \infty\right) + \left(1 - \frac{2x}{1!} + \frac{(2x)^2}{2!}\right) - \frac{(2x)^3}{3!} + \ldots \text{ to } \infty)$$

$$= 2 + 2\,\frac{(2x)^2}{2!} + 2\frac{(2x)^4}{4!} + \ldots \text{ to } \infty$$

$$= 2\left(1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \dots\right) = 2\left(1 + \frac{2^2.x^2}{2!} + \frac{2^4.x^4}{4!} + \frac{2^6.x^6}{6!} \dots + \infty\right)$$

b. $\frac{e^{7x} + e^x}{2\,e^{4x}} = \frac{1}{2}[e^{3x} + e^{-3x}] = \frac{1}{2}\left[\left(1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots\right)\right] + \left(1 - \frac{3x}{1!} + \frac{(3x)^2}{2!} - \frac{(3x)^3}{3!} + \dots\right)$

$= \frac{1}{2}\left[2 + \frac{2(3x)^2}{2!} + 2((3x)^4,4!) + \dots\right] = 1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} = 1 + \frac{3^2.x^2}{2!} + \frac{3^4.x^4}{4!} + \frac{3^6\,x^6}{6!} + \dots\ \infty$

3. Find the value of

a. $\sqrt{e}$ up to 4 places of decimals. b. $\frac{1}{\sqrt{e}}$ up to four places of decimals.

Solution:

a. $\sqrt{e} = e^{1/2}$

We know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

Put $x = \frac{1}{2}$

$$e^{1/2} = 1 + \frac{\frac{1}{2}}{1!} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \dots = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \dots$$

$= 1 + 0.5 + 0.125 + 0.0208 + \dots = 1.6458$

b. $\frac{1}{\sqrt{e}} = e^{-1/2} = 1 - \frac{\frac{1}{2}}{1!} + \frac{\left(\frac{1}{2}\right)^2}{2!} - \frac{\left(\frac{1}{2}\right)^3}{3!} + \dots = 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \dots$

$= 1 - 0.5 + 0.125 - 0.0208 + \dots = 0.6042$

4. Show that

a. $\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \text{ to } \infty\right)\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \text{ to } \infty\right) = 1$

b. $\dfrac{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots \text{ to } \infty}{1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \dots \text{ to } \infty} = \dfrac{e^2 + 1}{e^2 - 1}$ c. $\dfrac{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots\dots}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \dots\dots} = \dfrac{e - 1}{e + 1}$

d. $\frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \dots \text{ to } \infty = e$ e. $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots \text{ to } \infty = 1$

f $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \dots \text{ to } \infty = \frac{1}{e}$

Solution:

a. $(e^1) . (e^{-1})$
$e^0 = 1$ proved.

b. We know, $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \text{ to } \infty = e^x$

Putting $x = 1$ and $x = -1$, we get

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... = e$ ... ... (i)

and $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... = e^{-1}$ ... ... (ii)

Adding (i) and (ii), we get

$$2 + \frac{2}{2!} + \frac{2}{4!} + ... = e + e^{-1}$$

or, $2\left(1 + \frac{1}{2!} + \frac{1}{4!} + ...\right) = e + e^{-1}$

or, $1 + \frac{1}{2!} + \frac{1}{4!} + ... = \frac{e + e^{-1}}{2}$

$\therefore$ $1 + \frac{1}{2!} + \frac{1}{4!} + ... = \frac{e^2 + 1}{2e}$ ... ... (iii)

Subtracting (ii) from (i), we get

$2 + \frac{2}{3!} + \frac{2}{5!} + \frac{2}{7!} + ... = e–e^{-1}$

$1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + ... = \frac{e^2 - 1}{2e}$ ... ... (iv)

Dividing (iii) by (iv), we get,

$\frac{1 + \frac{1}{2!} + \frac{1}{4!} + ...}{1 + \frac{1}{3!} + \frac{1}{5!} + ...} = \frac{e^2 + 1}{e^2 - 1}$ Hence proved.

c. We know,

$\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ... = \frac{1}{2}(e + \frac{1}{e}) - 1$

and $1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + ... = \frac{1}{2}(e - \frac{1}{e})$

Now, $\frac{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ...}{1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + ...} = \frac{e^2 + 1 - 2e}{e^2 - 1} = \frac{(e-1)^2}{(e+1)(e-1)} = \frac{e-1}{e+1}$ proved.

d. $\frac{(1+1)}{1!} + \left(\frac{3+1}{3!}\right) + \frac{(5+1)}{5!} + ...$ to ∞ $= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + ... = e$

e. Here, $\frac{2-1}{2!} + \frac{3–1}{3!} + \frac{4-1}{4!} + ...$

$1 - \frac{1}{2!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{3!} - \frac{1}{4!} + ... = 1$ proved.

f. $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + ...$

$\frac{3-1}{3!} + \frac{5-1}{5!} + \frac{7-1}{7!} + ...$

$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + ...$

$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + ... = e^{-1}$ proved.

5. Find the sum to infinity of the series whose general terms are given by

a. $t_n = \dfrac{n^2}{(n+1)!}$ b. $t_n = \dfrac{1}{(n+1)!}$ c. $t_n = \dfrac{1}{(n+2)!}$

d. $t_n = \dfrac{n^3}{n!}$ e. $t_n = \dfrac{n(n+1)}{n!}$

Solution:

a. $t_n = \dfrac{n^2}{(n+1)!} = \dfrac{n^2 - 1 + 1}{(n+1)!} = \dfrac{(n^2-1)}{(n+1)!} + \dfrac{1}{(n+1)!} = \dfrac{(n-1)}{n!} + \dfrac{1}{(n+1)!}$

$= \dfrac{n}{n!} - \dfrac{1}{n!} + \dfrac{1}{(n+1)!} = \dfrac{1}{(n-1)!} - \dfrac{1}{n!} + \dfrac{1}{(n+1)!}$

$\Sigma t_n = \Sigma\dfrac{1}{(n-1)!} - \Sigma\dfrac{1}{n!} + \Sigma\dfrac{1}{(n+1)!}$

$= \left(1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots\right) - \left(\dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots\right) + \left(\dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \ldots\right)$

$= e - (e-1) + (e-2) = e - e + 1 + e - 2 = e - 1$

b. $t_n = \dfrac{1}{(n+1)!}$

$\Sigma t_n = \sum_{1}^{\infty} \dfrac{1}{(n+1)!}$

$= \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \ldots = e - 2$

c. $t_n = \dfrac{1}{(n+2)!}$

$\Sigma t_n = \sum_{1}^{\infty} \dfrac{1}{(n+2)!} = \dfrac{1}{3!} + \dfrac{1}{4!} + \dfrac{1}{5!} + \ldots = \left(1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \dfrac{1}{5!} + \ldots\right) - \dfrac{5}{2} = e - \dfrac{5}{2}$

d. $t_n = \dfrac{n^3}{n!} = \dfrac{n^2}{(n-1)!} = \dfrac{(n-1)(n+1)}{(n-1)!} + \dfrac{1}{(n-1)!}$

$= \dfrac{n+1}{(n-2)!} + \dfrac{1}{(n-1)!} = \dfrac{(n-2)}{(n-2)!} + \dfrac{3}{(n-2)!} + \dfrac{1}{(n-1)!} = \dfrac{1}{(n-3)!} + \dfrac{3}{(n-2)!} + \dfrac{1}{(n-1)!}$

$\Sigma t_n = \Sigma\dfrac{1}{(n-3)!} + 3\Sigma\dfrac{1}{(n-2)!} + \Sigma\dfrac{1}{(n-1)!}$

$= \left(\dfrac{1}{(-2)!} + \dfrac{1}{(-1)!} + \dfrac{1}{0!} + \dfrac{1}{1!} + \dfrac{1}{2!} + \ldots\right) + 3\left(\dfrac{1}{(-1)!} + \dfrac{1}{0!} + \dfrac{1}{1!} + \ldots\right) +$

$\left(\dfrac{1}{0!} + \dfrac{1}{1!} + \dfrac{1}{2!} + \ldots\right)$

$= e + 3e + e = 5e$

e. $t_n = \dfrac{n(n+1)}{n!} = \dfrac{n+1}{(n+1)!} = \dfrac{n-1}{(n-1)!} + \dfrac{2}{(n-1)!} = \dfrac{1}{(n-2)!} + \dfrac{2}{(n-1)!}$

$\Sigma t_n = \Sigma\dfrac{1}{(n-2)!} + \Sigma\dfrac{2}{(n-1)!} = e + 2e = 3e$

6. Show that

a. $1 + \dfrac{1+2}{2!} + \dfrac{1+2+2^2}{3!} + \dfrac{1+2+2^2+2^3}{4!} + \ldots\text{to } \infty$

b. $1 + \dfrac{1}{2!} + \dfrac{1.3}{4!} + \dfrac{1.3.5}{6!} + \ldots \text{ to } \infty$ c. $\dfrac{1.2}{1!} + \dfrac{2.3}{2!} + \dfrac{3.4}{3!} + \ldots \text{ to } \infty$

e. $\dfrac{1^2}{1!} + \dfrac{2^2}{2!} + \dfrac{3^2}{3!} + \ldots \text{ to } \infty$

f. $1 + \frac{1+2}{1!} + \frac{1+2+3}{2!} + \dots \text{ to } \infty = \frac{7e}{2}$

g. $3 + \frac{5}{1!} + \frac{7}{2!} + \frac{9}{3!} + \dots \text{ to } \infty = 5e$

Solution:

a. Let n^{th} term of above series be t_n

Then $t_n = \frac{1 + 2 + 2^2 + \dots + 2^{n-1}}{n!} = \frac{\frac{1(2^n - 1)}{2 - 1}}{n!} = \frac{2^n - 1}{n!}$

$t_n = \frac{2^n}{n!} - \frac{1}{n!}$

Let s_∞ be the required sum of the series.

Then, $s_\infty = \Sigma t_n = \Sigma\left(\frac{2^n}{n!} - \frac{1}{n!}\right) = \sum_1^\infty \frac{2^n}{n!} - \sum_1^\infty \frac{1}{n!} = \left(\frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \dots\right) - \left(\frac{1}{1!} + \frac{1}{2!} + ..\right)$

$= (e^2 - 1) - (e-1) = e^2 - e$

b. $1 + \frac{1}{2!} + \frac{1.3}{4!} + \frac{1.3.5}{6!} + \dots = 1 + \frac{1}{2} + \frac{1.3}{1.2.3.4} + \frac{1.3.5}{1.2.3.4.5.6} + \dots = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \dots$

$= 1 + \frac{\left(\frac{1}{2}\right)}{1!} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \dots = e^{1/2} = \sqrt{e}$

c. Let t_n be the n^{th} term of the given series

Then $t_n = \frac{n(n+1)}{n!} = \frac{n^2 + n}{n!} = \frac{n}{(n-1)!} + \frac{1}{(n-1)!}$

$= \frac{n+1}{(n-1)!} = \frac{(n-1) + 2}{(n-1)!} = \frac{1}{(n-2)!} + \frac{2}{(n-1)!}$

$\therefore$ The given series

$\Sigma t_n = \Sigma \frac{1}{(n-2)!} + \Sigma \frac{2}{(n-1)!} = \left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots\right) + 2\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots\right)$

but $(-1)! = \infty$ and $0! = 1$

$\therefore \Sigma t_n = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\right) + 2\left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots\right) = e + 2e = 3e$

e. Let t_n be the n^{th} term

$t_n = \frac{n^2}{n!} = \frac{n}{(n-1)!} = \frac{n - 1 + 1}{(n-1)!} = \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$

$\therefore$ Sum of the series

$\sum_1^\infty t_n = \sum_n^\infty \frac{1}{(n-2)!} + \Sigma\frac{1}{(n-1)!} = e + e = 2e$

f. n^{th} term of given series $(t_n) = \frac{1 + 2 + 3 + \dots + n}{(n-1)!} = \frac{\frac{n}{2}(n+1)}{(n-1)!} = \frac{n(n+1)}{2(n-1)!}$

g. $3 + \frac{5}{1!} + \frac{7}{2!} + \frac{9}{3!} + \dots \text{ to } \infty$

Let t_n be the n^{th} term of above series.

Then, $t_n = \frac{3 + (n-1)d}{(n-1)!} = \frac{3 + (n-1) . 2}{(n-1)!} = \frac{2n + 1}{(n-1)!}$

$t_n = \frac{2(n-1) + 3}{(n-1)!} = \frac{2}{(n-2)!} + \frac{3}{(n-1)!}$

$$s_n = \sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} \frac{2}{(n-2)!} + 3\sum_{n=1}^{\infty} \frac{1}{(n-1)!}$$

$$= 2\left(\frac{1}{(-1)!} + \frac{1}{0!} + \frac{1}{!} + \frac{1}{2!} + ...\right) + 3\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...\right) = 2e + 3e = 5e$$

7. Prove the following

a. $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots \text{ to } \infty = \ln 2$

b. $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots \text{ to } \infty = 2 - 2\ln 2$

c. $\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \ldots \text{ to } \infty = \frac{1}{2}\ln 2$

d. $1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \ldots \text{ to } \infty = \ln 3$

e. $\left(\frac{1}{3} - \frac{1}{2}\right) + \frac{1}{2}.\left(\frac{1}{3^2} - \frac{1}{2^2}\right) + \frac{1}{3}.\left(\frac{1}{3^3} - \frac{1}{2^3}\right) + \text{.... to } \infty = 0$

Solution:

a. We know that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \text{ to } \infty$

$\ln(1 + 1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

$\ln 2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \ldots$

$= \left(\frac{2-1}{1.2}\right) + \frac{(4-3)}{3.4} + \left(\frac{6-5}{5.6}\right) + \ldots = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots$

b. $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots \text{ to } \infty = 2 - 2\ln 2$

We have,

$\ln(1 + x) = x - x\frac{2}{2} + x\frac{3}{3} - x\frac{4}{4} + x\frac{5}{5} - x\frac{6}{6} + x\frac{7}{7} - \ldots$

Putting $x = 1$, we get,

$\ln(1 + 1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ldots$

$\Rightarrow \ln 2 = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \ldots$

$\Rightarrow \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \ldots = 1 - \ln 2$

$\Rightarrow \left(\frac{3-2}{2.3}\right) + \left(\frac{5-4}{4.5}\right) + \left(\frac{7-6}{6.7}\right) + \ldots = 1 - \ln 2$

$\Rightarrow \frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots = 1 - \ln 2$

Multiplying by 2 on both sides, we get

$\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots = 2(1 - \ln 2)$

Hence, $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \ldots = 2 - 2\ln 2$

c. We know that,

$$\ln_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots \text{ to } \infty \ldots \ldots \ldots \text{(i)}$$

$$\ln_e(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots \text{ to } \infty \ldots \ldots \ldots \text{(ii)}$$

Subtracting (ii) from (i)

$$\ln_e(1 + x) - \ln(1 - x) = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + 2\frac{x^7}{7} + \ldots$$

$$\text{or, } \ln_e\left(\frac{1 + x}{1 - x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)$$

$$\frac{1}{2}\ln_e\left(\frac{1 + x}{1 - x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots \text{ to } \infty$$

Put $x = \frac{1}{3}$

$$\frac{1}{2}\ln_e\left(\frac{1 + \frac{1}{3}}{1 - \frac{1}{3}}\right) = \frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} + \frac{\left(\frac{1}{3}\right)^7}{7} + \ldots$$

$$\frac{1}{2}\ln_e\left(\frac{\frac{4}{3}}{\frac{2}{3}}\right) = \frac{1}{3} + \frac{1}{3^3.3} + \frac{1}{3^5.5} + \frac{1}{3^7.7} + \ldots$$

$$\therefore \quad \frac{1}{2}\ln_e(2) = \frac{1}{3} + \frac{1}{3^3.3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \ldots$$

d. We know,

$$\ln_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots$$

$$\therefore \quad \ln(1 + x) = \ln(1 - x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)$$

$$\ln\left(\frac{1 + x}{1 - x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right)$$

Put $x = \frac{1}{2}$

$$\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) = 2\left(\frac{1}{2} + \frac{1}{3.2^3} + \frac{1}{5.2^5} + \frac{1}{7.2^7} + \ldots\right)$$

$$\ln 3 = 1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \ldots$$

e. LHS. $\left(\frac{1}{3} - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{3^2} - \frac{1}{2^2}\right) + \frac{1}{3}\left(\frac{1}{3^3} - \frac{1}{2^3}\right) + \ldots$

$$= \left(\frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3^2} + \frac{1}{3}\,\frac{1}{3^3} + \ldots\right) - \left(\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2^2} + \frac{1}{3} \cdot \frac{1}{2^3} - \ldots\right)$$

$$= -\left[-\left(\frac{1}{3}\right) - \frac{\left(\frac{1}{3}\right)^2}{2} - \frac{\left(\frac{1}{3}\right)^3}{3} - \ldots\right] - \left[\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \ldots\right]$$

$= -\ln_e\left(1 - \frac{1}{3}\right) - \ln_e\left(1 + \frac{1}{2}\right) = -\ln\frac{2}{3} - \ln\frac{3}{2} = -\left[\log\frac{2}{3} + \log\frac{3}{2}\right]$

$= -\ln\left(\frac{2}{3}, \frac{3}{2}\right) = -\ln 1 = 0$ Hence proved.

8. Sum to infinity the following series

a. $\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots$ to ∞. b. $\frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8} + \ldots$ to ∞.

Solution:

a. Sum to infinity the following series:

$\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \ldots$ to ∞

$\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) + \ldots$ to $\infty = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \ldots$ to ∞

$= 1 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \ldots\right) = 1 - \text{Ln}(1+1) = 1 - \ln 2$

b. We know that,

$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$ to ∞

Putting x = 1 on both sides, we get

$\ln(1 + 1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

or, $\ln 2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \ldots = \left(\frac{2-1}{2}\right) + \left(\frac{4-3}{3.4}\right) + \left(\frac{6-5}{5.6}\right) + \ldots$

$\ln 2 = \frac{1}{2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots$ to ∞

or, $\ln 2 - \frac{1}{2} = \frac{1}{3.4} + \frac{1}{5.6} + \ldots$

or, $\ln 2 - \frac{1}{2} = \frac{1}{3.4} + \frac{1}{5.6} + \ldots$

$\therefore \frac{1}{3.4} + \frac{1}{5.6} + \ldots$ to $\infty = \ln 2 - \frac{1}{2}$

9. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ to ∞, $|x| < 1$, prove that $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$ to ∞.

Solution:

Here, $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ to ∞

or, $y = \ln_e(1 + x)$

$\therefore e^y = 1 + x$

$1 + x = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$ to ∞

$\therefore x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$ to ∞

10. If $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$ to ∞, $|x| < 1$, prove that $x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \ldots$ to ∞

Solution:

Here, $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$ to ∞

$$y = -\left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \text{ to } \infty\right]$$

or, $y = -[\ln_e(1 - x)]$

or, $-y = \ln_e (1 - x)$

$\therefore$ $1 - x = e^{-y}$

$$1 - x = 1 - \frac{y}{1!} + \frac{y^2}{2!} - \frac{y^3}{3!} + \frac{y^4}{4!} - \dots$$

$$x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \dots \text{ to } \infty$$

11. If $y = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ to ∞, $|x| < 1$, prove that $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots$ to ∞

Solution:

Here, $y = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ to ∞

$1 + y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ to ∞

$1 + y = e^x$

Taking 'ln' on both sides

$\ln(1 + y) = x$

$\therefore$ $x = \ln(1 + y)$

$$x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots \text{ to } \infty$$

■

CHAPTER 3
ELEMENTARY GROUP THEORY

EXERCISE 3.1

1. Identify the binary operations from the followings
   a. Defined as $x \star y = x + y$ on the set of positive odd numbers O^+ for all $x, y \in O^+$.
   b. Defined as $x \star y = 2^{xy}$ on the set of real numbers $\mathbb{R}$, $\forall$ $x, y \in \mathbb{R}$
   c. Defined as $x \star y = 2x - y$ on the set of integers Z, $\forall x, y \in \mathbb{Z}$.
   d. Defined as $x \star y = x + y - xy$ on the set of natural numbers N, $\forall$ $x, y \in \mathbb{N}$.
   e. Defined as $A \star B = AB$ on the set of 2×2 matrix M, $\forall A, B \in M$.

Solution:

a. No, the operation $*$ on the set of positive odd numbers 0^+ defined by $x * y = x+y$ is not a binary operation because for all $x, y \in 0^+$, $x * y = x + y \notin 0^+$.
   e.g. $1, 3 \in 0^+$ but $1*3 = 1 + 3 = 4 \notin 0^+$.
b. Yes, since $\forall$ $x, y \in R$, $x + y = 2^{xy} \in R$
c. Yes, Here Z = {..., –3, –2, –1, 0, 1, 2, 3, ...}
   $\forall$ $x, y \in Z$, $2x - y$ is also an integer and uniquely belongs to Z. So, it is a binary operation.
d. No, let $2, 3 \in N$ then $2*3 = 2+3 - 2.3 = 5 - 6 = -1 \notin N$
   Therefore $*$ defined by $x*y = x + y - xy$ on the set of natural number is not a binary.
e. Yes, $\forall$ $A, B \in M$ = {set of 2×2 matrix}
   A*B = AB is also a 2×2 matrix and uniquery belongs to M. So, it is a binary.

3. Show that the multiplication is a binary operation on the set {-1,0,1}

Solution:

Let S = {–1, 0, 1}
For any $a, b \in S$ $\quad$ $a*b = a.b \in S$
$\therefore$ multiplication operation on S = {–1, 0, 1} is a binary operation.

4. Show that the binary operation $\star$ on the set S = {0, –1, 1} defined as $a \star b = a \times b$ $\forall a, b \in S$ is
   a. commutative $\qquad$ b. associative

Solution:

Given S = {–1, 0, 1}
Operation $*$ defined by $a*b = a \times b$
a. $\forall$ $a, b \in S$, $a*b = a.b = b.a = b*a$
   $\therefore$ $*$ is commutative on S.
b. $\forall$ $a, b, c \in S$
   $(a*b) * c = (a\times b) * c = a\times b\times c = a\times(b*c) = a*(b*c)$
   $\therefore$ '$*$' is an associative on S.

5. Test the existence of an identity element and inverses of the elements in the binary operation ($\star$) defined by $a \star b = 2a + b$ over the set of integers $\mathbb{Z}$ for $a, b \in \mathbb{Z}$

Solution:

Let e be an identify element of $a \in Z$ then

$a*e = a$	and $e*a = a$
$2a + e = a$	$2e + a = a$
$e = -a \in z$	$e = 0 \in z$

identify is not uniquely.
Let a' be inverse of $a \in z$ then $a * a' = e$
$2a + a' = -a$
$a^1 = -3a \in z$

6. The binary operation $\star$ on the set of rational numbers $\mathbb{Q}$ is defined as $a \star b = a + b + ab$ for every $a, b \in \mathbb{Q}$. Show that $\star$ satisfies the associative property.

Solution:

Let $a, b, c \in Q$ be any elements.

Then, $(a * b) * c = (a + b + ab) * c = a + b + ab + c + (a + b + ab) c$
$= a + b + ab + c + ac + bc + abc$
$= a + b + c + bc + ca + ab + abc$
$= a + (b + c + bc) + a(c + b + bc)$
$= (b + c + bc) + a + (b + c + bc) a$
$= a + (b + c + bc) + (b + c + bc) a$
$= a * (b + c + bc) = a * (b*c)$

$\therefore$ '$*$' is an associative.

7. Given that $a \star b = 3a + 2b$ $\forall a, b \in \mathbb{Z}$, set of integers. Verify that $\star$ is not a communicative binary operation on $\mathbb{Z}$.

Solution:

$\forall a, b \in z$

$a*b = 3a + 2b$ is also an integers and uniquely belongs to z. So, $*$ is a binary operation.

But $a*b = 3a + 2b \neq 3b + 2a = b*a$

$\therefore$ $a*b \neq b*a$

$\therefore$ '$*$' is not a commutative.

8. Let A and B are the subsets of the power set P of any non-empty set X. Show that $A \star B$ is defined as follows is a binary operation.
   a. Union b. Difference c. Intersection

Solution:

Given, P = power set of a non–empty set X.

a. Let $A, B \in P$ with $A*B = A \cup B$
   Here, $A \cup B$ must belong to set P. So, union operation on P is a binary.
b. Let $A, B \in P$ with $A*B = A - B$
   Here, $A-B$ or $B-A$ must belong to the power set P.
   $\therefore$ difference operation is a binary.
c. $\forall A, B \in P$ $A \cap B \in P$. So, intersection is a binary.

9. Show that the multiplicative operation on the set $S=\{1, \omega, \omega^2\}$ where ω is the cube root of unity is a binary operation. Is the operation
   a. Commutative b. Associative

Solution:

Given, set $S = \{1, \omega, \omega^2\}$ where ω is the cube root of unity operation; multiplication.

$1 \times \omega = \omega \in S$
$\omega \times \omega^2 = \omega^3 = 1 \in S$
$1 \times 1 = 1 \in S$
$\omega^2 \times \omega^2 = \omega^4 = \omega^3 . \omega = 1 . \omega = \omega \in S$
So, $\forall a, b \in S$ $a * b = a, b \in S$
$\therefore$ multiplication operation is binary on S.

a. Commutative

$1 \times \omega = \omega \times 1$

$\omega^2 \times \omega = \omega^3 = \omega \times \omega^2$

$\therefore$ $\forall$ a, b $\in$ S $\qquad$ $a*b = ab = ba = b*a$

$\therefore$ multiplication is commutative on S.

b. Associative

$1 \times (\omega \times \omega^2) = 1\times\omega^3 = 1\times\omega\times\omega^2 = (1 \times \omega) \times \omega^2$

$\therefore$ $\forall$ a, b, c $\in$ s. $(a*b)*c = (ab) * c = abc = a(bc) = a(b*c) = a*(b*c)$

$\therefore$ multiplication operation is association.

EXERCISE 3.2

1. What is congruent modulo 'n'? Illustrate with example.

Solution:

If x, y $\in$ z and n is positive integer

Then, x is said to be the congruent to y with modulo n if x–y is exactly divisible by n.

It can be expressed as $x \equiv y$ modulo n.

e.g. $7 \equiv 1$ modulo 3 $\Rightarrow$ 7 – 1 is divisible by 3

i.e. when 7 is divided by 3 the remainder is 1.

Similarly, $9 \equiv 1$ modulo 4 $\Rightarrow$ 9 – 1 is divisible by 4 i.e. when 9 is divided by 4 leaves remainder 1.

$a \equiv b$ modulo n $\Rightarrow$ a – b is divisible by n.

i.e. when a is divided by n, remainder is b.

2. Introduce one-one example of addition modulo 'n' and multiplication modulo 'n' to give a crystal clear concept of each other.

Solution:

Addition Modulo 'n'

Let x, y $\in$ z and n be a positive integer. The addition modulo 'n' is written as ($+_n$), defined as $x +_n y = r$ $(0 \le r < n)$ where r is remainder when x + y is divided by n.

e.g. $4 +_2 3 = 1$ i.e. when 4+3 = 7 is divided by 2, leaves remainder 1.

$12 +_3 4 = 1$ i.e. when 12+4 = 16 is divided by 3, remainder 1.

$18 +_4 4 = 2$ i.e. when 18+4 = 22 is divided by 4 remainder 2.

Multiplication Modulo 'n'

Let x, y $\in$ z and n is a positive integer. Then multiplication modulo n is denoted by (x_n), is defined by

$x \times_n y = r$, $(0 \le r < n)$ where r is remainder when $x \times y$ is divided by n.

e.g. $3 x_2 2 = 0$ i.e. when $3\times2 = 6$ is divided by 2, remainder is 0.

$7 \times_3 5 = 2$ when $7\times5 = 35$ is divided by 3, reminder is 2.

3. Prepare a Caley's table for the usual multiplication operation on the set S={1, -1, i, -i}. Is this operation a binary operation?

Solution:

x	1	–1	i	–i
1	1	–1	i	–i
–1	–1	1	–i	i
i	i	–i	–1	1
–i	–i	i	1	–1

$(i^2 = -1)$

Since, it is closed, the operation is a binary operation.

4. A residue class is given as $\mathbb{Z}_3$ = {0, 1, 2} prepare a Caley's table for multiplication modulo 3. Is the operation a binary operation?

Solution:

Given z_3 = {0, 1, 2}

We need to prepare a Caleys table for multiplication modulo 3.

x_3	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Since, it is closed, the multiplication modulo 3 on the set z = {0, 1, 2,} is a binary.

5. Define a binary operation. Use Caley's table to show that the multiplication operation on S= {0, 1} is binary operation.

Solution:

An operation '$*$' is said to be a binary on a set S if $\forall$ a, b $\in$ S then a $*$ b $\in$ S. In other words, an operation $*$ is said to be binary if it is closed.

x	0	1
0	0	0
1	0	1

Here, $0 \times 0 = 0$

$0 \times 1 = 0$

$1 \times 0 = 0$

$1 \times 1 = 1$

i.e. $\forall$ a, b $\in$ S

a$\times$b $\in$ S

'x' is a binary operation on S.

6. For all x, y$\in$ $\mathbb{Z}$, an operation is defined by x $*$ y=x+y-2 where $\mathbb{Z}$ is the set of integers. Is the operation communicative? Associative? Is it closure? Is the operation a binary one? Justify your answer.

Solution:

$\forall$ x, y $\in$ Z

$x * y = x + y - 2$ also belongs to Z

i.e. $\forall$ x, y $\in$ Z $\Rightarrow$ $x * y = x + y - 2 \in Z$

$\therefore$ '$*$' is closed.

Since it is closed, it is a binary.

$x * y = x + y - 2 = y + x - 2 = y * x$

$\therefore$ $\forall$ x, y $\in$ Z, x$*$y = y$*$x is proved.

'$*$' is commutative.

Finally, let x, y, z $\in$ Z then

$x*(y*z) = x * (y + z - 2) = x + y + z - 2 - 2 = x + y + z - 4 = x + y - 2 + z - 2$

$= (x*y) + z - 2 = (x*y) * z$

$\therefore$ This proves that '$*$' also associative.

7. A set M of all 3×2 matrices on which an operation defined by addition operation is performed. Is it a binary operation? Is it associative? If possible find its identity element. Does it have inverse? Justify it.

Solution:

Given, M = {set of all 3$\times$2 matrices}

Operation: addition

$\forall$ A, B $\in$ M, A + B $\in$ M

because addition of two matrices of order 3×2 is also 3×2 matrix.
Addition operation on set M is closed. It means it is a binary.
Let A, B, C $\in$ M then,
(A + B) + C = A + (B + C)
$\therefore$ Associative
Let I be an identify element of A$\in$M. Then,
A + I = A
I = A = A
I = null matrix

$\therefore$ $I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\qquad\qquad$ $\therefore$ $\qquad$ $A^1 = -A \in M$

8. If possible, solve 2x + 1 = 6 in $\mathbb{Z}_7$.

Solution:

$2x + 1 = 6$ in Z_7
or, $2 \times_7 x +_7 1 = 6$
or, $2 \times_7 x +_7 1 +_7 6 = 6 +_7 6$
or, $2 \times_7 x = 5$
or, $7 \times_7 (2 \times_7 x) = 4 \times_7 5$
or, $(4 \times_7 2) \times_7 x = 4 \times_7 5$ (By associative law)
or, $1 \times_7 x = 6$
or, $x = 6$

EXERCISE 3.3

1. Identify true and false statements from the followings:
   a. The set of natural numbers under multiplication is a group.
   b. The set of integers under addition is a group.
   c. The set of non-zero rational numbers is a group under multiplication.
   d. The set of 2×2 matrices under addition is a group.
   e. The set S = {0, 1, 2, 3, 4} under addition modulo 5 is a group.
   f. The set of fourth roots of 1 forms a group under multiplication.
   g. The set A = {1, -1} doesn't form a group under the addition.
   h. The set S = {0, 1, -1} forms a group under multiplication.
   i. The set of even positive numbers under addition is a group.
   j. The set S = {0, 1} under the addition modulo 2 is a group.
   k. The set S = {0, 1} under multiplication is a group.

Solution:

a. Set N (Natural number) Operation: Multiplication 'x'
   (N, x) is not a group because there doesn't exist inverse element.
b. (Z, +) is a group
c. (Q – {0}, X) is a group
d. Yes
e.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

It is closed, so binary
$(1*2)$ r $* 4 = 3*4 = 2$
$1*(2*4) = 1*1 = 2$
$\forall$ a, b, c $\in$ S, $(a*b) \times c = a*(b*c$
$\therefore$ associative holds
$0 \in s$ is an identity element $\forall$ $a \in S$.
$\forall$ x $\in$ S, $\exists$ inverse element $y \in$ S such that
$x +_5 y = 0$
Here, Inverse of 0 is 0
Inverse of 1 is 4
Inverse of 2 is 3
Inverse of 3 is 2
Inverse 4 is 1
$\therefore$ $(S, +_5)$ is a group

f. S = {1, –1, i, –i}
(S, $\times$) is a group

g. yes

h. yes

i. no, identity does not exist.

j. yes

k. yes

2. Construct a Caley's table for the multiplication modulo 3 for the set S = {0, 1, 2} and show that it is a group.

Solution:

$\times_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Closure Property: $\forall$ a, b $\in$ s $a x_3 b \in s$

$0x_2\ 0 = 9$	$0x_3\ 1 = 1$	$0x_3\ 2 = 0$
$1x_3\ 0 = 0$	$1x_3\ 1 = 1$	$1x_3\ –2 = –2$
$2x_3\ 0 = 0$	$2x_3\ 1 = 2$	$2x_3\ 2 = 1$

Associative Property
$0x_3\ (1\times_3 2) = 0\times_3 2 = 0$
$(0x_3 1) \times_3 2 = (0\times_3 2) = 0$
$(2x_3\ 1) \times_3 2) = 2\times_3 2 = 1$
$\therefore$ $\forall$ a, b, c $\in$ S, $(a*b) * c = a*(b*c)$
Existence of identity
$\forall$ a $\in$ S, $\exists$ $e \in$ S s.t. $a \times_3 e = a$
Existence of inverse:
$\forall$ $a \in S$, $\exists$ a' $\in$ S s.t. $a*a' = e$
$a * a^1 = e$
$\therefore$ $(S, \times_3)$ is a group.

3. Use Caley's table to show that the set S={1, -1, i, -i} is a group under multiplication.

Solution:

Given (S, $\times$) where s = {1, –1, 1, –1}
For closure
For any a, b $\in$ s $a*b = ab \in$ s

e.g. $1\times1 = 1\in s$ $\quad$ $-1\times-1 = 1 \in s$

$-1\times i = -i\in s$ $\quad$ $i\times i = i^2 = -1\in s$

$-i\times i = -i^2 = 1\in s$

So, for any two elements of S, the new element after operating also must belong to sets. So it is closed.

For associatively,

$(1\times1 \times - i = 1\times-i = -i$ $\quad$ $1\times(1 \times -i) = 1 \times - i = -1$

$\therefore (1 \times 1) \times -i = 1\times (1 \times - i)$

Similarly others follows

That is $\forall$ a, b, c $\in$ s $\Rightarrow (a\times b) \times c = a\times(b\times c)$

$\therefore$ It is associative.

For existence of identify:

Let $1\in s$ then $1\times1 = 1$

Let $-i\in s$ then $-i\times1 = -i$

Let $i\in s$ then $i\times1 = i$

Let $-1\in s$ then $-1\times1 = -1$

$\therefore$ -1 is an identify element of any element $\in s$.

For existence of inverse:

For $1\in s$	$1\times1 = 1$	$\therefore$ 1 is inverse of 1
For $-1\in s$	$-1\times-1 = 12$	$\therefore$ -1 is inverse of -1
For $i\in s$	$i \times -i = 1$	$\therefore$ $-i$ is inverse of i
For $-i\in s$	$-i\times i = 1$	$\therefore$ i is inverse of $-i$

Therefore, $\forall$ a $\in$ s $\exists$ a'$\in$ s.t. $a\times a' = e$

Hence, the algebraic structure $(S, \times)$ satisfies all the properties (i.e. closure, associativity, existence of identity and existence of inverse)

$\therefore$ $(S, \times)$ is a group

4. Define the followings:

a. Algebraic structure $\quad$ b. Semi-group

c. Group $\quad$ d. Monoid

e. Abelian group $\quad$ f. Trivial group

Solution:

a. Algebraic Structure:

An structure of the form $(G, *)$ is known as an algebraic structure. Where G is a non-empty set and '$*$' is a binary operation.

e.g. (G, +), (G, x), (Z, –) (Q, +) etc are some examples of an algebraic structure.

b. Semi-group: An algebraic structure $(G, *)$ is said to be a semi–group. It satisfies the associative property.

e.g. $(z^+, +)$ is a semi-group but $(z, +)$ is not.

c. Group: An algebraic structure $(G, *)$ is said to be a group if it satisfies the following for properties.

- Closure
- Associative
- Existence of identity
- Existence of inverse

d. Monoid: An algebraic structure $(G, *)$ is said to be a monoid if it satisfies associativity and existence of an identity. e.g. $(Z, \times)$

e. Abelian group: A group $(G, *)$ is said to be an abelian of it satisfies the commutative property.

f. Trivial group: A group $(G, *)$ is said to be a trivial group if G consists of a single element.

5. Show that the set of 2×2 non singular matrices under multiplication is a group.

Solution:

Let M = $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R \text{ and } ad - bc \neq 0\right\}$ be the set of 2×2 real non-singular matrices.

i. $\forall$ A, B $\in$ M, AB is again 2×2 real non-singular matrix. So, M is closed.

ii. $\forall$ A, B, C $\in$ M, A(BC) = (AB) C by matrix algebra. So M is associative under multiplication.

iii. $\forall$ A$\in$M, we get I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ even that AI = IA = A. So the identify matrix I exists.

iv. $\forall$ A$\in$M, we get A^{-1} (Since A is non–singular) set. $AA^{-1} - A^{-1}A = I$ where $A^{-1} = \frac{\text{Adj.}(A)}{|A|}$, is known as inverse of A. Hence M is a group.

6. Show that the set of integers $\mathbb{Z}$ forms a group under the operation addition.

Show that (Z, +) is a group.

i. Closure property: $\forall$ a, b $\in$ z, a + b $\in$ Z

$\therefore$ z is closed

ii. Associative: $\forall$ a, b, c $\in$ Z

(a + b) + c = a + (b + c)

$\therefore$ z is associative

iii. Existence of identify: $\forall$a $\in$ z, the must exist 0 $\in$ z s.t. a+0 = a

$\therefore$ 0 $\in$ Z is an identify element.

iv. Existence of inverse: $\forall$a$\in$Z there must –a$\in$Z s.t. a + (–a) = 0

$\therefore$ –a is inverse of a

Hence (Z, +) is a group.

7. Prepare a Cayley's table for the multiplication operation on the set S = {1, ω, ω²}, where ω is an imaginary cube root of unity and discuss the following:

a. Commutativity on the set
b. Associativity on the set
c. Closure property on the set
d. Existence of identity element
e. Existence of an inverse.

What can you conclude on the basis of the above all?

Solution:

$\times$	**1**	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

From the above table, s is closed

$(1\times\omega) \times \omega^2 = \omega\times\omega^2 = \omega^3 = 1$

$1\times(\omega\times\omega^2) = 1\times\omega^3 = 1\times1 = 1$

$\therefore\ (1\times\omega) \times \omega^2 = 1\times(\omega\times\omega^2)$

Again, $\omega\times(\omega^2 \times \omega^2) = \omega \times \omega^4 = \omega\times\omega = \omega^2$

$(\omega\times\omega^2) \times \omega^2 = \omega^3\times\omega^2 = 1\times\omega^2 = \omega^2$

$\therefore\ \omega\times(\omega^2 \times \omega^2) = (\omega \times \omega^2) \times \omega^2$

That is

$\forall$ a, b, c $\in$ S

(a×b) ×c = a×(b×c)

$\therefore$ S is associative.

From the table, 1 is an identity element of any element of S.

i.e. $1\times1 = 1$

$\omega\times1 = \omega$

$\omega^2\times1 = \omega^2$

$\therefore$ identity element 1 exist.

Since, $1\times1 = 1$

$\omega\times\omega^2 = 1$

$\omega^2\times\omega = 1$

$\therefore$ Inverse of 1 is 1 Inverse of w is ω^2

Inverse of w^2 is ω

So, there is an existence of an inverse element.

Finally, $1\times\omega = \omega\times1 = \omega$

$\forall$ a, b $\in$ s $a\times b = b \times a$

Comutative property satisifies.

Hence, $(S, \times)$ is an abelian group.

8. A binary operation on $\mathbb{Z}$ is defined as $a \star b = a + b + 2ab$ $\forall a, b\in \mathbb{Z}$, $a \neq -\frac{1}{2}$. Is $(\mathbb{Z}, \star)$ a group? Justify your answer.

Solution:

The set: Z

Operation '$*$' defined by $a*b = a+b+2ab$

a. Since, a, b $\in$ Z

$a + b + 2ab$ is also belongs to Z.

$\therefore$ z is closed

b. $\forall$ a, b, c $\in$ Z

$a * (b * c) = a * (b + c + 2bc) = a + b + c + 2bc + 2a(b + c + 2bc)$

$= a + b + c + 2bc + 2ab + 2ca + 4abc$

$= a + b + c + 2ab + 2bc + 2ca + 4abc$

Again, $(a*b) * c = (a + b + 2ab) * c = a + bb + 2ab + c + 2(a + b + 2ab) c$

$= a + b + 2ab + c + 2ca + 2bc + 4abc$

$= a + b + c + 2ab + 2bc + 2ca + 4abc$

$\therefore$ $a*(b*c) = (a*b) * c$

$\therefore$ z is associative

c. Since $a*0 = a + 0 + 2a0 = a$

$\forall$ a$\in$z, the identity element 0$\in$ Z exists.

d. Let d be inverse of a such that $a*d = 0$ (identify)

$a + d + 2ad = 0$

$d + 2ad = -a$

$d(1 + 2a) = -a$

$d = \frac{-a}{1 + 2a} \notin Z$

Even through

$a \neq -\frac{1}{2}$, if a = 1 then

$d = \frac{1}{3} \notin Z$

$\therefore$ Inverse element may not exist. Therefore, $(Z, *)$ is not a group.

9. Define a group. The binary operation $*$ is defined on a set S={a, ,b c} by the following Caley's table.

⋆	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Project your ideas about the identity element and inverse of the elements in the set. Is (S, ⋆) a group? Why?

Solution:

For definition of group look at 4(c)

From the given Calyes table,

S is closed.

$\forall$ a, b, c $\in$ S

$a * (b * c) = a * a = a$

$(a * b) * c = b * c = a$

$\therefore$ $a * (b * c) = (a * b) * c$

$\therefore$ S is associative

From the table, $a * a = a$ $b * a = b$ $c * a = c$

$\therefore$ a is identity element.

From the table,

$a * a = a$ $b * c = a$ $c * b = a$

$\therefore$ inverse of a is itself a

inverse of b is itself c

Therefore, inverse elements exists.

Since, S satisfies closure property, associative property, existence of identity and existence of inverse, (S, $*$) is a group.

10. Discuss whether the set of integers is a group

a. with respect to the subtraction? b. with respect to the multiplication?

Solution:

a. Set: Z

Operation: –

Now, we check (Z, –) is a group or not.

$\forall$ a, b $\in$ Z, $a * b = a - b \in Z$

$\therefore$ z is closed.

$\forall$ a, b, c $\in$ Z, $(a - b) - c \neq a - (b - c)$

e.g. let $a = -1$, $b = -3$ and $c = 5$

Then, $(a - b) - c = (-1 + 3) - 5 = 2 - 5 = -3$

$a - (b - c) = -1 - (-3 - 5) = -1 + 8 = 7$

$\therefore$ $(a - b) - c \neq a - (b - c)$

$\therefore$ Z is not associative

Since associative property is not satisfied.

The set of integers with subtraction operation is not a group.

b. (z, x) $\Rightarrow$ Group (check)

$\forall$ a, b $\in$ z, $a \times b \in z$ so, closure is satisfied.

$\forall$ a, b, c $\in$ z, $(a \times b) \times c = a \times (b \times c)$

$\therefore$ z is associative.

Let $a \in z$ then $a \times 1 = a$

$\therefore$ so there must exist $1 \in z$ s.t. $a \times 1 = a$ so identify element 1 exists.

If b is inverse of $a \in z$ then $a \times b = 1$

$b = \dfrac{1}{a} \notin Z$

Since, if a = 2 then $b = \dfrac{1}{2} \notin Z$

Therefore, there is no existence of inverse element.

$\therefore$ $(Z, \times)$ is not a group.

11. Prove that the set of all three dimensional vectors forms a group under the operation addition.

Solution:

Let $V = \{(a_1, a_2, a_3) : a_1\ a_2\ a_3 \in R\}$ be a set of 3 dimensional vectors.

Now, we have to show that $(v, +)$ is a group.

$\forall v_1\ v_2 \in v \qquad v_1 + v_2 \in V$

Since addition of two 3–dimention vectors is also 3–dimentional

$\therefore$ v is closed.

$\forall\ v_1, v_2, v_3 \in V$, then it is obvious that $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$

$\therefore$ associative property also holds.

$\forall\ v_1 \in v$ of 3 dimensional null vector

$(0, 0, 0)$ s.t. $v_1 + (0, 0, 0) = v_1$

$\therefore$ Identity element (0, 0, 0) exists.

$\forall v_1 \in V, \exists -v_1 \in V$ s.t. $v_1 + (-v_1) = (0, 0, 0)$

$\therefore$ inverse element also exists.

Hence, $(V, +)$ is a group.

12. Show that the set of all positive rational numbers form an Abelian group under the composition defined by $a \star b = \dfrac{ab}{4}$; $a, b \in \mathbb{Q}^+$

Solution:

i. Closure property:

$\forall\ a, b \in Q^+, \qquad a*b = \dfrac{ab}{4} \in Q^+$

$\therefore$ Q^+ is closed.

ii. Associative property:

$\forall\ a, b, c \in Q^+$ then $(a*b) * c = \left(\dfrac{ab}{4}\right) * c = \dfrac{\frac{abc}{4}}{4} = \dfrac{abc}{16}$

$a*(b*c) = a*\left(\dfrac{bc}{4}\right) = \dfrac{\frac{abc}{4}}{4} = \dfrac{abc}{16}$

$\therefore$ $(a*b) * c = (b * c)$

$\therefore$ associative property holds

iii. Existence of identity

Let e be an identify of $a \in Q^+$

Then, $a*e = a$

$\dfrac{ae}{4} = a$

$ae = 4a$

$ae - 4a = 0$

$a(e - 4) = 0$

$e = 4 \in Q^+$ since $a \neq 0$

Identify element exists.

iv. Existence of inverse:

let b be an inverse of $a \in Q^+$

such that, $a * b = e$

$\frac{ab}{4} = 4$

$ab = 16$

$b = \frac{16}{a} \in Q^+$

$\therefore$ inverse element $b \in Q^+$ exists.

Hence, $(Q^+, *)$ is a group.

Where $*$ is defined by $a*b = \frac{ab}{4}$

Further, $\forall$ a, $b \in Q^+$

$a*b = \frac{ab}{4} = \frac{ba}{4} = b*a$

$\therefore$ commutative property is also satisfied. Therefore, $(Q^+, *)$ is an abelian group.

13. If P be the set of all non-empty sub sets of X under the binary operation $\star$ defined by the relation $A \star B = A \cup B$ $\forall A, B \in X$. Is P a group?

Solution:

Given,

P = {non empty subsets of X}

Is (P, U) is a group?

$\forall P_1 P_2 \in P$ then $P_1 * P_2 = P_1 U P_2 \in P$

$\therefore$ P is closed.

$\forall P_1, P_2, P_3 \in P$, $(P_1 U P_2) U P_3 = P_1 U (P_2 U P_3)$

$\therefore$ P is associative.

$\forall P_1 \in P$ then $P_1 U \phi = P_1$ but $\phi \notin P$.

$\therefore$ identity element does not exist.

$\therefore$ this is not a group.

■

CHAPTER 4

COMPLEX NUMBER

EXERCISE 4.1

1. Find cube roots of

a. –1 b. 8

Solution:

a. Let, z be the cube root of –1

$z^3 = -1$

or, $z^3 + 1 = 0$

or, $(z)^3 + (1)^3 = 0$

or, $(z + 1)\ (z)^2 - z + 1) = 0$

Either,

$z = -1$

$z^2 - z + 1 = 0$ ... ... (i)

Comparing equation (i) with $ax^2 + bx + c = 0$

$\therefore$ a = 1, $\therefore$ b = –1, c = +1

Now,

$$x = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = 1 \pm \frac{\sqrt{1 - 4 \times 1 \times 1}}{2 \times 1} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

Taking positive

$$x = \frac{1 + \sqrt{3}i}{2}$$

Taking negative

$$x = \frac{1 - \sqrt{3}i}{2}$$

Here, z is the value of x.

Hence, the cube root of unity is –1, $\frac{1 + \sqrt{3}i}{2}$ and $\frac{1 - \sqrt{3}i}{2}$

b. Here,

Let, z be the cube root of 8

So, $z^3 = 8$

or, $(z)^3 - (2)^3 = 0$

or, $(z - 2)\ (z^2 + 2z + 4) = 0$

Either,

$z = 2$

$z^2 + 22 + 4 = 0$ ... ... (i)

Comparing equation (i) with $az^2 + bz + c = 0$

So, a = 1, b = 2, c = 4

Now,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times 4}}{2 \times 1} = \frac{-2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}\ i$$

Taking positive,

$z = -1 + \sqrt{3}\ i$

Taking negative

$z = -1 - \sqrt{3}\ i$

Hence, The required cube roots of 8 are 2, $-1 + \sqrt{3}i$ and $-1 - \sqrt{3}i$

2. Solve:

a. $z^4 = 1$ b. $z^4 = -1$ c. $z^6 = 1$

Solution:

a. Here, $z^4 = 1$
or, $(z)^4 - (1)^4 = 0$
or, $(z^2)^2 - (1^2)^2 = 0$
or, $(z^2 - 1)(z^2 + 1) = 0$
or, $(z - 1)(z + 1)(z^2 + 1) = 0$
Either,
or, $z = 1$,
or, $z = -1$
or, $z^2 + 1 = 0$ ... ... (i)
or, Comparing equation (i) with $az^2 + bz + c = 0$
$\therefore$ $a = 1, b = 0, c = 1$

Now, $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{0 \pm \sqrt{0 - 4\times1\times1}}{2\times1} = \dfrac{0 \pm \sqrt{-4}}{2} = \dfrac{0 \pm 2i}{2} = \pm i$

Taking positive Taking negative
$z = i$ $z = -i$
Hence, The required value of z is ±1 and $\pm i$.

b. Here,
$z^4 = -1$
or, $z^4 = -1 + i \times 0$
or, $z^4 = \cos180^o + i\sin180^o$
or, $z^4 = \{\cos(k.360 + 180^o) + i\sin(k.360^o + 180)\}$
or, $z = \{\cos(k.360^o + 180^o) + i\sin(k.360^o + 180^o)\}^{1/4}$

$= \cos\left(\dfrac{k.\ 360 + 180}{4}\right) + i\sin\left(\dfrac{k.\ 360 + 180^o}{4}\right)$

where, $k = 0, 1, 2, 3$
When $k = 0$ then, $z = \cos(k.90^o + 45^o) + i\sin(k.90^o + 45^o)$

$= \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

when, $k = 1$, $z = \cos135^o + i\sin135^o = -\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

when, $k = 2$, $z = \cos225^o + i\sin225^o = \dfrac{-1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i$

when, $k = 3$, $z = \cos315^o + i\sin315^o = \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i$

$\therefore$ $z = \pm\left(\dfrac{1 + i}{\sqrt{2}}\right), \pm\left(\dfrac{1 - i}{\sqrt{2}}\right)$

c. Here, $z^6 = 1$
$z^6 = 1^6 = 0$
or, $(z^2)^3 - (1)^3 = 0$
or, $(z^2 - 1)(z^4 + z^2 + 1) = 0$
Either,
$z = \pm1$
$z^4 + z^2 + 1 = 0$
or, $(z^2)^2 + (1)^2 + z^2 = 0$
or, $(z^2 + 1)^2 - 2z^2 + z^2 = 0$
or, $(z^2 + 1)^2 - (z)^2 = 0$
or, $(z^2 + 1 - z)(z^2 + 1 + z) = 0$
Either,
$z^2 + z + 1 = 0$ ... ... (i)

$z^2 - z + 1 = 0$ ... ... (ii)

Comparing equation (i) with $az^2 + bz + c = 0$

$\therefore$ $a = 1, b = 1, c = 1$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 3\times1\times1}}{2\times1} = \frac{-1 \pm \sqrt{3}i}{2}$$

Taking positive

$$z = \frac{-1 + \sqrt{3}i}{2}$$

Taking negative

$$z = \frac{-1 - \sqrt{3}i}{2}$$

Again, comparing equation (ii) with $az^2 + bz + c = 0$

$\therefore$ $a = 1, b = -1, c = 1$

Now,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1^2 - 4\times1\times1}}{2\times1} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

Taking positive

$$z = \frac{1}{2} + \frac{\sqrt{3}i}{2}$$

Taking negative

$$z = \frac{1}{2} - \frac{\sqrt{3}i}{2}$$

$$\therefore \quad z = \pm 1, \pm\left(\frac{-1 - \sqrt{3}i}{2}\right), \pm\left(\frac{1 - \sqrt{3}i}{2}\right)$$

3. If w be complex cube roots of unity, show that

a. $(1 + \omega^2)^3 - (1 + \omega)^3 = 0$

b. $(2 + \omega)(2 + \omega^2)(2 - \omega^2)(2 - \omega^4) = 21$

c. $(1 - \omega + \omega^2)^4 (1 + \omega - \omega^2)^4 = 256$

d. $(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6 = 128$

e. $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5) = 9$

f. $\dfrac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} = \omega$

g. $\dfrac{1}{1 + 2\omega} + \dfrac{1}{2 + \omega} - \dfrac{1}{1 + \omega} = 0.$

Solution:

a. $(1 + \omega^2)^3 - (1-\omega)^3 = (-\omega)^3 - (-\omega^2)^3 = -\omega^3 - (-\omega^6) = -1 - (-(\omega^3)^3)$
$= -1 - (-1) = -1 + 1 = 0$

b. $(2 + \omega)(2 + \omega^2)(2 - \omega^2)(2 - \omega^4) = (1 + 1 + \omega)(1 + 1 + \omega^2)(1 + 1 - \omega^2)(1 + 1 - \omega^4)$
$= (1 - \omega^2)(1 - \omega)(1 + 1 - \omega^2)(1 + 1 - \omega) \quad (\because \omega^3 = 1)$
$= (1 - \omega^2)(1 - \omega)(2 - \omega^2)(2 - \omega)$
$= 1 - \omega - \omega^2 + \omega^3)(4 - 2\omega - 2\omega^2 + \omega^3) = (1 - \omega - \omega^2 + 1)(4 - 2\omega - 2\omega^2 + 1)$
$= (2 + 1)(4 + 1 + 2) = 3 \times 7 = 21$

c. $(1 - \omega + \omega^2)^4 . (1 + \omega - \omega^2)^4 = (-2\omega)^4 . (-2\omega^2)^4$
$= 16\omega^3. \omega . 16\omega^3 . \omega^3 . \omega^2 = 16. \omega \times 16\omega^2 \; [\because \omega^3 = 1] = 256 \times \omega^3 = 256 \times 1 = 256$

d. $(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6 = (-2\omega)^6 + (-2\omega^2)^6$
$= 64\,\omega^3. \omega^3 + 64\,\omega^3 . \omega^3 . \omega^3 . \omega^3$
$= 64 + 64 \; [\because \omega^3 = 1] = 128$

e. $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5) = (1 - \omega)(1 - \omega^2)(1 - \omega^3. \omega)(1 - \omega^3 . \omega^2)$
$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2) = (1 - \omega^2)^2 (1 - \omega)^2$
$= (1 - 2\omega^2 + \omega^4)(1 - 2\omega + \omega^2) = (1 - 2\omega^2 + \omega)(1 - 2\omega + \omega^2) = (-3\omega^2)(-3\omega)$
$= 9\omega^3 = 9 \times 1 \; [\because \omega^3 = 1] = 9$

f. $\dfrac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} = \dfrac{a\omega^3 + b\omega.\omega^3 + c\omega^2 . \omega^3}{a\omega^2 + c\omega + b} = \dfrac{\omega(a\omega^2 + b\omega^3 + c\omega^4)}{(a\omega^2 + c\omega + b)}$
$= \omega \dfrac{(a\omega^2 + b + c\omega)}{(a\omega^2 + c\omega + b)} \; [\because \omega^3 = 1] = \omega$

g. $\dfrac{1}{1+2\omega}+\dfrac{1}{3+\omega}-\dfrac{1}{1+\omega}=\dfrac{1}{1+\omega+\omega}+\dfrac{1}{1+1+\omega}-\dfrac{1}{1+\omega}$

$=\dfrac{1}{-\omega^2+\omega}+\dfrac{1}{-\omega^2+1}-\dfrac{1}{1+\omega}$

$=\dfrac{1}{\omega(1-\omega)}+\dfrac{1}{(1-\omega^2)}-\dfrac{1}{1+\omega}=\dfrac{1}{\omega(1-\omega)}+\dfrac{1}{(1-\omega)(1+\omega)}-\dfrac{1}{(1+\omega)}$

$=\dfrac{1+\omega+\omega-\omega+\omega^2}{\omega.(1+\omega)(1-\omega)}=\dfrac{1+\omega+\omega^2}{\omega(1-\omega^2)}=0\ [\because 1+\omega+\omega^2=0]$

4. If α and β are complex cube roots of unity show that

a. $\alpha^4+\beta^4+\alpha^{-1}\beta^{-1}=0$ b. $\alpha^4+\alpha^2\beta^2+\beta^4=0$

Solution:

a. If $\alpha=\omega$, $\beta=\omega^2$

$\alpha^4+\beta^4=\dfrac{1}{\alpha\beta}=\omega^4=(\omega^2)^4+\dfrac{1}{\omega.\omega^2}=\omega+(\omega^3)^2.\omega^2+1\ [\because\omega^3=1]$

$=\omega+\omega^2+1=0\ [\because 1+\omega+\omega^2=0]$

b. Here,

$\alpha^4+\alpha^2\beta^2+\beta^4=\omega^4+\omega^2.(\omega^2)^2+(\omega^2)^4=\omega+\omega^2.\omega^4+\omega^8$

$=\omega+1+\omega^2\ [\because\omega^3=1]=0\ [\because 1+\omega+\omega^2=0]$

5. If $x=a+b$, $y=a\omega+b\omega^2$ and $z=a\omega^2+b\omega$, show that

a. $xyz=a^3+b^3$ b. $x+y+z=0$

c. $x^3+y^3+z^3=3(a^3+b^3)$ d. $x^2+y^2+z^2=6ab$

Solution:

Given,

$x=a+b$ $y=a\omega+b\omega^2$ $z=a\omega^2+b\omega$

a. $xyz=(a+b)(a\omega+b\omega^2)(a\omega^2+b\omega)=(a+b)(a^2\omega^3+ab\omega^2+ab\omega^4+b^2\omega^3)$

$=(a+b)\{a^2.1+ab(\omega^2+\omega^4)+b^2.1\}$

$=(a+b)\{a^2+ab(\omega+\omega^2)+b^2\}\ [\because\omega^4=\omega^3.\omega=1.\omega=\omega]$

$=(a+b)\{a^2-ab+b^2\}=a^3+b^3\ [\because\omega^2+\omega=-1)$

b. $x+y+z=(a+b)+(a\omega+b\omega^2)+(a\omega^2+b\omega)=a+b+a\omega+b\omega^2+a\omega^2+b\omega$

$=a(1+\omega+\omega^2)+b(1+\omega^2+\omega)=a\times0+b\times0=0$

c. $x^3+y^3+z^3=x^3+y^3+z^3-3xyz+3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)+3xyz$

$=0+3(a^3+b^3)\ [\because\text{ from (a) \& (b)}]=3(a^3+b^3)$

d. $x^2+y^2+z^2=(x+y+z)^2-2xy-2yz-2xz=0-2(xy+yz+xz)$

$=-2\{(a+b)(a\omega+b\omega)^2+(a\omega+b\omega^2).(a\omega^2+b\omega)+(a+b).(a\omega^2+b\omega)\}$

$=-2\{a^2\omega+ab\omega^2+ab\omega+b^2\omega^2+a^2\omega^3+ab\omega^2+ab\omega+b^2+a^2\omega^2+ab\omega+a\omega^2+b^2\omega\}$

$=-2\{a^2\omega+a^2\omega^2+a^2\omega^3+3ab\omega^2+3ab\omega+b^2\omega^2+b^2+b^2\omega\}$

$=-2\{a^2(\omega+\omega^2+1)+3(-1)ab+b^2(\omega^2+\omega+1)\}$

$=-2\{a^2(\omega+\omega^2+1)+3(-1)ab+b^2(\omega^2+\omega+1)\}=-2\{0-3ab+0\}=6ab$

6. Prove that,

a. $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^6+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^{12}=2$ b. $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^8+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^8=-1$

c. $1+\left(\dfrac{-1+\sqrt{-3}}{2}\right)^n+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^n=\begin{cases}3 \text{ if n is a multiple of 3}\\ 0 \text{ if n an integer which is not a multiple of 3}\end{cases}$

Solution:

a. We know,

$\omega = \dfrac{-1 + \sqrt{3}i}{2}$

$\omega^2 = \dfrac{-1 - \sqrt{3}i}{2}$

Now, $\left(\dfrac{-1 + \sqrt{-3}}{2}\right)^6 + \left(\dfrac{-1 - \sqrt{-3}}{2}\right)^{12} = \omega^6 + (\omega^2)^{12} = (\omega^3)^2 + \omega^{24}$

$= 1^2 + (\omega^3)^8 = 1 + 1^8 = 2$

b. $\left(\dfrac{-1 + \sqrt{-3}}{2}\right)^8 + \left(\dfrac{-1 - \sqrt{-3}}{2}\right)^8 = \omega^8 + (\omega^2)^8 = (\omega^3)^2 . \omega^2 + (\omega^3)^5 . \omega = \omega^2 + \omega = -1$

c. Let, $\omega = \dfrac{-1 + \sqrt{-3}}{2}$ $\qquad \omega^2 = \dfrac{-1 - \sqrt{-3}}{2}$

Case-I: If n is multiple of 3 i.e. n = 3k, k is on integer.

$= 1 + \left(\dfrac{-1 + \sqrt{-3}}{2}\right)^n + \left(\dfrac{-1 - \sqrt{-3}}{2}\right)^n = 1 + \omega^{3k} + (\omega^2)^{3k} = 1 + (\omega^3)^k + (\omega^3)^{2k}$

$= 1 + 1^k + 1^{2k} = 1 + 1 + 1 = 2 + 1 = 3$ proved.

Case II: n is not a multiple of 3 i.e. n = 3k + 1

$= 1 + \left(\dfrac{-1 + \sqrt{-3}}{2}\right)^n + \left(\dfrac{-1 - \sqrt{-3}}{2}\right)^n = 1 + (\omega)^{3k+1} + (\omega^2)^{3k+1}$

$= 1 + (\omega)^{3k+1} + \omega^{6k} - \omega^2 = 1 + (\omega^3)^k . \omega + (\omega^3)^{2k} . \omega^2 = 1 + \omega + \omega^2 = 0$ proved.

EXERCISE 4.2

1. Express the following complex numbers in polar form

a. $2 + 2i$ b. $-\sqrt{2} + i\sqrt{2}$ c. -1 d. $3i$

e. $-5i$ f. $i - \sqrt{3}$ g. $-3 - \sqrt{3}i$ h. $1 - \sqrt{3}\,i$

i. $(2, 2\sqrt{3})$ j. $\dfrac{1}{1 - i}$ k. $\sqrt{\dfrac{1 + i}{1 - i}}$

Solution:

a. Here, 2 + 2i

x = 2, y = 2

$r = \sqrt{2^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8} \;\; = 2\sqrt{2}$

$\text{Tan}\theta = \dfrac{y}{x} = \dfrac{2}{2} = 1$

$\therefore \;\; \theta = 45^o$

It can be written in polar form as $2\sqrt{2}\,(\cos 45^o + i\sin 45^o)$

b. Here,

$-\sqrt{2} + \sqrt{2}i$

Here, $x = -\sqrt{2}$ $\qquad y = \sqrt{2}$

$r = \sqrt{(-\sqrt{2})^2 + (\sqrt{2})^2}$

$\text{Tan}\theta = \dfrac{y}{x} = \sqrt{2 + 2}$

$= \dfrac{\sqrt{2}}{-\sqrt{2}} = -1 = \sqrt{4} = 2$

$\therefore \;\; \theta = 135^o$

In polar form = $2(\cos 135^o$ + is in $135^o)$

c. Here, Let, z = –1 + 0i

Here, x = –1

y = 0

$r = \sqrt{(-1)^2 + 0} = -\sqrt{1} = 1$

$\text{Tan}\theta = \dfrac{y}{x} = \dfrac{0}{-1} = 0$

d. Here,

Let, z = 0 + 3i

Here, x = 0, y = 3

$\text{Tan}\theta = \dfrac{y}{x}$ $r = \sqrt{(0)^2 + (3)^2}$

$= \dfrac{3}{0} = \infty = \sqrt{9} = 3$

$\therefore \theta = 180^o$

In polar form $= 1(\cos180^o + i\sin180^o)$
$= \cos180^o + i\sin180^o$

$\theta = 90^o$

In polar form $= 3(\cos90^o + i\sin90^o)$

e. Here,

Let $z = 0 - 5i$

Here, $x = 0$,

$y = -5$

$r = \sqrt{x^2 + y^2} = \sqrt{0 + 25} = \sqrt{25} = 5$

$\text{Tan}\theta = \dfrac{y}{x} = \dfrac{-5}{0} = \infty$

$\therefore \theta = 270^o$

Now, In polar form $-5i$

$= 5(\cos270^o + i\sin270^o)$

f. Here, Let, $z = -\sqrt{3} + i$

Here, $x = -\sqrt{3}$

$y = 1$

$r = \sqrt{(-\sqrt{3})^2 + (1)^2} = \sqrt{4} = 2$

$\text{Tan}\theta = \dfrac{y}{x}$

or, $\text{Tan}\theta = \dfrac{1}{-\sqrt{3}}$

or, $\text{Tan}\theta = \text{Tan } 150^o$

$\therefore \theta = 150^o$

In polar form $i-\sqrt{3}$

$= 2(\cos150^o + i\sin150^o)$

g. Here,

Let, $z = -3 - \sqrt{3}i$

Here, $x = -3$

$y = -\sqrt{3}$

$\text{Tan}\theta = \dfrac{y}{x} = \dfrac{-\sqrt{3}}{-3} = \dfrac{1}{\sqrt{3}}$

$\therefore \theta = 210^o$

$r = \sqrt{(-3)^2 + (-\sqrt{3})^2} = \sqrt{9 + 3} = \sqrt{12}$
$= 2\sqrt{3}$

In polar form,

$-3-\sqrt{3}i = 2\sqrt{3}\ (\cos210^o + i\sin210^o)$

h. Here,

Let, $z = 1 - \sqrt{3}i$

Here, $x = 1$, $y = -\sqrt{3}$

$\text{Tan}\theta = \dfrac{y}{x}$

or, $\text{Tan}\theta = \dfrac{-\sqrt{3}}{1}$

or, $\text{Tan}\theta = -\sqrt{3}$

or, $\text{Tan}\theta = \text{Tan}300^o$

$r = \sqrt{(1)^2 + (-\sqrt{3})^2} = \sqrt{1 + 3} = \sqrt{4}$
$= 2$

In polar form,

$1 - \sqrt{3}i = 2(\cos300^o + i\sin300^o)$

i. Here,

Let, $z = 2 + 2\sqrt{3}\ i$

Here, $z = 2$, $y = 2\sqrt{3}$

$\text{Tan}\theta = \dfrac{y}{x} = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

$\therefore \theta = 60^o$

$r = \sqrt{4 + 4\times3} = \sqrt{16} = 4$

In polar form,

$(2, 2\sqrt{3}) = 4(\cos60^o + i\sin60^o)$

j. Here,

Let, $z = \dfrac{1}{1-i} = \dfrac{1}{1-i} \times \dfrac{1+i}{1+i} = \dfrac{1+i}{1+1}$

$\therefore z = \dfrac{1}{2} + \dfrac{1}{2}i$

Here, $x = \dfrac{1}{2}$, $y = \dfrac{1}{2}$

Now, $\text{Tan}\theta = \dfrac{y}{x}$, $r = \sqrt{\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^2}$

$= \dfrac{\frac{1}{2}}{\frac{1}{2}} = 1 = \sqrt{\dfrac{1}{4} + \dfrac{1}{4}} = \sqrt{\dfrac{2}{4}} = \dfrac{1}{\sqrt{2}}$

$\therefore \theta = 45^o$

In polar form, $\dfrac{1}{1-i} = \dfrac{1}{\sqrt{2}}(\cos45^o + i\sin45^o)$

k. Here,

Let, $z = \sqrt{\dfrac{1+i}{1-i}} = \sqrt{\dfrac{1+i}{1-i}} \times \dfrac{1+i}{1+i} = \dfrac{1+i}{\sqrt{2}}$ $\therefore z = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

Here, $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$

$$r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

Now, $\text{Tan}\theta = \frac{y}{x} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 \quad \therefore \theta = 45^o$

In polar form, $\sqrt{\frac{1+i}{1-i}} = \cos 45^o + i\sin 45^o$

2. Express the following complex numbers in the form x + iy
   a. $2(\cos 30° + i \sin 30°)$
   b. $3 (\cos 150° + i \sin 150°)$
   c. $4 (\cos 240° + i \sin 240°)$
   d. $2\sqrt{2} (\cos 270° + i \sin 270°)$

Solution:

a. Here,
Let, $2(\cos 30^o + i\sin 30^o) = x + iy$
Equating real and imaginary parts;
$x = 2 \cos 30^o$ $\quad$ $y = 2\sin 30^o$
$= 2 \times \frac{\sqrt{3}}{2}$ $\quad$ $= 2 \times \frac{1}{2}$
$= \sqrt{3}$ $\quad$ $= 1$
$\therefore\ 2(\cos 30^o + i\sin 30^o) = \sqrt{3} + i.$

b. Here, Let, $3(\cos 150^o + i\sin 150^o) = x + iy$
Equating real and imaginary parts;
$x = 3 \cos 150^o$, $\quad$ $y = 3\sin 150^o$
$= 3 \times \left(\frac{-\sqrt{3}}{2}\right)$ $\quad$ $= 3 \times \frac{1}{2}$
$= \frac{-3\sqrt{3}}{2}$ $\quad$ $= \frac{3}{2}$
$\therefore\ 3(\cos 150^o + i\sin 150^o) = \frac{-3\sqrt{3}}{2} + \frac{3}{2} i$

c. Here, Let, $4(\cos 240^o + i\sin 240^o) = x + iy$
Equating real is imaginary parts;
$x = 4 \cos 240^o$,
$= 4 \times \left(\frac{-1}{2}\right) = -2$
$y = 4 \sin 240^o$
$= 4 \times \left(\frac{-\sqrt{3}}{2}\right) = \frac{-\sqrt{3} \times 2}{2} = -2\sqrt{3}$
$\therefore\ 4(\cos 240^o + i\sin 240^o) = -2 - 2\sqrt{3}i$

d. Here, Let, $2\sqrt{2} (\cos 270^o + i\sin 270^o) = x + iy$
Equating real and imaginary parts;
$x = 2\sqrt{2} \cos 270^o = 2\sqrt{2} \times 0 = 0$
$y = 2\sqrt{2} \sin 270^o = -2\sqrt{2}$
$\therefore\ 2\sqrt{2} (\cos 270^o + i\sin 270^o) = -2\sqrt{2}i$

3. Simplify
   a. $2(\cos 53° + i \sin 53°) . 3(\cos 7° + i \sin 7°)$
   b. $(\cos 5\theta + i\sin 5\theta). (\cos 3\theta + i \sin 3\theta)$
   c. $(\cos 72° + i \sin 72°) (\cos 12° - i \sin 12°)$
   d. $\frac{\cos 50° + i \sin 50°}{\cos 20° + i \sin 20°}$
   e. $\frac{(\cos 4\theta + i \sin 4\theta). (\cos 3\theta - i \sin 3\theta)}{\cos 3\theta + i \sin 3\theta}$
   f. $\frac{\cos 5\theta + i \sin 5\theta}{(\cos 2\theta + i \sin 2\theta)^2}$
   g. $\frac{(\cos 3\theta + i \sin 3\theta)^5}{(\cos \theta + i \sin \theta)^7}$

Solution:

a. Here,
$2(\cos 53^o + i\sin 53^o).3(\cos 7^o + i\sin 7^o) = 2\times 3 \{\cos(53^o + 7) + i\sin (53^o + 7^o)\}$

$= 6P\{\cos60^o + i\sin60^o\} = 6\left(\frac{1}{2} + i \times \frac{\sqrt{3}}{2}\right) = 3 + 3\sqrt{3}i$

b. $(\cos5\theta + i\sin\theta)(\cos3\theta + i\sin3\theta) = \cos(5\theta + 3\theta) + i\sin(5\theta + 3\theta) = \cos8\theta + i\sin8\theta$

c. $(\cos72^o + i\sin72^o)(|\cos12 - i\sin12^o) = (\cos72^o + i\sin72^o)\{\cos(-12) + i\sin(-12)\}$

$= \cos(72 - 12) + i\sin(72 - 12) = \cos60^o + i\sin60^o = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

d. $\frac{\cos50^o + i\sin50^o}{\cos20^o + i\sin20^o} = \cos(50 - 20) + i\sin(50 - 20) = \cos30^o + i\sin30^o = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

e. $\frac{(\cos4\theta + i\sin4\theta)(\cos3\theta - i\sin3\theta)}{\cos3\theta + i\sin3\theta} = \frac{(\cos4\theta + i\sin4\theta)(\cos(-3\theta) + i\sin(-3\theta))}{\cos3\theta + i\sin3\theta}$

$= \frac{\cos(4\theta - 3\theta) + i\sin(4\theta - 3\theta)}{\cos3\theta + i\sin3\theta} = \frac{\cos\theta = i\sin\theta}{\cos3\theta + i\sin3\theta}$

$= \cos(\theta - 3\theta) + i\sin(\theta - 3\theta) = \cos2\theta - i\sin2\theta$

f. $\frac{\cos5\theta + i\sin5\theta}{(\cos2\theta + i\sin2\theta)^2} = \frac{\cos5\theta + i\sin5\theta}{(\cos4\theta + i\sin4\theta)} = \cos(5\theta - 4\theta) + i\sin(5\theta - 4\theta) = \cos\theta + i\sin\theta$

g. $\frac{(\cos3\theta + i\sin3\theta)^5}{(\cos\theta + i\sin\theta)^7} = \frac{(\cos15\theta + i\sin15\theta)}{(\cos7\theta + i\sin7\theta)} = \cos(15\theta - 7\theta) + i\sin(15\theta - 7\theta)$

$= \cos8\theta + i\sin8\theta$

4. Using De Moivre's theorem, simplify

a. $\left[3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{16}$ b. $[2(\cos 50° + i\sin 50°)]^3$

c. $[4(\cos 6° + i\sin 6°)]^{30}$

d. $(\cos 70° + i\sin 70°)^6$ e. $(1 + i)^{15}$ f. $(1 - i)^{10}$

g. $(2i)^4$ h. $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^7$

Solution:

a. Here, $\left[3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{16} = 3^{16} \times \left[\cos\left(16 \times \frac{\pi}{4}\right) + i\sin\left(16 \times \frac{\pi}{4}\right)\right]$

$= 3^{16}[\cos4\pi + i\sin4\pi] = 3^{16}[1 + 0] = 3^{16}$

b. Here, $[2(\cos50^o + i\sin50^o)]^3 = 2^3[\cos150^o + i\sin150^o]$

$= 8\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right) = -4\sqrt{3} + 4i$

c. $[4(\cos6^o + i\sin6^o)]^{30} = 4^{30}[\cos(6\times30) + i\sin(6\times30)] = 4^{30}[-1+0] = -4^{30}$

d. $(\cos70^o + i\sin70^o)^6 = \cos(70\times6) + i\sin(70\times6) = \cos420^o + i\sin420^o = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

e. $(1 + i)^{15}$

Here, $x = 1, y = 1$

$\tan\theta = \frac{y}{x}$, $r = \sqrt{1^2 + 1^2} = \sqrt{2}$

or, $\tan\theta = \frac{1}{2}$

or, $\tan\theta = 1$

$\therefore\ \theta = 45^o$

In polar form,

$(1 + i) = \sqrt{2}(\cos45^o + i\sin45^o)$

f. $(1 - i)^{10}$

Let, $z = 1 - i$

Here, $x = 1, y = -1, r = \sqrt{(1)^2 + (-1)^2}$

$\tan\theta = \frac{y}{x} = \sqrt{1 + 1} = \sqrt{2}$

or, $\tan\theta = \frac{-1}{1}$

or, $\tan\theta = -1$

or, $\theta = 315^o$

In polar form,

Now,

$= \sqrt{2}\,(\cos 45^o + i\sin 45^o)\}^{15}$

$= 128\times\sqrt{2}\,\{\cos(45\times15) + i\sin(45\times15)\}$

$= 128\sqrt{2}\,(\cos 675^o + i\sin 675^o)$

$= 128\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 128\,(1 - i)$

g. $(2i)^4$

Let, $z = 0 + 2i$

Here,

$x = 0,\ y = 2$

$\text{Tan}\theta = \frac{y}{x} = \frac{2}{0} = \infty$

$\therefore\ \theta = 90^o$

$r = \sqrt{0 + 2^2}$

$= \sqrt{4} = 2$

In polar form,

$2i = 2(\cos 90^o + i\sin 90^o)$

Now,

$= \{2(\cos 90^o + i\sin 90^o\}^4$

$= 2^4\,(\cos 360^o + i\sin 360^o)$

$= 16\,(1 + 0) = 16$

$(1 - i) = \sqrt{2}\,(\cos 315^o + i\sin 315^o)$

Now,

$\left\{\begin{matrix}\sqrt{2}\,(\cos 315^o + i\sin 315^o) \\ 25\,(\cos 630^o + i\sin 630^o)\end{matrix}\right\}^{10}$

$= 2^5\,(0 + (-1)i) = 25 \times (-1)i = -32i$

h. Here,

Let, $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

Here, $x = \frac{1}{x}$, $y = \frac{\sqrt{3}}{2}$

$\text{Tan}\theta = \frac{y}{x}$, $\quad r = \sqrt{\left(\frac{1}{1}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \quad = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$

$\therefore\ \theta = 60^o$

In polar form, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$= \cos 60^o + i\sin 60^o$

Now, $(\cos 60^o + i\sin 60^o)^7$

$= \cos 420^o + i\sin 720^o = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

5. Using De Moivre's theorem, find square roots of

a. $-2 - 2\sqrt{3}i$ b. $4 + 4\sqrt{3}i$ c. $4i$ d. $-i$

e. -1 f. $4 - 4\sqrt{3}$

Solution:

Let, $z = -2 - 2\sqrt{3}i$

Here, $x = -2$

$y = -2\sqrt{3}$

$r = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$

$\text{Tan}\theta = \frac{y}{x} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}$

$\therefore\ \theta = 240^o$

In polar form, $z = 4(\cos 24^o + i\sin 240^o)$

In general polar form;

$\sqrt{z} = 4\,\{\cos(240 + 360.k) + i\sin(240 + 360.k)\}^{½} =$

$2\left\{\cos\left(\frac{240 + 360.k}{2}\right) + i\sin\left(\frac{240 + 360.k}{2}\right)\right\}$

where, $k = 0$ and 1

When, $k = 0$

$\sqrt{z} = 2(\cos 120^o + i\sin 120^o) = 2\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i$

when $k = 1$

$\sqrt{z} = 2\,(\cos 300^o + i\sin 300^o) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}\,i$

$\therefore \sqrt{-2 - 2\sqrt{3}i} = \pm\,(-1 + \sqrt{3}i)$

b. Let, $z = 4 + 4\sqrt{3}i$

Here, $x = 4$, $y = 4\sqrt{3}$

$\text{Tan}\theta = \frac{y}{x} = \frac{4\sqrt{3}}{4} = \sqrt{3}$

$\therefore\ \theta = 60^o$

$r = \sqrt{(4)^2 + (4\sqrt{3})^2} = \sqrt{16 + 48} = \sqrt{64} = 8$

In polar form, $4 + 4\sqrt{3}\,i = 8(\cos 60^o + i\sin 60^o)$

In general polar form i

$z = 8\{\cos(60 + 360.k) + i\sin(60 + 360.k)\}$

where, k = 0 and 1

when, k = 0

$\sqrt{z} = 2\sqrt{2}\left\{\cos\left(\frac{60 + 360k}{2}\right) + i\sin\left(\frac{60 + 360k}{2}\right)\right\}$

$= 2\sqrt{2}\,(\cos 30^o + i\sin 30^o) = 2\sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{6} + \sqrt{2}\,i$

when, k = 1

$\sqrt{z} = 2\sqrt{2}\,\{\cos 210^o + i\sin 210^o\} = 2\sqrt{2}\,(\cos 210^o + i\sin 210^o) = 2\sqrt{2}\left(\frac{-\sqrt{3}}{2} \frac{-1}{2}i\right)$

$= -\sqrt{6} - \sqrt{2}i = -(\sqrt{6} + \sqrt{2}i)$

$\therefore \sqrt{4 + 4\sqrt{3}\,i} = \pm\,(\sqrt{6} + \sqrt{2}i)$

c. Let, $z = 0 + 4i$

Here, $x = 0$, $y = 4$

$\text{Tan}\theta = \frac{y}{x} = \frac{4}{0} = \infty$

$\therefore\ \theta = 90^o$

$r = \sqrt{0 + (4)^2} = \sqrt{16} = 4$

In polar form, $z = 4\,(\cos 90^o + i\sin 90^o)$

In general polar formi

$z = 4\{\cos(90 + 360.k) + i\sin(90 + 360k)\}$

where, k = 0, 1

$\sqrt{z} = \sqrt{4}\ \left\{\cos\left(\frac{90 + 360.k}{2}\right) + i\sin\left(\frac{90 + 360k}{2}\right)\right\}$

$= 2\left\{\cos\left(\frac{90 + 360k}{2}\right) + i\sin\left(\frac{90 + 360k}{2}\right)\right\}$

where, k = 0

$\sqrt{z} = 2\{\cos 45^o + i\sin 45^o\}$

$= 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2} + \sqrt{2}\,i$

when k = 1

$\sqrt{z} = 2\{\cos 225^o + i\sin 22.5^o\} = 2\left(\frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = \sqrt{2} - \sqrt{2}\,i$

$\therefore \sqrt{4i} = \pm\sqrt{2}\,(1 + i)$

d. Let, $z = -i$

Here, $x = 0$, $y = -1$

$\text{Tan}\theta = \frac{y}{x} = \frac{-1}{0} = -\infty$

$\therefore \quad \theta = 270^o$

$r = \sqrt{0 + (-1)^2} = \sqrt{1} = 1$

In polar form;

$z = \cos(270^o + i\sin 270^o$

In general polar form;

$z = \cos(270 + 360.k) + i\sin(270 + 360.k)$

Now, k = 0 and 1

$\sqrt{z} = \cos\left(\frac{270 + 360.k}{2}\right) + i\sin\left(\frac{270 + 360.k}{2}\right)$

when k = 0

$\sqrt{z} = \cos 135^o + i\sin135^o = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i = \frac{-1}{\sqrt{2}}(1 - i)$

when, k = 1

$\sqrt{z} = \cos 315^o + i\sin315^o = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i = \frac{1}{\sqrt{2}}(1 - i)$

$\therefore \quad \sqrt{-i} = \pm\frac{1}{\sqrt{2}}(1 - i)$

e. Let, $z = -1 + 0i$

Here, $x = -1$, $y = 0$

$\text{Tan}\theta = \frac{y}{x} = \frac{0}{-1} = 0, \qquad r = \sqrt{(-1)2 + 0} = \sqrt{1} = 1$

$\therefore \theta = 180^o$

In polar form,

$z = \cos180^o + i\sin180^o$

In general polar form;

$z = \cos(180 + 360k) + i\sin(180 + 360.k)$

where, k = 0 and 1

$\sqrt{z} = \cos\left(\frac{180 + 360.k}{2}\right) + i\sin\left(\frac{180 + 360.k}{2}\right)$

when, k = 0

$\sqrt{z} = \cos90^o + i\sin90^o = 0 + i$

when, k = 1

$\sqrt{z} = a\cos270^o + i\sin 270^o = 0 - 1i$

$\therefore \sqrt{-1} = \pm i$

f. Let, $z = 4 - 4\sqrt{3}$

Here, $x = y$, $y = -4\sqrt{3}$

$\text{Tan}\theta = \frac{y}{x} = \frac{-4\sqrt{3}}{4} = -\sqrt{3}$ $, r = \sqrt{(4)^2 + (-4\sqrt{3^2})} = \sqrt{16 + 48} = 8$

$\therefore \quad \theta = 300^o$

In polar form,

$z = 8(\cos300^o + i\sin300^o)$

In general form;

$z = 8\{\cos(300 + 360.k) + i\sin(300 + 360.k)\}$

where, k = 0 and 1

$\sqrt{z} = 2\sqrt{2}\left\{\cos\left(\frac{300 + 360k}{2}\right) + i\sin\left(\frac{300 + 360.k}{2}\right)\right\}$

when, k = 0

$\sqrt{z} = 2\sqrt{2}\ (\cos150^o + i\sin150^o) = 2\sqrt{2}\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right) = -\sqrt{6} + \sqrt{2}\ i$

when, k = 1

$\sqrt{z} = 2\sqrt{2}\ (\cos330^o + i\sin330^o) = 2\sqrt{2}\left(\frac{\sqrt{3}}{2} \frac{-1}{2}\ i\right) = \sqrt{6} - \sqrt{2}\ i$

$\therefore \sqrt{4 - 4\sqrt{3}i} = \pm\ (\sqrt{6} - \sqrt{2}\ i)$

6. Using De Moivre's theorem, find

a. Cube roots of unity

b. Cube roots of –1

c. Cube roots of i

d. Cube roots of –i

e. fourth roots of unity

f. fourth roots of $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

Solution:

a. Let, z = 1 + 0i

Here, x = 1, y = 0

$\text{Tan}\theta = \frac{y}{x} = \frac{0}{1} = 0$ and $r = \sqrt{(1)^2 + 0} = \sqrt{1} = 1$

$\therefore \theta = 0^o$

In polar form,

$z = \cos0^o + i\sin0^o$

In general polar form;

$z = \cos(360.k + 0^o) + i\sin(0 + 360.k)$

where, k = 0 and 2

$z^{1/3} = \{\cos(0 + 360.k) + i\sin(0 + 360.k)\}^{1/3} = \cos(0 + 120k) + i\sin(0 + 120k)$

when, k = 0

$z^{1/3} = \cos 0 + i\sin0 = 1 + 0 = 1$

when, k = 1

$z^{1/3} = \cos120 + i\sin 120^o = \frac{-1}{2} + \frac{\sqrt{3}}{2}\ i$

when, k = 2

$z^{1/3} = \cos240^o + i\sin240^o = \frac{-1}{2} \frac{-\sqrt{3}}{2}\ i$

$\therefore$ Cube roots of 1 = 1, $\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}\ i\right)$, $\left(\frac{-1-\sqrt{3}}{2}\ i\right)$

b. Let, z = –1 + 0i

Here, x = –1, y = 0

$\text{Tan}\theta = \frac{y}{x} = \frac{0}{-1} = 0$, $r = \sqrt{(-1)^2 + 0} = \sqrt{1} = 1$

$\therefore \theta = 180^o$

In polar form, $z = \cos180^o + i\sin180^o$

In general polar form;

$z = \cos(180^o + 360.k) + i\sin(180^o + 360.k)$

where, k = 0, 1, 2

$z^{1/3} = \cos\left(\frac{180 + 360.k}{3}\right) + i\sin\left(\frac{180 + 360.k}{3}\right)$

when, k = 0

$z^{1/3} = \cos60^o + i\sin60^o = \frac{1}{2} + \frac{\sqrt{3}}{2}\ i$

when, k = 1,

$z^{1/3} = \cos180^o + i\sin180^o = -1 + 0 = -1$

when, k = 2

$z^{1/3} = \cos300^o + i\sin300^o = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\therefore$ Hence, the required cube roots of unity are

$-1, \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\frac{1}{2} - \frac{\sqrt{3}}{2}i$,

c. Let, z = 0 + 1i

Here, z = 0, y = 1

$\text{Tan}\theta = \frac{y}{x} = \frac{1}{0} = \infty$

$\therefore \theta = 90^o$

$r = \sqrt{0 + 1^2} = 1$

In polar form, $z = \cos90^o + i\sin90^o$

In general polar form;

$z = \cos(90 + 360.k) + i\sin(90 + 360.k)$

$z^{1/3} = \cos\left(\frac{90 + 360.k}{3}\right) + i\sin\left(\frac{90 + 360.k}{3}\right)$

when k = 0, where, k = 0, 1 and 2

$z^{1/3} = \cos30^o + i\sin30^o = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

when k = 1

$z^{1/3} = \cos150^o + i\sin150^o = \frac{-\sqrt{3}}{2} + \frac{1}{2}i$

when k = 2

$z^{1/3} = \cos270^o + i\sin270^o = 0 - 1i = -i$

Hence, the required cube roots of unity are $-i, \frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $\frac{-\sqrt{3}}{2} + \frac{1}{2}i$

d. Let, z = 0 – i

Here, x = 0, y = –1

$\text{Tan}\theta = \frac{y}{x} = \frac{-1}{0} = \infty$

$\therefore \theta = 270^o$

$r = \sqrt{0 + (-1)^2} = \sqrt{1} = 1$

In polar form, $z = \cos270^o + i\sin270^o$

In general polar form;

$z = \cos(270 + 360.k) + i\sin(270 + 360.k)$

$z^{1/3} = \cos\left(\frac{270 + 360.k}{3}\right) + i\sin\left(\frac{270 + 360.k}{3}\right)$

where, k = 0, 1 and 2

when, k = 0

$z^{1/3} = \cos 90^o + i\sin90^o = 0 + 1i = i$

when k = 1,

$z^{1/3} = \cos210^o + i\sin210 = \frac{-\sqrt{3}}{2} - \frac{1}{2}i = \frac{-\sqrt{3} - 1i}{2}$

when k = 2

$z^{1/3} = \cos.330^o + i\sin330^o = \frac{\sqrt{3}}{2} - \frac{i}{2} = \frac{\sqrt{3} - i}{2}$

Hence, the required cube roots of i are i, $\frac{-\sqrt{3}-i}{2}$ and $\frac{\sqrt{3}-i}{2}$

e. Let, z = 1 + 0i

Here, x = 1, y = 0

$\text{Tan}\theta = \frac{y}{x} = \frac{0}{1} = 0$, $r = \sqrt{1+0} = \sqrt{1} = 1$

$\therefore\ \theta = 0^o$

In polar form;

In general polar form, z = cos(0 + 360.k) + isin (0 + 360k)

$z^{1/4} = \cos\left(\frac{0 + 360.k}{4}\right) + i\sin\left(\frac{0 + 360.k}{4}\right)$

when k = 0, where, k = 0, 1, 2 and 3

$z^{1/4} = \cos 0^o + i\sin 0^o = 1 + 0 = 1$

when k = 1

$z^{1/4} = \cos 90^o + i\sin 90^o = 0 + 1i = i$

when k = 2

$z^{1/4} = \cos 180^o + i\sin 180^o = -1 + 0 = -1$

when k = 3

$z^{1/4} = \cos 270^o + i\sin 270^o = 0 - 1i = -i$

Hence, the required forth roots of unity are ± 1 and $\pm$ i

f. Let, $z = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$

Here, $x = \frac{-1}{2}$, $y = \frac{\sqrt{3}}{2}$

$\text{Tan}\theta = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}} = \frac{\sqrt{3}}{2} \times \frac{2}{-1} = -\sqrt{3}$, $r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$

$\therefore\ \theta = 120^o$

In polar form, $z = \cos 120^o + i\sin 120^o$

In general polar form; $z = \cos(120 + 360.k) + i\sin(120^o + 360.k)$

$z^{1/4} = \cos\left(\frac{120 + 360.k}{4}\right) + i\sin\left(\frac{120 + 360.k}{4}\right)$

where, k = 0, 1, 2, 3

when k = 0 $\quad z^{1/4} = \cos 30^o + i\sin 30^o = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

when k = 1 $\quad z^{1/4} = \cos 120^o + i\sin 120^o = \frac{-1}{2} + \frac{\sqrt{3}}{2}i = \frac{-1+\sqrt{3}i}{2}$

when k = 2 $\quad z^{1/4} = \cos 210^o + i\sin 210^o = \frac{-\sqrt{3}}{2} - \frac{1}{2}i = \frac{-\sqrt{3}-1i}{2}$

when k = 3 $\quad z^{1/4} = \cos 300^o + i\sin 300^o = \frac{1}{2}\frac{-\sqrt{3}i}{2} = \frac{1-\sqrt{3}i}{2}$

Hence, the required fourth roots of unity are $\pm\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$ and $\pm\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

7. Using De Moivre's theorem, solve

a. $z^3 + 8i = 0$ b. $z^4 = -1$ c. $z^6 = 1$

Solution:

a. Here, $z^3 + 8i = 0$

Let, $z^3 = -8i$

$z^3 = 0 - 8i$

Here, $x = 0$, $y = -8$

$\text{Tan}\theta = \frac{y}{x} = \frac{-8}{0} = \frac{-8}{0} = \infty$, $r = \sqrt{x^2 + y^2} = \sqrt{0 + (-8)^2} = \sqrt{64} = 8$

$\therefore \theta = 270^o$

In polar form, $z = 8(\cos 270^o + i\sin 270^o)$

In general polar form;

$z = 8\{\cos(270 + 360.k) + i\sin(270 + 360.k)\}$

$z^{1/3} = 2\left\{\left(\frac{270 + 360.k}{3}\right) + i\sin\left(\frac{270 + 360.k}{3}\right)\right\}$

where, k = 0, 1, 2

when k = 0

$z^{1/3} = \{\cos 90^o + i\sin 90^o\} = 2(0 + 1i) = 2i$

when k = 1 $\quad z^{1/3} = 2(\cos 210^o + i\sin 210^o) = 2\left(\frac{-\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$

when k = 2 $\quad z^{1/3} = 2(\cos 330^o + i\sin 330^o) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$

Hence, the required cube roots of $-8i$ are $2i$, $\sqrt{3} - i$ and $-(\sqrt{3} + i)$

b. Let, $z^4 = -1 = -1 + 0i$

Here, $x = -1$, $y = 0$

$\text{Tan}\theta = \frac{y}{x} = \frac{0}{-1} = 0$, $r = \sqrt{(-1)^2 + 0} = \sqrt{1} = 1$

$\therefore \theta = 180^o$

In polar form;

$z = \cos(180^o + i\sin 180^o)$

In general polar form;

$z = \cos(180 + 360.k) + i\sin(180 + 360.k)$

$z^{1/4} = \cos\left(\frac{180 + 360.k}{4}\right) + i\sin\left(\frac{180 + 360.k}{4}\right)$

when k = 0 $\quad z^{1/4} = \cos 45^o + i\sin 45^o = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

when k = 1 $\quad z^{1/4} = \cos 135^o + i\sin 135^o = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

when k = 2 $\quad z^{1/4} = \cos 225^o + i\sin 225^o = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

when k = 3 $\quad z^{1/4} = \cos 315^o + i\sin 315^o = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

Hence, the required fourth roots of -1 is $\pm\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$ and $\pm\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

c. $z^6 = 1$

We have, $z^6 = 1 = 1 + i(0) = \cos 30^o + i\sin 0^o$

$\Rightarrow \quad z^6 = \cos 2n\pi + i\sin 2n\pi$

$\Rightarrow \quad z = [\cos 2n\pi + i\sin 2n\pi]^{1/6}$

By De-moivre's theorem

$z = \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}$

where n = 0, 1, 2, 3, 4, 5

When n = 0 then the first root of z is,

$z = \cos 0 + i\sin 0 = 1 + 0 = 1$

When n = 1 then the 2nd root of z is,

$z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{1 + i\sqrt{3}}{2}$

When n = 2 then the 3rd root of z is,

$z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{-1 + i\sqrt{3}}{2}$

When n = 3 then the 4th root of z is

$z = \cos\pi + i\sin\pi = -1 + i.0 = -1$

When n = 4 then the 5th root of z is,

$z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{1 - i\sqrt{3}}{2}$

When n = 5 then the 6th root of z is,

$z = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{1 - i\sqrt{3}}{2}$

Hence, the required six roots of z are

$1, -1, \frac{1 + i\sqrt{3}}{2}, \frac{1 - i\sqrt{3}}{2}, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$

8. If $z = \cos\theta + i\sin\theta$, show that

a. $z^n + \frac{1}{z^n} = 2\cos n\theta$

b. $z^n - \frac{1}{z^n} = 2i\sin n\theta$

Solution:

a. Here, $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$ $\qquad$ $z^{-n} = \cos n\theta - i\sin n\theta$

LHS $z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta$ proved.

b. Here, $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

LHS $z^n - \frac{1}{z^n} = z^n - z^{-n} = \cos n\theta + i\sin\theta - \cos n\theta + i\sin n\theta = 2i\sin n\theta$ RHS

9. If z_1 and z_2 are two complex number, prove that

a. $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

b. $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

c. $\arg\left(\overline{z_1}\right) = 2\pi - \arg(z_1)$

Solution:

a. Let, z_1 and z_2 be $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ respectively.

Then, $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1).\ r_2(\cos\theta_2 + i\sin\theta_2) = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$\therefore \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$ proved.

b. Let z_1 and z_2 be $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ respectively with $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

Now, $\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

So, arg $\left(\frac{z_1}{z_2}\right)$ = $\theta_1 - \theta_2$ = arg (z_1) – arg (z_2) proved

c. Let, $z = r(\cos\theta + i\sin\theta)$
where, Ag . $z = \theta$
Then, $\overline{z} = r(\cos\theta - i\sin\theta)$
$\overline{z} = r\{\cos(2\pi - \theta) + i\sin(2\pi - \theta)\}$ ∴ Arg $(\overline{z}) = 2\pi - \theta = 2\pi -$ Arg (z)

10. Express the following into the form of x + iy

a. $e^{\frac{i\pi}{2}}$ b. $e^{\frac{-i\pi}{6}}$ c. $-5e^{\frac{-i\pi}{3}}$

Solution:

a. $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$ b. $e^{-i\pi/6} = \cos\frac{\pi}{6} - i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - i\frac{1}{2}$

c. $-5e^{-i\pi/3} = -5[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}] = -5\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right] = -\frac{5}{2} + i\frac{\sqrt{3}}{2}$

11. Express the following into the form of re^{ix}

a. $3 + 4i$ b. $3i$ c. $-2 - 2i$ d. $1 + i\sqrt{3}$

Solution:

a. To express the complex form into re^{ix} form firstly, we change into polar form,
Let $3 + 4i = r(\cos\theta + i\sin\theta)$ ... ... (i)
$\Rightarrow$ $r\cos\theta = 3$ and $ir\sin\theta = 4i \Rightarrow r\sin\theta = 4$
Squaring and adding these two
We get, $r^2 = 25$ ∴ $r = 5$
Also, $\tan\theta = \frac{4}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.927$
∴ The complex number in exponential form is $re^{i\theta}$ i.e. $5e^{0.927i}$

b. 3i
Let $0 + 3i = r(\cos\theta + i\sin\theta)$
$\Rightarrow$ $r\cos\theta = 0$ and $r\sin\theta = 3$
$r^2 = 9$ ∴ $r = 3$
And, $\tan\theta = \frac{3}{0} = \infty = \tan\frac{\pi}{2}$
∴ The complex number in exponential form is $re^{i\theta}$ i.e. $3e^{i\pi/2}$

c. $-2 - 2i$
Let $-2 - 2i = r(\cos\theta + i\sin\theta)$
$\Rightarrow$ $r\cos\theta = -2$ and $r\sin\theta = -2$
$r^2 = 4 + 4 \Rightarrow r^2 = 8$ ∴ $r = 2\sqrt{2}$
And, $\tan\theta = \frac{-2}{-2} = 1 = \tan\frac{5\pi}{4}$ ∴ $\theta = 5\frac{\pi}{4}$
∴ The complex number is exponential form is, $re^{i\theta}$ i.e. $2\sqrt{2}\,e^{i5\pi/4}$

d. $1 + i\sqrt{3}$
Let $1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$
$\Rightarrow$ $r\cos\theta = 1$ and $r\sin\theta = \sqrt{3}$
$r^2 = 4 \Rightarrow r = 2$
And, $\tan\theta = \sqrt{3} = \tan\frac{\pi}{3}$
∴ The complex number in exponential form is $re^{i\theta}$ i.e. $2e^{i\pi/3}$

■

CHAPTER 5

QUADRATIC EQUATIONS

EXERCISE 5.1

1. Determine the nature of the roots of the following equations
   a. $x^2 - 12x + 40 = 0$ b. $x^2 - 4x - 3 = 0$ c. $2x^2 - 12x + 18 = 0$
   d. $4x^2 + 8x - 5 = 0$ e. $x^2 - 16 = 0$

Solution:

a. Here, $x^2 - 12x + 40 = 0$ ... ... (i)
   Comparing equation (i) with $ax^2 + bx + c = 0$, we get
   $\therefore$ $a = 1, b = -12, c = 40$
   Now, $b^2 - 4ac = (-12)^2 - 4 \times 1 \times 40 = 144 - 160 = -16 < 0$
   Hence, roots are imaginary and unequal.

b. Here, $x^2 - 4x - 3 = 0$ ... ... (i)
   Comparing equation (i) with $ax^2 + bx + c = 0$, we get
   $\therefore$ $a = 1, b = -4, c = -3$
   Now, $b^2 - 4ac = (-4)^2 - 4\times1\times(-3) = 16 + 12 = 28 > 0$
   Hence, roots are unequal, real and irrational.

c. Here, $2x^2 - 12x + 18 = 0$ ... ... (i)
   Comparing equation (i) with $ax^2 + bx + = 0$, we get
   $\therefore$ $a = 2, b = -12, c = 18$
   Now, $b^2 - 4ac = (-12)^2 - 4\times2\times18 = 144 - 144 = 0$
   Hence, roots are real, equal and rational.

d. Here, $4x^2 + 8x - 5 = 0$ ... ... (i)
   Comparing equation (i) with $ax^2 = bx + c = 0$, we get,
   $\therefore$ $a = 4, b = 8, c = -5$
   Now, $b^2 - 4ac = (8)^2 - 4 \times 4 \times (-5) = 64 + 80 = 144 > 0$ and perfect square
   Hence, Roots are real, unequal, rational.

e. Here, $x^2 - 16 = 0$ ... ... (ii)
   Comparing equation (i) with $ax^2 + bx + c = 0$
   $\therefore$ $a = 1, b = 0, c = -16$
   Now, $b^2 - 4ac = 0 - 4 \times 1 \times (-16) = 64 > 0$ and perfect square
   Hence, roots are real, unequal is rational.

2. For what value of the p will the equation $5x^2 - px + 45 = 0$ have the equal roots?

Solution:

Given equation is $5x^2 - px + 45 = 0$ ... (i)
Comparing equation (i) with
$ax^2 + bx + c = 0$
$\therefore$ $a = 5, b = -p, c = 45$
Now, for being equal roots;
$b^2 - 4ac = 0$
or, $(-p)^2 - 4\times5\times45 = 0$
or, $p^2 = 900$
or, $(p)^2 = (\pm 30)^2 \therefore p = \pm 30$

3. Find the value of k so that the equation
   a. $x^2 + (k+2)x + 2k = 0$ has equal roots
   b. $x^2 - (2k - 1)x - (k - 1) = 0$ has equal roots

Solution:

a. Here,
Comparing equation $x^2 + (k + 2) x + 2k = 0$ with $ax^2 + bx + c = 0$
$\therefore$ $a = 1, b = k+2, c = 2k$
Now, for being equal roots;
$b^2 - 4ac = 0$
or, $(k+2)^2 - 4\times1\times2k = 0$
or, $k^2 + 4k + 4 - 8k = 0$
or, $k^2 - 4k + 4 = 0$
or, $(k - 2)^2 = 0$
$\therefore$ $k = 2$

b. Here, Comparing equation $x^2 - (2k-1). x - (k - 1) = 0$ with $ax^2 + bx + c = 0$. we get,
$\therefore$ $a = 1, b = -(2k -1), c = -(k - 1)$
Now, for being equal roots;
$b^2 - 4ac = 0$
or, $\{-(2k - 1)\}^2 - 4\times1\times\{- (k - 1)\} = 0$
or, $4k^2 - 4k + 1 + 4k - 4 = 0$
or, $4k^2 - 3 = 0$
or, $k^2 = \frac{3}{4}$ $\therefore$ $k = \pm\frac{\sqrt{3}}{2}$

4. If the equation $(1+ m^2)x^2+ 2mcx + c^2 - a^2 = 0$ has equal roots, show that $c^2 = a^2(1 + m^2)$

Solution:

a. Here, comparing equation $(1 + m^2). x^2 + 2mc. x + (c^2 - a^2) = 0$ with $ax^2 + bx + c = 0$,
we get,
$\therefore$ $a = 1+m^2, b = 2mc, c = c^2 - a^2$
Now,
For being equal roots;
$b^2 - 4ac = 0$
or, $(2mc)^2 - 4(1 + m^2) . (c^2 - a^2) = 0$
or, $4m^2c^2 - 4\{1(c^2 - a^2) + m^2 (c^2 - a^2)\} = 0$
or, $m^2c^2 - (c^2 - a^2) - m^2c^2 + m^2a^2 = 0$
or, $-(c^2 - a^2) = - m^2a^2$
or, $c^2 - a^2 = m^2a^2$
or, $c^2 = m^2a^2 + a^2$
or, $c^2 = a^2(1 + m^2)$ proved.

5. Show that the roots of the equation $(a^2 - bc)x^2 + 2(b^2 - ca)x + (c^2 - ab) = 0$ will be equal if either $b = 0$ or $a^3+ b^3 +c^3 - 3abc = 0$

Solution:

Here, comparing $(a^2 - bc) . x^2 + 2(b^2 - ca) . x + c^2 - ab = 0$ with $Ax^2 + Bx + C = 0$
$\therefore$ $A = a^2 - bc, B = 2(b^2 - ca), c = c^2 - ab$
For equal roots,
$B^2 - 4AC = 0$
or, $\{2(b^2 - ca)\}^2 - 4(a^2 - bc) . (c^2 - ab) = 0$
or, $(b^2 - ca)^2 - (a^2 - bc) (c^2 - ab) = 0$
or, $b^4 - 2ab^2c + c^2a^2 - a^2c^2 + a^3b + bc^3 - ab^2c = 0$
or, $a^3b + b^4 + bc^3 - 3ab^2c = 0$

or, $b(a^3 + b^3 + c^3 - 3abc) = 0$
Either, $b = 0$,
$a^3 + b^3 + c^3 - 3abc = 0$

6. Prove that the roots of the equation $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$ are real. Also, prove that the roots are equal if $a = b = c$.

Solution:

Here, given equation is
$(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$
or, $x^2 - bx - ax + ab + x^2 - cx - bx + bc + x^2 - ax - cx + ca = 0$
or, $3x^2 - 2(a + b + c).\ x + (ab + bc + ca) = 0$ ... ... (i)
Comparing equation (i) with $Ax^2 + Bx + C = 0$
$A = 3$, $B = -2(a + b + c)$, $c = ab + bc + ca$
Now, $B^2 - 4ac = 0$
or, $\{-2(a + b + c)\}^2 - 4\times3(ab + bc + ca) = 0$
or, $4(a^2 + b^2 + c^2 + ab + bc + ca) - 12(ab + bc + ca) = 0$
or, $(a^2 + b^2 + c^2 + ab + bc + ca - 3ab - 3bc - 3ca) = 0$
or, $(a^2 + b^2 + c^2 - 2ab - 2bc - 2ca) = 0$
or, $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$
Either, $a = b$, $b = c$, $c = a$ $\therefore$ $a = b = c$

7. If the roots of the equation $(a^2 + b^2)x^2 - 2(ac + bd)x + (c^2 + d^2) = 0$ are equal then prove that $\frac{a}{b} = \frac{c}{d}$

Solution:

Here, comparing $(a^2 + b^2)x^2 - 2(ac + bd)x + (c^2 + d^2) = 0$ with $Ax^2 + BX + C = 0$, we get,
$A = a^2 + b^2$ $\quad$ $B = -2(ac + bd)$ $\quad$ $C = c^2 + d^2$
The roots are equal if
$B^2 - 4AC = 0$
or, $\{-2(ac + bd)\}^2 - 4\times(a^2 + b^2)(c^2 + d^2) = 0$
or, $4(ac + bd)^2 - 4(a^2 + b^2)(c^2 + d^2) = 0$
or, $a^2c^2 + 2abcd + b^2d^2 - a^2c^2 - a^2d^2 - b^2c^2 - b^2d^2 = 0$
or, $-a^2d^2 + 2abcd - b^2c^2 = 0$
or, $-(ad - bc)^2 = 0$
or, $ad - bc = 0$
or, $ad = bc$
or, $\frac{a}{b} = \frac{c}{d}$ $\therefore$ $\frac{a}{b} = \frac{c}{d}$ proved.

8. If a, b, c are rational and $a + b + c = 0$, show that the roots of $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ are rational.

Solution:

Here, given equation is $(b + c - a).x^2 + (c + a - b).\ x + (a + b - c) = 0$ ... ... (i)
If $a + b + c = 0$
Comparing equation (i) with $Ax^2 + Bx + C = 0$
$A = (b + c - a)$ $\quad$ $B = (c + a - b)$ $\quad$ $C = (a + b - c)$
Now, $B^2 - 4AC = (c + a - b)^2 - 4(b + c - a).(a + b - c)$
$= (-b - b)^2 - 4(-2a).(-2c) = 4b^2 - 16ac = 4(b^2 - 4ac)$
$= 4\{b^2 - 4a(-a - b)\} = 4(b^2 + 4a^2 + 4ab)$
$= 4(b + 2a)^2 > 0$ and a perfect square
Hence, roots are rational.

9. Prove that the roots of the equation $(x - a)(x - b) = k^2$ are real for all value of k.

Solution:

Here, given equation is $(x – a)\ (x – b) = k^2$ ... ... (i)
or, $x^2 – bx – ax + ab – k^2 = 0$
or, $x^2 – (a + b) . x + (ab – k^2) = 0.$
Comparing equation (i) with $Ax^2 = BX + C = 0$, we get,
$A = 1, B = –(a + b), C = ab – k^2$
Now,
$B^2 – 4AC = \{–(a + b)\}^2 – 4\times1(ab – k^2) = a^2 + 2ab + b^2 – 4(ab – k^2)$
$= a^2 – 2ab + b^2 + 4ab + 4k^2 = a^2 – 2ab + b^2 + 4k^2 = (a – b)^2 + 4k^2 > 0$ for all k
Hence, roots are real.

10. Show that the roots of the quadratic equation $(b - c)x^2 + 2(c - a)x + (a - b) = 0$ are always real.

Solution:

Comparing equation $(b – c)\ x^2 + 2(c – a) . x + (a – b) = 0$ with $Ax^2 + BX + C = 0.$
$A = (b – c)$ $B = 2(c – a)$ $C = (a – b)$
Now,
$B^2 – 4AC = 4(c – a)^2 – 4(b – c)\ (a – b) = 4\{(c – a)^2 – (b – c)\ (a – b)^2\}$
$= 4\{c^2 + a^2 – 2ca – ab + b^2 + ca – bc\} = 4\{a^2 + b^2 + c^2 – ab – bc – ca\}$
$= 2\{2a^2 + 2b^2 + 2c – 2ab – 2bc – 2ca^2\}$
$= 2\{(a – b)^2 + (b – c)^2 + (c – a)^2\} > 0$
Hence, roots are always real.

11. Show that the roots of the equation $x^2+(2m - 1)x + m^2 = 0$ are real if $m \le \frac{1}{4}$

Solution:

Here, given equation is $x^2 + (2m – 1). x + m^2 = 0$ .. .. (i)
Comparing equation (i) with $ax^2 + bx + c = 0$, we get,
$a = 1, b = (2m – 1), c = m^2$
Now, $b^2 – 4ac$
or, $(2m – 1)^2 – 4\times1\times m^2$
or, $4m^2 – 4m + 1 – 4m^2$
or, $–(um – 1)$
or, $–(4m – 1)$
The roots will be real if $b^2 –4ac$
or, $– 4m + 1 \ge 0$
or, $1 \ge 4m \quad \therefore\ m \le \frac{1}{4}$

12. Show that the roots of the equation $x^2+ 4abx + (a^2+2b^2)^2 = 0$ are imaginary.

Solution:

Comparing $x^2 + 4abx + (a^2 + 2b^2)^2 = 0$ with $Ax^2 + Bx + C = 0$. We get,
$A = 1, B = 4ab, C = (a^2 + 2b^2)^2$
Now, $B^2 – 4AC = (4ab)^2 = 4\times1\times(a^2 + 2b^2)^2 = 16a^2b^2 – 4(a^4 + 2a^2b^2 + 4b^4)$
$= 4(4a^2b^2 – a^4 – 2a^2b^2 – 4b^4) = 4(–a^4 + 2a^2b^2 – 4b^4)$
$= –4(a^4 – 2a^2b^2 + 4b^4) = –4(a^2 – 2b^2)^2 < 0$
Hence, roots are imaginary.

13. If the roots of the equation $qx^2 + 2px + 2q = 0$ are real and unequal, prove that the roots of the equation $(p + q)x^2+ 2qx+ (p - q) = 0$ are imaginary.

Solution:

Here, $qx^2 + 2px + 2q = 0$

$b^2 - 4ac = (2p)^2 - 4.q.\ 2q] = 4p^2 - 8q^2 = 4(p^2 - 2q)^2 > 0$ ... ... (i)

$(p + q)\ x^2 + 2qx + (p - q) = 0$

$b^2 - 4ac = (2q)^2 - 4(p + q)\ .\ (p - q) = 4q^2 - 4(p^2 - q^2) = 4(q^2 - p^2 + q^2)$

$= -4(p^2 - 2q^2) < 0$ ... ... (ii)

The roots of second equation (ii) are imaginary if the roots of first equations are real.

14. If the roots of equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ be equal then show that a, b, c are in HP. (i.e.; b(a + c) = 2ac)

Solution:

Here, comparing $(ab - ac)\ x^2 + (bc - ab)\ x + ca - ab = 0$ with $Ax^2 + BX + C = 0$

$\therefore$ A = (ab – ac) $\therefore$ B = (bc – ab)

$\therefore$ C = (ca – ab)

Now, $B^2 - 4AC = 0$

or, $(bc - ab)^2 - 4(ab - ac)\ (ca - ab) = 0$

or, $b^2c^2 - 2ab^2c + a^2b^2 - 4\{a^2bc - a^2b^2 - a^2c^2 + a^2bc\} = 0$

or, $b^2c^2 - 2ab^2c + a^2b^2 - 4a^2bc + 4a^2b^2 + 4a^2c^2 - 4a^2bc$

or, $(bc + ab)^2 - 4abc\ (a + c) + (2ac)^2 = 0$

or, $(bc + ab - 2ac)^2 = 0$

or, $bc + ab - 2ac = 0$

or, $bc + ab = 2ac$

or, $b(a + c) = 2ac$

This shows that a, b, c are in H.P.

EXERCISE 5.2

1. Form a quadratic equation whose roots are

a. 3, –5 b. $2, \frac{1}{2}$ c. 2 – 3i, 2 + 3i

Solution:

a. Let, α and β be the two roots i.e. $\alpha = 3$, $\beta = -5$

Now, x^2 – (sum of roots) . x + product of roots = 0

or, $x^2 - (3 - 5)\ .\ x + 3 \times (-5) = 0$

or, $x^2 + 2x - 15 = 0$

Hence, The required quadratic equation is $x^2 + 2x - 15 = 0$.

b. Here, let, α and β be the two roots i.e. $\alpha = 2$, $\beta = \frac{1}{2}$.

Now, x^2 – (sum of roots) x + product of roots = 0

or, $x^2 - \left(2 + \frac{1}{2}\right) x + 2 \times \frac{1}{2} = 0$

or, $x^2 - \frac{5x}{2} + 1 = 0$

or, $2x^2 - 5x + 2 = 0$

Hence, the required equation is $2x^2 - 5x + 2 = 0$

c. Here, let α and β be the two roots i.e. $\alpha = 2 - 3i$, $\beta = 2 + 3i$

Now, x^2 – (sum of roots) x + product of roots = 0

or, $x^2 - (2 - 3i + 2 + 3i)\ x + (2 - 3i)\ (2 + 3i) = 0$

or, $x^2 - 4x + 4 + 9 = 0$

or, $x^2 - 4x + 13 = 0$

Hence, the required quadratic equation is $x^2 - 4x + 13 = 0$

2. Form a quadratic equation whose one root is

a. $3 - \sqrt{5}$ b. $2i$ c. $i\sqrt{3} + 1$

d. $\dfrac{1}{3 + \sqrt{5}}$ e. $\dfrac{1}{3i}$

Solution:

a. Here, one root (α) = $3 - \sqrt{5}$

Other root (β) = $3 + \sqrt{5}$

Quadratic equation is x^2 – (sum of roots) x + product of roots = 0

or, $x^2 - (3 - \sqrt{5} + 3 + \sqrt{5})\, x + 9 - 5 = 0$

or, $x^2 - 6x + 4 = 0$

$\therefore$ $x^2 - 6x + 4 = 0$

b. Here, one root (α) = $-2i$

Other root (β) = $2i$

The required equation is $x^2 - (\alpha + \beta)\, x + \alpha\beta = 0$

or, $x^2 - (2i - 2i) \,.\, x - 4i^2 = 0$

or, $x^2 + 4 = 0$

c. Here, one root (α) = $1 + \sqrt{3}i$

Other root (β) = $1 - \sqrt{3}i$

The required quadratic equation is

$x^2 - (\alpha + \beta)\, x + \alpha\beta = 0$

or, $x^2 - (1 + \sqrt{3}i + 1 - \sqrt{3}i)\, x + 1^2 + (\sqrt{3})^2 = 0$

or, $x^2 - 2x + 1 + 3 = 0$

or, $x^2 - 2x + 4 = 0$

d. Here, One root (α) = $\dfrac{1}{3 + \sqrt{5}}$

Other root (β) = $\dfrac{1}{3 - \sqrt{5}}$

The required quadratic equation is,

$x^2 - (\alpha + \beta)\, x + \alpha\beta = 0$

or, $x^2 - \left(\dfrac{1}{3 + \sqrt{5}} + \dfrac{1}{(3 - \sqrt{5})}\right) x + \dfrac{1}{3 + \sqrt{5}}\,\dfrac{1}{3 - \sqrt{5}} = 0$

or, $x^2 - \dfrac{(3 - \sqrt{5} + 3 + \sqrt{5})}{9 - 5} \,.\, x + \dfrac{1}{9 - 5} = 0$

or, $x^2 - \dfrac{6x}{4} + \dfrac{1}{4} = 0$

or, $4x^2 - 6x + 1 = 0$

e. Here,

One root (α) = $\dfrac{1}{3i} = \dfrac{1}{3i} \times \dfrac{i}{i} = \dfrac{i}{3i^2} = -\dfrac{i}{3}$

Other root (β) = $\dfrac{i}{3}$

The required quadratic equation is,

$x^2 - (\alpha + \beta)\, x + \alpha\beta = 0$

or, $x^2 - \left(\dfrac{-i}{3} + \dfrac{i}{3}\right) x + \left(\dfrac{-i}{3} \cdot \dfrac{i}{3}\right) = 0$

$\Rightarrow$ $X^2 - \dfrac{i^2}{9} = 0$

$\Rightarrow\ x^2 + \frac{1}{9} = 0$

$\therefore\ 9x^2 + 1 = 0$

3. Form a quadric equation whose roots are
   a. Square the roots of $4x^2 + 8x - 5 = 0$
   b. Reciprocal of the roots of $3x^2 - 5x - 2 = 0$
   c. m–times the roots of $x^2 - bx + c = 0$
   d. greater by h than the roots of $x^2 - px + q = 0$.

Solution:

a. Here, let α and β be the roots of $4x^2 + 8x - 5 = 0$

$\alpha + \beta = \frac{-8}{4} = -2,\ \alpha\beta = \frac{-5}{4}$

and α^2 and β^2 be the roots of required equation.

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 4 - 2.\frac{-5}{4} = \frac{13}{2}$ $\qquad \alpha^2\beta^2 = (\alpha\beta)^2 = \left(\frac{-5}{4}\right)^2 = \frac{25}{16}$

The required quadratic equation is, $x^2 - \frac{13}{2}.x + \frac{25}{16} = 0$

or, $16x^2 - 104x + 25 = 0$

b. Here, let α and β be the two roots of $3x^2 - 5x - 2 = 0$

$\alpha + \beta = \frac{5}{3},\ \alpha\beta = \frac{-2}{3}$

and $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ be the two roots of required equation,

$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{5/3}{-2/3} = -\frac{-5}{2}$

and $\frac{1}{\alpha}.\frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{(-2/3)} = -\frac{3}{2}$

The required quadratic equation is, $x^2 - \frac{5}{2}.x - \frac{3}{2} = 0$

or, $2x^2 + 5x - 3 = 0$

c. Here, let, α and β be the two roots of $x^2 - bx + c = 0$

$\alpha + \beta = \frac{b}{1},\ \alpha\beta = c$

or, $\alpha + \beta = b$

and $m\alpha$ and $m\beta$ be the two roots of required equation,

$m\alpha + m\beta = m(\alpha + \beta) = mb,\ m\alpha\ .\ m\beta = m^2\alpha\beta = m^2 \times c$

The required quadratic equation is $x^2 - mbx + m^2c = 0$

d. Here, let α and β be the roots of $x^2 - px + q = 0$

Then, sum of roots $= \alpha + \beta = \frac{-p}{-1} = p$ Product of roots $= \alpha\beta = \frac{q}{1} = q$

Since, The roots of the required equation are by h, so

Sum of roots $= (\alpha + h) + \beta + h = (\alpha + \beta) + 2h = p + 2h$

Product of roots $= (\alpha + h)(\beta + h) = \alpha\beta + (\alpha + \beta)h + h^2 = q + ph + h^2$

The required equation is $x^2 - (p + 2h).x + (q + ph + h^2) = 0$

4. If one root of the equation $ax^2 + bx + c = 0$ is thrice the other, then show that $3b^2 = 16ac$.

Solution:

a. Here, let α and 3α be the two roots of $ax^2 + bx + c = 0$

Sum of roots, $\alpha + 3\alpha = \frac{b}{a}$

or, $4\alpha = \frac{b}{a}$ or, $\alpha = \frac{b}{4a}$

Production of roots, $\alpha \cdot 3\alpha = \frac{c}{a}$

or, $3\alpha^2 = \frac{c}{a}$

or, $\frac{3b^2}{16a^2} = \frac{c}{a}$

or, $3b^2 = 16ac$

$\therefore$ $3b^2 = 16ac$ proved.

5. If the roots of the equation $ax^2 + bx + b = 0$ are in the ratio p:q, show that

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{b}{a}} = 0$$

Solution:

Here, α and β be the roots of $ax^2 + bx + b = 0$ and $\frac{\alpha}{\beta} = \frac{p}{q}$

Then, $\alpha + \beta = \frac{-b}{a}$

$\alpha\beta = \frac{b}{a}$

Now,

$$\text{LHS} = \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{b}{a}}$$

$$= \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{b}{a}}$$

$$= \frac{\alpha + \beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{b}{a}}$$

$$= \frac{-\frac{b}{a}}{\sqrt{\frac{b}{a}}} + \sqrt{\frac{b}{a}} = -\sqrt{\frac{b}{a}} + \sqrt{\frac{b}{a}} = 0$$

6. If α, β be the roots of the equation $px^2 + qx + q = 0$, show that

a. $\frac{1}{\alpha} + \frac{1}{\beta} + 1 = 0$

b. $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}} = 0$

Solution:

a. Here,

If α, β be the roots of $px^2 + qx + q = 0$

Then, $\alpha + \beta = \frac{-q}{p}$

$\alpha\beta = \frac{q}{p}$

b. Here, $\alpha + \beta = -\frac{q}{p}$

$\alpha\beta = \frac{q}{p}$

Now, $\text{LHS} = \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}}$

L.H.S. $\frac{1}{\alpha} + \frac{1}{\beta} + 1 = \frac{\beta + \alpha}{\alpha\beta} + 1$

$= \frac{\frac{-q}{p}}{\frac{q}{p}} + 1 = -1 + 1 = 0$ R.H.S.

$= \frac{\alpha + \beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{q}{p}}$

$= \frac{-\frac{q}{p}}{\sqrt{\frac{q}{p}}} + \sqrt{\frac{q}{p}}$

$= -\sqrt{\frac{q}{p}} + \sqrt{\frac{q}{p}} = 0$

7. If one root of the equation $ax^2 + bx + c = 0$ is square of other, prove that $b^3 + a^2c + ac^2 = 3abc$

Solution:

Let, α be the one root of $ax^2 + bx + c = 0$ then other root be α^2.

$\alpha + \alpha^2 = \frac{-b}{a}$ .... (i)

or, $\alpha \,.\, \alpha^2 = \frac{c}{a}$

or, $\alpha^3 = \frac{c}{a}$ .... (ii)

Cubing on both side of equation (i)

$(\alpha + \alpha^2)^3 = \left(\frac{-b}{a}\right)^3$

or, $\alpha^3 + (\alpha^2)^3 + 3\alpha \,.\, \alpha^2 (\alpha + \alpha^2) = \frac{-b^3}{a^3}$

or, $\alpha^3 + (\alpha^3)^2 + 3\alpha^3 (\alpha + \alpha^2) = \frac{-b^3}{a^3}$

$\Rightarrow \frac{c}{a} + \frac{c^2}{a^2} + 3\frac{c}{a} \times -\frac{b}{a} = -\frac{b^3}{a^3}$

$\Rightarrow a^2c + ac^2 - 3abc = -b^3$

$\therefore b^3 + a^2c + ac^2 - 3abc = 0$

8. Let α, β be the roots of the equation $x^2 + px + q = 0$, then find the equation whose roots are

a. $\alpha\beta^{-1}$, $\alpha^{-1}\beta$ b. $(\alpha - \beta)^2$, $(\alpha+\beta)^2$ c. $\alpha^2\beta^{-1}$, $\beta^2\alpha^{-1}$

Solution:

Let, α and β be the two roots of $x^2 + px + q = 0$ $\alpha + \beta = -p$, $\alpha\beta = q$

a. The roots of required equation are
$\alpha\beta^{-1}$ and $\beta\alpha^{-1}$

Sum of roots $= \alpha\beta^{-1} + \beta\alpha^{-1} = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{p^2 - 2q}{q}$

Product of roots $= \alpha\beta^{-1} \times \beta\alpha^{-1} = \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$

The required equation is

$x^2 -$ (sum of roots) . x + product of roots = 0

or, $x^2 - \frac{(p^2 - 2q)}{q} \,.\, x + 1 = 0$

or, $qx^2 - (p^2 - 2q)\,x + q = 0$

b. Here, $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$ are the roots of required equation,
Sum of roots $= (\alpha - \beta)^2 + (\alpha + \beta)^2$

$= (\alpha + \beta)^2 - 4\alpha\beta + (\alpha + \beta)^2 = 2(\alpha + \beta)^2 - 4\alpha\beta = 2p^2 - 4q$

Product of roots $= (\alpha - \beta)^2 \,.\, (\alpha + \beta)^2 = \{(\alpha + \beta)^2 - 4\alpha\beta\}\,(\alpha + \beta)^2 = (p^2 - 4q)\,.\,p^2$

The required equation is

$x^2 -$ (Sum of roots) $x +$ product of roots $= 0$

or, $x^2 - (2p^2 - 4q)\,.\,x + (p^4 - 4p^2q)\,.\,(2p^2 - 4q) = 0$

or, $x^2 - (2p^2 - 4q)\,.\,x + p^2\,(p^2 - 4q) = 0$

or, $x^2 - 2(p^2 - 2q)\,.\,x + p^2\,(p^2 - 4q) = 0$

c. $\alpha^2\beta^{-1}$and $\beta^2\alpha^{-1}$ be the two roots of required equation,

Sum of roots $= \alpha^2\beta^{-1} + \beta^2\alpha^{-1}$

$$= \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{(\alpha + \beta)\,(\alpha^2 + \beta^2 - \alpha\beta)}{\alpha\beta} = \frac{-p\,\{(\alpha + \beta)^2 - 3\alpha\beta}{q}$$

$$= \frac{-p\,\{p^2 - 3q\}}{q} = \frac{-p^3 + 3pq}{q}$$

Product of roots $= \dfrac{\alpha^2}{\beta} \times \dfrac{\beta^2}{\alpha} = \dfrac{(\alpha\beta)^2}{\alpha\beta} = \alpha\beta = q$

The required equation is,

$x^2 -$ (sum of roots) $x +$ product of roots $= 0$

or, $x^2 + \dfrac{P(p^2 - 3q)}{q}\,x + q = 0$

or, $qx^2 + p^3 - 3pqx + q^2 = 0$

9. Find the value of k so that the equation
   a. $2x^2 + kx - 15 = 0$ has one root 3.
   b. $3x^2 + kx - 2 = 0$ has two roots whose sum is 6
   c. $2x^2 + (4 - k)x - 17 = 0$ has roots equal but opposite in sign.
   d. $7kx^2 - 12x - 21 = 0$ has reciprocal roots.
   e. $x^2 - kx + 1 = 0$ has a root square of another.

Solution:

a. Let, the other root be α then,

$\alpha\,.\,3 =$ Product of roots $= \dfrac{-15}{2}$

or, $\alpha = -\dfrac{5}{2}$

Sum of roots $= \dfrac{-k}{2}$

or, $\alpha + 3 = \dfrac{-k}{2}$

or, $\dfrac{-k}{2} + 3 = \dfrac{-k}{2}$

or, $\dfrac{-5 + 6}{2} = \dfrac{-k}{2}$

or, $k = -1 \;\therefore\; k = -1$

b. Given, equation is $3x^2 + kx - 2 = 0$

Sum of roots $= \dfrac{-k}{3}$

or, $6 = \dfrac{-k}{3}$

$\therefore\; k = -18$

c. Given,

equation is $2x^2 + (4 - k).\,x - 17 = 0$

If one root $= \alpha$ then other root $= -\alpha$

So that sum of the roots $= 0$

Sum of roots $= -\dfrac{4 - k}{2}$

d. Let, one root $= \alpha$

Another root $= \dfrac{1}{\alpha}$

Product of roots $= \dfrac{c}{a}$

or, $\alpha\,.\,\dfrac{1}{\alpha} = \dfrac{-21}{7k}$

or, $0 = -\frac{4-k}{2}$

or, $0 = -4 - k$

$\therefore$ $k = 4$

or, $7k = -21$

$\therefore$ $k = -3$

e. Let, α and α^2 be the two roots of $x^2 - kx + 1 = 0$

Sum of roots $= \frac{k}{1}$

or, $\alpha + \alpha^2 = k$

Products of roots $= \frac{1}{1}$

or, $\alpha . \alpha^2 = 1$

or, $\alpha^3 = 1$, or, $\alpha = 1$, i.e. $1 \times 1^2 = k \Rightarrow k = 2$

10. The sum of the roots of a quadratic equation is 1 and sum of their square is 13, find the equation.

Solution:

Here, if α and β be the two roots of equations, then,

$\alpha + \beta = 1$, and $\alpha^2 + \beta^2 = 13$

or, $(\alpha + \beta)^2 - 2\alpha\beta = 13$

or, $1 - 2\alpha\beta = 13$

or, $2\alpha\beta = -12$

or, $\alpha\beta = -6$

Now, Sum of roots $= \alpha + \beta = 1$

Products of roots $= -6$

The required equation is

$x^2 -$ (sum of roots) . x + product of roots = 0

or, $x^2 - 1.x - 6 = 0$

$\therefore$ $x^2 - x - 6 = 0$

11. If the roots of the equation $x^2 + px + q = 0$ are in the same ratio as those of the equation $x^2 + lx + m = 0$. Show that $p^2m = l^2q$.

Solution:

Let, α and β be the equation of $x^2 + px + q = 0$

$\alpha + \beta = -p$ $\alpha\beta = q$

If the roots of $x^2 + lx + m = 0$ are in same ratio. Let $k\alpha$ and $k\beta$ be the roots of $x^2 + \ell x + m = 0$

Then, $k\alpha + k\beta = -\ell$, $\Rightarrow k = \frac{-\ell}{-p} = \frac{\ell}{p}$

$k\alpha . k\beta = m$

or, $k^2 = \frac{m}{q}$

Now, $\frac{\ell^2}{p^2} = \frac{m}{q}$

or, $p^2m = \ell^2q$

$\therefore$ $p^2m = \ell^2q$ proved.

12. If the ratio of roots of the equations $lx^2 + mx + n = 0$ be equal to that of the roots

$l_1x^2 + m_1x + n_1 = 0$ prove that $\frac{m^2}{m_1^2} = \frac{l\,n}{l_1n_1}$.

Solution:

Let, α and β be the roots of $\ell x^2 + mx + n = 0$

Then, $\alpha + \beta = \frac{-m}{\ell}$

$\alpha\beta = \frac{n}{\ell}$

Again, Let, α' and β' be the roots of $\ell_1 x^2 + m_1 x + n_1 = 0$

Then, $\alpha'\beta' = \frac{-m_1}{\ell_1}$ $\qquad$ $\alpha'\beta' = \frac{n_1}{\ell_1}$

By the question, $\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'}$

By componendo and dividendo,

$\frac{\alpha + \beta}{\alpha - \beta} = \frac{\alpha' + \beta'}{\alpha' - \beta'}$

or, $\frac{(\alpha + \beta)^2}{(\alpha - \beta)^2} = \frac{(\alpha' + \beta')^2}{(\alpha' - \beta')^2}$

or, $\frac{(\alpha + \beta)^2}{(\alpha + \beta)^2 - (\alpha - \beta)^2} = \frac{(\alpha' + \beta')^2}{(\alpha' + \beta')^2 - (\alpha' - \beta')^2}$

or, $\frac{(\alpha + \beta)^2}{4\alpha\beta} = \frac{(\alpha' + \beta')^2}{4\alpha'\beta'}$

or, $\frac{\left(\frac{-m}{\ell}\right)^2}{4\frac{n}{\ell}} = \frac{\left(\frac{-m_1}{\ell_1}\right)^2}{\frac{n_1}{\ell_1}}$

or, $\frac{m^2}{4\ell n} = \frac{m1^2}{4\ell_1 n_1}$

or, $\frac{m^2}{ln} = \frac{m\ell^2}{\ell_1 n_1}$, or, $\frac{m^2}{m\ell^2} = \frac{\ell n}{\ell_1 n_1}$ proved.

EXERCISE 5.3

1. Show that the following pair of equations has a common root.
   a. $2x^2 + x - 3 = 0$, $\qquad$ $3x^2 - 4x + 1 = 0$
   b. $3x^2 - 8x + 4 = 0$, $\qquad$ $4x^2 - 7x - 2 = 0$

Solution:

a. Given, equations are

$2x^2 + x - 3 = 0$ and $3x^2 - 4x + 1 = 0$

Writing the coefficients of order and repeating the first one.

$$\begin{array}{cccc} 2 & 1 & -3 & 2 \\ 3 & -4 & 1 & 3 \end{array}$$

The left hand expression of the condition

$(2 \times (-4) - 3\times1) . (1\times1 - (-4) \times (-3)) = (-8 - 3) . (1 - 12) = -11 -11 = 121$

The right hand expression of the condition,

$\{(-3\times3) - 1\times2\}^2 = (-9 - 2) = (-11)^2 = 121$

Since, two results are equal, they have common root.

b. Here, given equations are $3x^2 - 8x + 4 = 0$ and $4x^2 - 7x - 2 = 0$

Writing the coefficients of order and repeating the first one

$$\begin{array}{cccc} 3 & -8 & 4 & 3 \\ 4 & -7 & -2 & 4 \end{array}$$

The left hand expression of the condition,

$= (3\times(-7) - 4\times(-8) . (-8) . (-2) - (-7) \times 4 = (-21 + 32) . (16 + 28) = 11.44 = 484$

The right hand expression of the condition

$(4\times4 - (-2) \times 3)^2 = (16 + 6)^2 = (22)^2 = 484$

Since, two results are equal, they have common root.

2. Determine the value of m for which the equations

$3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.

Solution:

Here, given equations are

$3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$

Writing the coefficients of order and repeating the first one

$$\begin{array}{cccc} 3 & 4m & 2 & 3 \\ 2 & 3 & -2 & 2 \end{array}$$

The left hand expression of the condition,

$= (3\times3 - 2\times4m) . (4m . (-2)) - (3\times2) = (9 - 8m) . (-8m - 6)$

$= -72m - 54 + 64m^2 + 48m = -24m + 64m^2 - 54$

The right hand expression of the condition,

$(2\times2 - (-2) \times 3)^2 = (4 + 6)^2 = 100$

$\therefore\ 64m^2 - 24m - 54 = 100$

or, $64m^2 - 24m - 154 = 0$

or, $32m^2 - 12m - 77 = 0$ ... ... (i)

or, Comparing equation (i) with $ax^2 + bx + c = 0$

$\therefore\ a = 32, b = -12, c = -77$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{12 \pm \sqrt{144 - 4 \times 32 \times (-77)}}{2\times32} = \frac{12 \pm 100}{64}$$

Taking +ve,

$x = \frac{12 + 100}{64} = \frac{7}{4}$

Taking –ve,

$x = \frac{12 - 100}{64} = \frac{-11}{8}$

Here, x is the value of m

So, $m = \frac{7}{4}$ and $\frac{-11}{8}$

3. Find the value of p so that each pair of equations may have a common root

a. $4x^2 + px - 12 = 0, 4x^2 + 3px - 4 = 0$ b. $2x^2 + px - 1 = 0, 3x^2 - 2x - 5 = 0$

Solution:

a. Here, given equation are $4x^2 + px - 12 = 0$ and $4x^2 + 3px - 4 = 0$

Writing the coefficients of order and repeating the first one.

$$\begin{array}{cccc} 4 & p & -12 & 4 \\ 4 & 3p & -4 & 4 \end{array}$$

The left hand expression of the condition,

$= (4 \times 3p - 4p) . (-4p + 36p) = (12p - 4p) . (32p) = 8p . 32p = 256p^2$

The right hand expression of the condition,

$= (-12\times4 - (-4) \times 4)^2 = (-48 + 16)^2 = (32)^2 = 1024$

Now, $256p^2 = 1024$

or, $p^2 = 4$

$\therefore\ p = \pm2$

b. Here, Given equations are

$2x^2 + px - 1 = 0$ and $3x^2 - 2x - 5 = 0$

Writing the coefficients of order and repeating the first one,

$$\begin{array}{cccc} 2 & p & -1 & 2 \\ 3 & -2 & -5 & 3 \end{array}$$

The left hand expression of the condition,

$= (2\times(-2) - 3p) . (-5p - 2) = (-4 - 3p) . (-5p - 2)$

$= 20p + 8 + 15p^2 + 6p = 26p + 8 + 15p^2$

The right hand expression of the condition,
$= ((-1) \times 3 - (-5) \times 2)^2 = (-3 + 10)^2 = 49$
Now,
$15p^2 + 26p + 8 = 49$
or, $15p^2 + 26p - 41 = 0$
or, $15p^2 + 41p - 15p - 41 = 0$
or, $p(15p + 41) - 1p(15p + 41) = 0$
or, $(15p + 41)(p - 1) = 0$
either, $p = 1$,
or, $p = \dfrac{-41}{15}$

4. If the quadratic equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root show that it must be either $\dfrac{pq' - p'q}{q - q'}$ or $\dfrac{q - q'}{p' - p}$

Solution:
Let, α be the common root of the given equations,
$\alpha^2 + p\alpha + q = 0$
$\alpha^2 + p'\alpha + q' = 0$
By using cross multiplication method;
$$\frac{\alpha^2}{pq' - qp'} = \frac{\alpha}{q - q'} = \frac{1}{p' - p}$$
$\therefore \quad \alpha = \dfrac{pq' - p'q}{q - q'}$, $\alpha = \dfrac{q - q'}{p' - p}$
$\therefore$ The common root is $\dfrac{pq' - p'q}{q - q'}$ or $\dfrac{q - q'}{p' - p}$

5. If the equations $x^2 + qx + pr = 0$ and $x^2 + rx + pq = 0$ have a common root, prove that $p + q + r = 0$

Solution:
Let, α be the common root of the given equations,
$\alpha^2 + q\alpha + pr = 0$
$\alpha^2 + r\alpha + pq = 0$
By the rule of cross–multiplication method'
$$\frac{\alpha^2}{pq^2 - pr^2} = \frac{\alpha}{pr - pq} = \frac{1}{r - q}$$
$\alpha = \dfrac{pq^2 - pr^2}{pr - pq} = \dfrac{p(q - r)(q + r)}{p(r - q)} = -q - r$ $\qquad \alpha = \dfrac{p(r - q)}{(r - q)} = p$
Now,
$-q - r = p$
or, $p = -q - r$
or, $p + q + r = 0$
$\therefore \quad p + q + r = 0$

6. If $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have a common root show that their other roots will satisfy the equation $x^2 + ax + bc = 0$

Solution:
Let, α be the common root of the given equations,
$\alpha^2 + b\alpha + ca = 0$
$\alpha^2 + c\alpha + ab = 0$
By the rule of cross multiplication method,

$$\frac{\alpha^2}{ab^2 - ac^2} = \frac{\alpha}{ca - ab} = \frac{1}{c - b}$$

or, $\dfrac{\alpha^2}{a(b + c)\,(b - c)} = \dfrac{\alpha}{-a(b - c)} = \dfrac{1}{-(b - c)}$

$\therefore \quad \alpha = \dfrac{-a(b - c)}{-(b - c)}$

Also, $\alpha = \dfrac{a(b + c)\,(b - c)}{-a\,(b - c)} = -(b + c)$

$\therefore \quad a = -(b + c)$

or, $a + b + c = 0$

If β be the other roots of $x^2 + bx + ca = 0$, then $\alpha\beta = \dfrac{ca}{1}$.

or, $a\beta = ca$

$\therefore \quad \beta = c$

Again, If γ be the other root of $x^2 + cx + ab = 0$, then $\alpha \,.\, \gamma = \dfrac{ab}{1} = ab$

or, $a \,.\, \gamma = \dfrac{1}{ab}$

$\therefore \quad \gamma = b$

The quadratic equation whose roots are β and γ is

$x2 - (\beta + \gamma)\, x + \beta\gamma = 0$

or, $x^2 - (c + b) \,.\, x + cb = 0$

or, $x^2 - (-a) \,.\, x + bc = 0 \quad [\because a + b + c = 0]$

or, $x^2 + ax + bc = 0$

$\therefore \quad x^2 + ax + bc = 0$

7. If $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0$ have a common root then show that $a + 4b + 4c = 0$

Solution:

Let, α be the common root of the equation,

$\alpha^2 + 2b\alpha + c = 0$

$a\alpha^2 + 2c\alpha + b = 0$

By the rule of cross multiplication method,

$$\frac{\alpha 2}{2b^2 - 2c^2} = \frac{\alpha}{ac - ab} = \frac{1}{2ac - 2ab}$$

$\alpha = \dfrac{2(b - c)\,(b + c)}{a(c - b)} = \dfrac{2(-b - c)}{a \times 1} = \dfrac{2(-b - c)}{a}, \quad \alpha = \dfrac{a(c - b)}{2a(c - b)} = \dfrac{1}{2}$

Now, $\dfrac{-2b - 2c}{a} = \dfrac{1}{2}$

or, $-4b - 4c = a$

or, $a = -4b - 4c$

or, $a + 4b + 4c = 0$

8. If $x^2 + px + q = 0$ and $x^2 + qx + p = 0$ have a common root, prove that either $p = q$ or $p + q + 1 = 0$.

Solution:

Here, α be the common roots of the given equations, then

$\alpha^2 + p\alpha + q = 0$

$\alpha^2 + q\alpha + p = 0$

By the rule of cross multiplication,

$$\frac{\alpha^2}{p^2 - q^2} = \frac{\alpha}{q - p} = \frac{1}{q - p}$$

$\alpha = \dfrac{q - p}{q - p}$ and $\alpha = \dfrac{p^2 - q^2}{q - p}$

$\therefore \quad \dfrac{p^2 - q^2}{p - p} = \dfrac{q - p}{q - p}$

or, $p^2 - q^2 = -(p - q)$

or, $(p + q)(p - q) + (p - q) = 0$

or, $(p - q)(p + q + 1) = 0$

either,

$p - q = 0 \therefore p = q$

$p + q + 1 = 0$ P

CHAPTER 6

MATHEMATICAL INDUCTION

EXERCISE 6.1

1. Find the n^{th} term and then the sum of the first n terms of each of the following series.

a. 1.3 + 2.4 + 3.5 + …
b. 1 + 4 + 9 + 16 + …
c. 1.3 + 3.5 + 5.7 + …
d. 1.2.3 + 2.3.4 + 3.4.5 + …
e. 1 + (1 + 2) + (1 + 2 + 3) + …

Solution:

a. Here,

Now, n^{th} term of given series

t_n = (n^{th} term of 1, 2, 3, ...) × (n^{th} term of 3, 4, 5, ...)

$= [1 + (n – 1). 1] \times [3 + (n–1).1] = n \times (n + 2) = n(n + 2)$

$\therefore\ t_n = n(n + 2)$

Again, the sum of first n terms of the given series

$$s_n = \sum t_n = \sum n(n + 2) = \sum(n^2 + 2n) = \sum n^2 + 2\sum n = \frac{n(n + 1)(2n + 1)}{6} + \frac{2n(n + 1)}{2}$$

$$= \frac{n(n + 1)(2n + 1 + 6)}{6} = \frac{n(n + 1)(2n + 7)}{6}$$

b. Here, $1 + 4 + 9 + 16 + ... = 1^2 + 2^2 + 3^2 + 4^2 + ...$

n^{th} term of given series

$t_n = [a + (n–1)d]^2 = [1 + (n – 1).1]^2 = n^2$

Again, let the sum of n natural number

$$s_n = \sum t_n = \sum n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

c. Here, n^{th} term of given series

t_n = (nth term of 1, 3, 5, ...) × (n^{th} term of 3, 5, 7, ...)

$= [1 + (n – 1).2] + p3 + (n – 1).2] = (2n – 1)(2n + 1) = 4n^2 – 1$

$\therefore\ t_n = 4n^2 – 1$

Again, the sum on of n natural number is

$s_n = \sum t_n = \sum(n^2 – 1) = 4\sum n^2 – \sum 1$

$$= \frac{4n(n + 1)(2n + 1)}{6} – n = n\left[\frac{2n(n + 1)(2n + 1) – 3}{3}\right] = \frac{n}{3}[4n^2 + 6n – 1]$$

d. Here, n^{th} term of given series

t_n = (n^{th} term of 1, 2, 3, 4, ...)×(n^{th} term of 2, 3, 4, 5, ...)×(n^{th} term of 3, 4, 5, ...)

$= [1 + (n – 1).1] \times [2 + (n–1).1] \times [3 + (n–1).1]$

$= n(n + 1)(n + 2) = n(n^2 + 2n + n + 2) = n^3 + 3n^2 + 2n$

Again, the sum of first n natural number

$s_n = \sum t_n = \sum(n^3 + 3n^2 + 2n)$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{3n(n + 1)(2n + 1)}{6} + \frac{2n(n + 1)}{2} = \frac{n^2(n + 1)^2}{4} + \frac{n(n + 1)(2n + 1)}{2} + n(n + 1)$$

$$= \frac{n^2(n + 1)^2 + 2n(n + 1)(2n + 1) + 4n(n + 1)}{4} = \frac{n}{4}[(n + 1)(n + 2)(n + 3)]$$

e. Here, 1 + (1 + 2) + (1 + 2 + 3) + ...

The n^{th} term is $t_n = 1 + 2 + 3 + ... = \frac{n(n + 1)}{2} = \frac{n^2}{2} + \frac{n}{2}$

Now, sum of n term is

$$S_n = \frac{1}{2}(\Sigma n^2 + \Sigma n) = \frac{1}{2}\left\{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right\} = \frac{n(n+1)(n+2)}{6}$$

2. Sum to n terms of the following series
   a. $(x + a) + (x^2 + 2a) + (x^3 + 3a) + \ldots$
   b. $5 + 55 + 555 + \ldots$ to n terms.
   c. $0.3 + 0.33 + 0.333 + \ldots$ to n terms.
   d. $1\times n + 2 \times (n - 1) + 3 \times (n - 2) + \ldots$.
   e. $1 + 3 + 6 + 10 + \ldots$
   f. $3 + 6 + 11 + 18 + \ldots$

Solution:

a. $(x + a) + (x^2 + 2a) + (x^3 + 3a) + \ldots$

Here,

Let $s_n = (x + a) + (x^2 + 2a) + (x^3 + 3a) + \ldots$ to n term

$= (x + x^2 + x^3 + \ldots + x^n) + (a + 2a + 3a + \ldots na)$

$= \frac{n(x^n - 1)}{x -} + a(1 + 2 + 3 + \ldots \text{th}) = \frac{x(x^n - 1)}{x - 1} + \frac{a.n(n+1)}{2}$

b. Let $s_n = 5 + 55 + 555 + \ldots$ to n

$= 5(1 + 11 + 111 + \ldots \text{ to } n) = \frac{5}{9}(9 + 99 + 999 + \ldots \text{ to } n)$

$= \frac{5}{9}[(10 - 1) + (100 - 1) + (1000 - 1) + \ldots \text{ to } n]$

$= \frac{5}{9}[(10 + 100 + 1000 + \ldots \text{ to } n) - (1 + 1 + 1 \ldots \text{ to } n)] = \frac{5}{9}\left[\frac{10(10^n - 1)}{10 - 1} - n\right]$

$S_n = \frac{5}{9}\left[\frac{10}{9}(10^n - 1) - n\right]$

c. Here, Let $s_n = 0.3 + 0.33 + 0.333 + \ldots$ to n

$$= \frac{3}{10} + \frac{33}{100} + \frac{333}{1000} + \ldots \text{ to } n) = 3\left(\frac{1}{10} + \frac{11}{100} + \frac{111}{1000} + \ldots \text{ to } n\right)$$

$$= \frac{3}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots \text{ to } n\right]$$

$$= \frac{1}{3}\left[\frac{(10-1)}{10} + \frac{(100-1)}{100} + \frac{(1000-1)}{1000} + \ldots \text{ to } n\right]$$

$$= \frac{1}{3}\left[(1 + 1 + 1 \ldots n) - \left(\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots n\right)\right]$$

$$= \frac{1}{3}\left[n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}}\right] = \frac{1}{3}\left[n - \frac{10}{90}\left(1 - \frac{1}{10^n}\right)\right]$$

$$= \frac{1}{3}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right] = \frac{n}{3} - \frac{1}{27}\left(1 - \frac{1}{10^n}\right)$$

d. Here, r^{th} term of 1, 2, 3, ... = r and r^{th} term of n, n – 1, n – 2, ...

$= n - (r - 1) = n - r + 1$

So, the r^{th} term of the series is $r(n - r + 1)$

$\therefore \quad t_r = nr - r^2 + r$

So, sum $S_n = \sum_{r=1}^{n} tr = n\Sigma r - \Sigma r^2 + \Sigma r = \frac{n.n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$$= \frac{n(n+1)}{2}.\left\{n - \frac{2n+1}{3} + 1\right\} = \frac{n(n+1)}{2}.\frac{3n - 2n - 1 + 3}{3} = \frac{n(n+1)(n+2)}{6}$$

e. Let t_n be the n^{th} term and S_n the sum of the first n terms of 1 + 3 + 6 + 10 + ...
Then, $S_n = 1 + 3 + 6 + 10 + \dots\dots\dots + t_{n-1} + t_n$
Also, $S_n = \quad 1 + 3 + 6 + \dots\dots\dots + t_{n-2} + t_{n-1} + t_n$
$-\quad -\quad -\quad -\quad -\quad -\quad -\quad -$

Subtraction yields, $0 = 1 + 2 + 3 + \dots\dots\dots + (t_n - t_{n-1}) - t_n$

or, $t_n = 1 + 2 + 3 + \dots\dots\dots$ to n terms $= \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$

Hence, $S_n = \frac{1}{2}\Sigma n^2 + \frac{1}{2}\Sigma n = \frac{1}{2}(1^2 + 2^2 + 3^3 + \dots\dots + n^2) + \frac{1}{2}(1 + 2 + 3 + \dots\dots + n)$

$$= \frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2}$$

$$= \frac{1}{4}n(n+1)\left\{\frac{(2n+1)}{3} + 1\right\} = \frac{1}{4}n(n+1)\frac{(2n+1+3)}{3}$$

$$= \frac{n(n+1)(n+2)}{6}$$

f. We have,
$3 + 6 + 11 + 18 + \dots = (2^2 - 1) + (3^2 - 3) + (4^2 - 5) + (5^2 - 7) + \dots$
$= (2^2 + 3^2 + 4^2 + 5^2 +$ to n terms$) - (1 + 3 + 5 + 7 + \dots$ to n terms$)$
$= (n+1)^2 - (2n-1) = n^2 + 2n + 1 - 2n + 1$
$\therefore \quad t_n = n^2 + 2$
Now, $\Sigma t_n = \Sigma n^2 + \Sigma 2$

$$s_n = \frac{n(n+1)(2n+1)}{6} + 2n = \frac{(n^2+n)(2n+1) + 12n}{6} = \frac{2n^3 + n^2 + 2n^2 + n + 12n}{6}$$

$$= \frac{2n^3 + 3n^2 + 13n}{6} = \frac{n(2n^2 + 3n + 13)}{6}$$

EXERCISE 6.2

1. a. If P(n) is the statement "$n^3 + n$ is divisible by 2", prove that P(1), P(2), P(3) and P(4) are true.

b. If P(n) is the statement "$n^2 + n$ is even", Prove that P(1), P(2), P(3) and P(4) are true.

c. If P(n) is the statement "$n^3 \geq 2^n$" show that P(1) is false and P(2), P(3) are true.

d. Let P(n) denote the statement " $\frac{n(n+1)}{6}$ is a natural number". Show that P(2) and P(3) are true but P(4) is not true.

Solution:

a. Here, P(n) = $(n^3 + n)$ is divisible by 2...(i)
Putting n =1, 2, 3, and 4 in (i) we get,
$P(1) = 1^3 + 1 = 2$
$P(2) = 2^3 + 1 = 9$
$P(3) = 3^3 + 1 = 28$
$P(4) = 4^3 + 1 = 65$
from above, P(n) is false.

b. Here, P(n) = '$n^2 + n$' is even
Put n = 1, 2, 3 and 4
$P(1) = 1^2 + L = 2$
$P(2) = 2^2 + 1 = 5$
$P(3) = 3^2 + 3 = 12$
$P(4) = 4^2 + 3 = 19$
$\therefore$ from above, P(n) is false.

c. Here, $P(n) = n^3 \geq 2^n$
Put $P(1) = 1^3 \geq 2^1 = 1 \geq 2$ which is false.
Put n = 2 and 3
$P(2) = 2^3 \geq 2^2 = 8 \geq 4$

d. Here,
P(n) : $\frac{n(n+1)}{6}$ is natural number
Putting n = 28384

$P(3) = 3^3 \geq 2^3 = 27 \geq 8$
From above P(1) is false and P(2) and P(3) is true.

$\therefore$ P(2) = $\frac{2(n+1)}{6} = \frac{2\times3}{6}$ = 1 true

$\therefore$ P(4) = $\frac{4(4+1)}{6} = \frac{4\times6}{5} = \frac{10}{3}$ is false.

$\therefore$ P(3) = $\frac{3(3+1)}{6} = \frac{3\times4}{6}$ = 2 true

Hence, from above, P(n) is natural number.

2. Prove by the method of induction that

a. $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$

b. $1^2+3^2+5^2+\ldots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

c. $4 + 8 + 12 + \ldots + 4n = 2n(n+1)$ d. $1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n-1)}{2}$

e. $1.2 + 2.3 + 3.4 + \ldots$ to n terms $= \frac{n(n+1)(n+2)}{3}$

Solution:

a. If P(n) denotes the given statement, then;

P(n) = $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$

When n = 1 then (HS : P(2) = 2

RHS : $\frac{1(3\times1+1)}{2} = 2$

$\therefore$ LHS = RSH i.e. P(1) is true.
Suppose that P(n) is true for some $n = k \in N$

Then P(k) = $2 + 5 + 8 + \ldots (3k - 1) = \frac{k(3k+1)}{2}$ ... (i)

Here, we shall prove that P(k + 1) is true.
Whenever P(k) is true.
For this, adding 3(k + 1) – 1 = 3k + 2 on both sides of (i), we get

$$2 + 5 + 8 + \ldots + (3k - 1) + (3k + 2) = \frac{k(3k+1)}{2} + 3k + 2 = \frac{3k^2 + k + 6k + 4}{2}$$

$$= \frac{3k^2 + 7k + 4}{2} = \frac{3k^2 + 3k + 4k + 4}{2}$$

$$= \frac{(3k+4)(k+1)}{2} = \frac{(k+1)[3(k+1)+1]}{2}$$

This shows that P(k+1) is true whenever P(k) is true. Hence by the principle of mathematical inclusion, P(n) is true for all $n \in N$.

b. Here, suppose P(n) denotes the given st.

Then, P(n) = $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)2 = \frac{n(2n-1)(2n+1)}{3}$

When, n = 1, then LHS = P(1) = 1

RHS = $\frac{L(2\times1 - 1)(2\times1 + 1)}{3} = \frac{3}{3} = 1$

Hence, LHS = RHS. This shows that P(n) is true for n = 1. So suppose P(n) is true for $n = k \in N$. so that

P(k) = $1^2 + 3^2 + 5^2 + \ldots + (2k - 1)^2 = \frac{k(2k-1)(2k+1)}{3}$

Here, we shall prove that the statement P(k+1) is true whenever P(k) is true.

For this, adding $(2k+1)^2$ on both sides of (1), we get

$$1^2 + 3^2 + 5^2 + ... + (2k-1)^2 + (2k+1)^2 = \frac{k(2k-1)(2k+1)}{2} + (2k+1)^2$$

$$= \frac{(2k+1)(2k^2+5k+3)}{3} = \frac{(2k+1)(2k+3)(k+1)}{3}$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$= \frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}$$

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all $n \in N$.

c. Suppose P(n) denotes the given st.

$P(n) = 4 + 8 + 12 + ... + 4n = 2n(n + 1)$

When n = 1 LHS: P(1) = 4 and RHS: $P(1) = 2 \times 1(1 + 1) = 4$

This shows that P(n) is true for n = 1, so suppose P(n) is true for some integer $n = k \in N$, then

$P(k) = 4 + 8 + 12 + ... + 4k = 2k(k + 1)$ ... ... (i)

Here, we shall show that P(k+1) is true whenever P(k) is true.

For this adding 4(k + 1) on both sides of (i), we get,

$$4 + 8 + 12 + ... + 4k + 4k(k+1) = 2k(k+1) + 4(k+1) = 2(k+1)[k+2]$$

$$= 2(k+1)[(k+1)+1]$$

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all $n \in N$.

d. Here, Suppose P(n) denotes the given st.

$$P(n) = 1 + 4 + 7 + ... + (3n-2) = \frac{n(3n-1)}{2}$$

When n=1, LHS : $P(1) = 3 \times 1 - 2 = 1$

$$\text{RHS: } P(1) = \frac{1(3 \times 1 - 1)}{2} = 1$$

$\therefore$ LHS = RHS

This shows that P(n) is true for n = L, so suppose P(n) is true for some integer $n = k \in N$, then

$$P(k) = 1 + 4 + 7 + ... (3k-2) = \frac{k(3k-1)}{2} \text{ ... ... (i)}$$

Here, we shall show that P(k + 1) is true whenever P(k) is true

$$1 + 4 + 7 + ... + (3k-2) + (3k+1) = \frac{k(3k-1)}{2} + (3k+1) = \frac{k(3k-1) + 2(3k+1)}{2}$$

$$= \frac{3k^2 - k + 6k + 2}{2} = \frac{3k^2 + 5k + 2}{2}$$

$$= \frac{3k^2 + 3k + 2k + 2}{2} = \frac{(3k+2)(k+1)}{2}$$

$$= \frac{(k+1)[3(k+1)-1]}{2}$$

This shows that P(k+1) is true whenever P(k) is true. Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

e. Here, Suppose P(n) denotes the given st.

$$P(n) = 1.2 + 2.3 + 3.4 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

When n = 1, then LHS: P(1) = 1(1 + 1) = 2

$$\text{RHS: } P(1) = \frac{1(1+1)(1+2)}{3} = 2$$

$\therefore$ LHS = RHS

This shows that P(n) is true for n = 1. So suppose P(n) is true for some integer n = k$\in$N. then,

$$P(k) = 1.2 + 2.3 + 3.4 + ... + k(k + 1) = \frac{k(k + i)\ (k + 2)}{3} \text{ ... ... (i)}$$

Here, we shall show that P(k + 1) is true whenever P(k) is true for k$\in$N for this purpose, adding, (k+1) (k+2) on both sides (i) we get

$$1.2 + 2.3 + 3.4 + ... + (k+1)\ (k+2) + (k+1) = \frac{k(k+1)\ (k+2)}{3} + (k+1)\ (k+2)$$

$$= (k + 1)\ (k - 12) \left[1 + \frac{k}{3}\right]$$

$$= \frac{(k + 1)\ (k + 2)\ (k + 3)}{3}$$

This shows that P(k + 1) is true whenever P(k) is true. Hence, by the principle of mathematical induction P(n) is true for all n$\in$N.

3. Prove by the method of induction that

a. $\dfrac{1}{1.3} + \dfrac{1}{3.5} + \ldots + \dfrac{1}{(2n-1)\ (2n+1)} = \dfrac{n}{2n+1}$

b. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

c. $2 + 2^2 + \ldots + 2^n = 2\ (2^n - 1)$

d. $3 + 3^2 + \ldots + 3^n = \dfrac{3(3^n - 1)}{2}$

e. $\dfrac{1}{5} + \dfrac{1}{5^2} + \dfrac{1}{5^3} + \ldots$ to n terms $= \dfrac{1}{4}\left(1 - \dfrac{1}{5^n}\right)$

Solution:

a. Suppose P(n) denotes the given st. then

$$P(n) = \frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2n-1)\ (2n+1)} = \frac{n}{2n+1}$$

When n = 1, then LHS: $P(1) = \dfrac{1}{(2\times1 - 1)\ (2\times1 + 1)} = \dfrac{1}{3}$

RHS: $P(1) = \dfrac{1}{3} \Rightarrow$ LHS = RHS

This show that P(n) is true for n=1, so suppose P(n) is true for some integer n = k$\in$N. then

$$P(k) = \frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2k-1)\ (2k+1)} = \frac{k}{2k+1}$$

Here, we shall show that P(k + 1) is true whenever P(k) is true.

For this adding $\dfrac{1}{(2k + 1)\ (2k + 3)}$ on both sides of (i), we get

$$\frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)\ (2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)\ (2k+3)}$$

$$= \frac{k(2k+3) + 1}{(2k+1)\ (2k+3)}$$

$$= \frac{2k^2 + 3k + L}{(2k+1)\ (2k+3)} = \frac{2k^2 + 2k + k + 1}{4k^2 + 8k + 3} = \frac{(2k+1)\ (k+1)}{(2k+1)\ [2(k+1)+1]} = \frac{k+1}{2(k+1)+1}$$

This shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction, P(n) is true for all n$\in$N.

b. Here, Suppose P(n) denotes the given st. then

$P(n) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

When n = 1, then LHS: $P(1) = \frac{1}{2'} = \frac{1}{2}$ RHS: $1 - \frac{1}{2'} = \frac{1}{2}$

This shows that P(n) is true for n = 1, so suppose P(n) is true for some integer n = k∈N. Then

$P(k) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^k} = 1 - \frac{1}{2^k}$ ... ... (i)

We shall show that P(k + 1) is true whenever P(k) is true for this adding $\frac{1}{2^{(k+1)}}$ on both side of (i), we get

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k}\left(1 - \frac{1}{2}\right) = 1 - \frac{1}{2^k} : \frac{1}{2} = 1 - \frac{1}{2^{k+1}}$

This show that P(k+1) is true whenever P(k) is true. Hence by the principle of mathematical induction P(n) is true for all n∈k.

c. Here, Suppose P(n) denotes the given st. then

$P(n) = 2 + 2^n + 2^3 + ... + 2^n = 2(2^n - 1)$

When, n = 1, then LHS = P(1) = 2 and RHS = 2

∴ LHS = RHS. This shows that P(n) is true for n = 1. So suppose P(n) is true for some integer n=k∈N. Then,

$P(k) = 2 + 2^2 + 2^3 + ... + 2k = 2(2k - 1)$ ... ... (i)

Here, we shall prove that P(k + 1) is true whenever P(k) is true.

For this, adding 2^{k+1} on both sides of (i), we get

$2 + 2^2 + 2^3 + ... + 2^k + 2^{k+1} = 2(2^k - 1) + 2^{k+1} = 2^k.2 - 2 + 2^k.2 = 2.2^{k+1} - 2 = 2(2^{k+1} - 1)$

This shows that P(k + 1) is true whenever P(k) is true for all k∈N. Hence by the principle of mathematical induction P(n) is true for all n∈N.

d. Here, Suppose P(n) denotes the given st. then

$P(n) = 3 + 3^2 + ... 3^n = \frac{3(3^2 - 1)}{2}$

When, n = 1, LHS = 3 and RHS 3

∴ LHS = RHS. This shows that P(n) is true for n = 1. So, suppose P(n) is true for some integer n = k∈N. then

$P(k) = 3 + 3^2 + ... + 3^k = \frac{3(3^k - 1)}{2}$ ... ... (i)

Here, we shall prove that P(k+1) is also true whenever P(k) is true for this adding 3^{k+1} on both side of (i) $3+3^2 + ... 3^k + 3^{k+1} = \frac{3(3^k - 1)}{2} + 3^{k+1}$

$= \frac{3.3^k - 3 + 2.3^k.3}{2} = k.\frac{3.3^{k+1} - 3}{2} = \frac{3(3^{k+1} - 1)}{2}$

This shows that P(k + 1) is true whenever P(k) is true for all k∈N. Hence by the principle of mathematical induction P(n) is true for all n∈N.

e. Suppose P(n) denotes the given st. then

$P(n) = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ...$ to n terms $= \frac{1}{4}\left(1 - \frac{1}{5^n}\right)$

i.e. to $= ar^{n-1} = \frac{1}{5}\left(\frac{1}{5}\right)^{n-1} = \frac{1}{5^n}$

When, n = 1, LHS $= \frac{1}{5}$ RHS $= \frac{1}{4}\left(1 - \frac{1}{5'}\right) = \frac{1}{4} \cdot \frac{4}{5} = \frac{1}{5}$

∴ LHS = RHS this show that P(n) is true for n = 1. So suppose P(n) is true for some integer n = k∈N. Then,

$$P(k) = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ... + \frac{1}{5^k} = \frac{1}{4}\left(1 - \frac{1}{5^k}\right)$$

Here, we shall prove that P(k+1) is true whenever P(k) is true. For this adding 5^{k+1} on both sides of (i)

$$\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + ... + \frac{1}{5^k} + \frac{1}{5^{k+1}} = \frac{1}{4}\left(1 - \frac{1}{5^k}\right) + \frac{1}{5^{k+1}} = \frac{1}{4} - \frac{1}{4.5^k} + \frac{1}{5.5^k}$$

$$= \frac{1}{4} + \frac{1}{5.5^k} - \frac{1}{4.5^k} = \frac{1}{4} + \frac{4-5}{4.5.5^k} = \frac{1}{4} + \frac{-1}{4.5^{k+1}} = \frac{1}{4}\left[1 - \frac{1}{5^{k+1}}\right]$$

This shows that P(k+1) is also true whenever P(k) is true for all k∈N. Hence by the principle of mathematical induction P(n) is true for all n∈N.

4. Prove by the method of induction that
   a. $4^n - 1$ is divisible by 3.
   b. $3^{2n} - 1$ is divisible by 8.
   c. $10^{2n-1} + 1$ is divisible by 11.
   d. $x^n - y^n$ is divisible by $x - y$.
   e. $n(n + 1)(n + 2)$ is a multiple of 6.

Solution:

a. Here, suppose P(n) denotes the given st. then
   P(n) = $4^n - 1$ is divisible by 3
   When n = 1, P(1) = $4^1 - 1 = 3$ is divisible by 3. So P(1) is true
   Let P(k) be true for k∈N. That is
   P(k) : $4_k - 1$ is divisible by 3 ... ... (i)
   Now we shall show that P(k+1) is true when P(k) is true.
   P(k+1): $4^{k+1} - 1$ is divisible by 3
   Now, $(4^{k+1} - 1)$ is divisible by 3. Therefore P(k+1) is true whenever P(k) is true.
   Hence by induction method, P(n) is true for all n∈N.= $1^k . 4 - 4 + 3 = 4(4^k - 1) + 3$

b. Here, Suppose P(n) be the given st. then P(n): $3^{2n} - 1$ is divisible by 8.
   If n = 1. P(1) : $3^2 - 1 = 8$ which is divisible by 8.
   So, the statement P(n) is true for n = 1
   Let P(k) be true for k∈N, that is
   P(k) = $3^{2k} - 1$ is divisible by 8 ... ... (i)
   Now, we shall show that P(k+1) is true when P(k) is true i.e. P(k+1): $3^{2(k+1)} - 1$
   $= 3^{2k+2} - 1 = 3^{2k} . 3^2 - 1$
   $= 9.3^{2k} - 1 = 9.32k - 9+8 = 9(3^{2k} - 1) + 8$ is divisible by 8.
   Thus, P(k + 1) is true whenever P(k) is true. Hence by induction method, P(n) is true for all n∈N.

c. Here, Let P(n) be given st. then
   P(n) : $10^{2n-1} + 1$ is divisible by 11
   When n = 1, P(L): $10^{2-1} + 1 = 11$ which is divisible by 11. So P(1) is true.
   Let P(k) be true for K∈N. That is
   P(k) : $10^{2k-1} + 1$ ... ... (i)
   We shall show that P(k+1) is true when P(k) is true i.e. P(k+1): $10^{2(k+1)-1} + 1$
   $= 10^{2k+1} + 1 = 10^{2k-1} . 10^2 + 1 = (10^{2k-1} + 1 - 1)10^2 + 1 = 100(10^{2k-1} + 1) - 99$
   which is divisible by 11.

d. Here, let P(n) be given st.
   i.e. P(n): $x^n - y^n$ is divisible by x–y
   When n = 1 P(1): x – y is divisible y x – y. So P(1) is true.
   Let P(k) be true for k∈N. i.e.
   P(k) : $x^k - y^k$ is divisible by x–y ... ... (i)
   Now, we shall show that P(k+1) is true when P(k) is true i.e. P(k+1): $x^{k+1} - y^{k+1}$
   $= x(x^k - y^k) + y(x^k - y^k) - xy(x^{k-1} - y^{k-1})$
   $= (x + y)(x^k - y^k) - xy(x^{k-1} - y^{k-1})$ is divisible by x – y.

Therefore, P(k+1) is true whenever P(k) is true. Hence by induction method, P(n) is true for all $n \in N$.

e. Here, Let P(n) be given st. then

P(n): n(n+1) (n+2) is multiple of 6.

When n=1, P(1): 1(1+1) (1+2) = 6 is multiple of 6. So P(1) is true

Let P(k) is true for $k \in N$. i.e.

$\therefore$ P(k): k(k+1) (k+2) is multiple of 6 ... ... (i)

Now, we shall show that P(k+1) is true when P(k) is true i.e. P(k+1) : (k+1) (k+2) (k+3) i.

= k(k+1) (k+2) +3(k+1) (k+2) is multiple of 6.

Therefore, P(k+1) is true whenever P(k) is true. Hence by induction method, P(n) is true for all $n \in N$.

■

CHAPTER 7

MATRIX BASED SYSTEM OF LINEAR EQUATIONS

EXERCISE 7.1

1. By drawing graph or otherwise, classify each of the following system of the equations.
   a. $4x - 3y = -6$
      $-4x + 2y = 16$
   b. $2x - y = 3$,
      $-4x + 2y = 6$
   c. $-6x + 4y = 10$
      $3x - 2y = -5$
   d. $7x + 2y = 15$,
      $x + y = 5$

Solution:

a. Here,
Given equations are $4x - 3y = -6$ ... (i) and $-4x + 2y = 16$ ... (ii)
Adding equation (i) and (ii), we get
$$\begin{array}{l} 4x - 3y = -6 \\ \underline{-4x + 2y = 16} \\ -y = 10 \end{array}$$
$\therefore$ $y = -10$
Putting in equation (i),
$4x - 3x - 10 = -6$
or, $4x = -6 - 30$
$\therefore$ $x = -9$
Hence, $(-9, -10)$ is the solution of the system. This kind of system where we get only one solution is known as consistent and independent.

b. Here,
Given equation of system are,
$2x - y = 3$ ... ... (i)
$-4x + 2y = 6$ ... ... (ii)
Multiplying by 2 in equation (i) and adding with (ii), we get
$$\begin{array}{l} 4x - 2y = 6 \\ \underline{-4x + 2y = 6} \\ 0 = 12 \end{array}$$
This is impossible result. In other word, the system has no solution. This is an inconsistent and independent.

c. Here,
Given, $-6x + 4y = 10$ ... ... (i)
$3x - 2y = -5$ ... ... (ii)
Multiplying by 2 in equation (ii)and adding with (i), we get
$$\begin{array}{l} 6x + 4y = 10 \\ \underline{6x - 4y = -10} \\ 0 = 0 \end{array}$$
So, we do not get particular value of x and y. However, the result $0 = 0$ is true. In this situation, whatever be the solution of one equation satisfies the other equation as well. This kind of system, where we get infinitely many solution is known as consistent and dependent.

d. Here,
Given, $7x + 2y = 15$ ... ... (i)
$x + y = 5$ .. .. (ii)
Multiplying with 7 in equation (ii) and subtracting (i) from (ii),

$7x + 2y = 15$
$\underline{(-)\ (-)\quad\ (-)}$
$5y = 20$
$\therefore\ y = 4$
Putting y = 4 in equation (ii), we get
$x + 4 = 5$
$\therefore\ x = 1$
Hence, (1, 4) is the solution of the system. This kind of system of solution where only one solution we get is known consistent and independent.

2. Solve the following systems by using row – equivalent matrix method

a. $x + y = 5$
$2x + 3y = 12$

b. $2x + 12y = 16$
$3x + 10y = 8$

c. $x - 3y = -1$
$4x - y = 7$

d. $8x - 3y = -31$
$2x + 6y = 26$

e. $5x - 3y = -2$
$4x + 2y = 5$

f. $2/x + 3/y = 2$
$4/x - 5/y = 7$

Solution:

a. Here,
$x + y = 5$
$2x + 3y = 12$
The augmented matrix is
$$\left[\begin{array}{cccc} 1 & 1 & : & 5 \\ 2 & 3 & : & 12 \end{array}\right]$$
Multiplying by 2 in R_1 and subtracting from R_2.
$$\sim\left[\begin{array}{cccc} 1 & 1 & : & 5 \\ 0 & 1 & : & 2 \end{array}\right]$$
Applying $R_2 \to R_1 - R_2$
$$\sim\left[\begin{array}{cccc} 1 & 0 & : & 3 \\ 0 & 1 & : & 2 \end{array}\right]$$
Hence the solution is x = 3 and y = 2

b. Here, Augmented matrix is
$$\left[\begin{array}{cccc} 2 & 12 & : & 16 \\ 3 & 10 & : & 8 \end{array}\right]$$
$$\sim\left[\begin{array}{cccc} 3 & 10 & : & 8 \\ 2 & 12 & : & 16 \end{array}\right] \text{ Applying } R_1 \leftrightarrow R_2$$
Applying $R_1 \to R_1 - R_2$
$$\sim\left[\begin{array}{cccc} 1 & -2 & : & -8 \\ 2 & 12 & : & 16 \end{array}\right]$$
Applying $R_2 \to R_2 - 2R_1$
$$\sim\left[\begin{array}{cccc} 1 & -2 & : & -8 \\ 0 & 16 & : & 32 \end{array}\right]$$
$$\sim\left[\begin{array}{cccc} 1 & -2 & : & -8 \\ 0 & 1 & : & 2 \end{array}\right] \text{ Applying } R_2 \to \frac{1}{16} R_2$$
$$\sim\left[\begin{array}{cccc} 1 & 0 & : & -4 \\ 0 & 1 & : & 2 \end{array}\right] \text{ Applying } R_1 \to R_1 + 2R_2$$
Hence, the required solution is x = –4 and y = 2

c. Here, Augmented matrix is
$$\left[\begin{array}{cccc} 1 & -3 & : & -1 \\ 4 & -1 & : & 7 \end{array}\right]$$
Applying $R_2 \to R_2 - 4R_1$
$$\sim\left[\begin{array}{cccc} 1 & -3 & : & -1 \\ 0 & 11 & : & 11 \end{array}\right]$$
Applying $R_2 \to \frac{1}{11} R_2$
$$\sim\left[\begin{array}{cccc} 1 & -3 & : & -1 \\ 0 & 1 & : & 1 \end{array}\right]$$
Applying $R_1 \to R_1 + 3R_2$

d. Here, The augmented matrix is
$$\left[\begin{array}{cccc} 8 & -3 & : & -31 \\ 2 & 6 & : & 26 \end{array}\right]$$
$$\sim\left[\begin{array}{cccc} 2 & 6 & : & 26 \\ 8 & -3 & : & -31 \end{array}\right] \text{ Applying } R_1 \leftrightarrow R_2$$
$$\sim\left[\begin{array}{cccc} 1 & 3 & : & 13 \\ 8 & -3 & : & -31 \end{array}\right] \text{ Applying } R_1 \to \frac{1}{2} R_1$$
Applying $R_2 \to R_2 - 8R_1$
$$\sim\left[\begin{array}{cccc} 1 & 3 & : & 13 \\ 0 & -27 & : & -13 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & : & 2 \\ 0 & 1 & : & 1 \end{bmatrix}$$

Hence, the required solution is
$x = 2$ and $y = 1$

e. Here,
The augmented matrix is

$$\begin{bmatrix} 5 & -3 & : & -2 \\ 4 & 2 & : & 5 \end{bmatrix}$$

Applying $R_1 \to \frac{1}{5} R_1$

$$\sim \begin{bmatrix} 1 & -3/5 & : & -2/5 \\ 4 & 2 & : & 5 \end{bmatrix}$$

Applying $R_2 \to R_2 - 4R_1$

$$\begin{bmatrix} 1 & -3/5 & : & -2/5 \\ 0 & 22/5 & : & 33/5 \end{bmatrix}$$

Applying $R_2 \to \frac{5}{22} R_2$

$$\sim \begin{bmatrix} 1 & -3/5 & : & -2/5 \\ 0 & 1 & : & 3/2 \end{bmatrix}$$

Applying $R_1 \to R_1 + \frac{3}{5} R_2$

$$\sim \begin{bmatrix} 1 & 0 & : & 1/2 \\ 0 & 1 & : & 3/2 \end{bmatrix}$$

Hence the required solution is
$x = \frac{1}{2}$ and $y = \frac{3}{2}$

Applying $R_2 \to -\frac{1}{27} R_2$

$$\sim \begin{bmatrix} 1 & 3 & : & 13 \\ 0 & 1 & : & 5 \end{bmatrix}$$

Applying $R_1 \to R_1 - 3R_2$

$$\sim \begin{bmatrix} 1 & 0 & : & -2 \\ 0 & 1 & : & 5 \end{bmatrix}$$

Hence, the required solution is
$x = -2$ and $y = 5$

f. Here, The augmented matrix is

$$\begin{bmatrix} 2 & 3 & : & 2 \\ 4 & -5 & : & 7 \end{bmatrix}$$

Applying $R_1 \to \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 1 & 3/2 & : & 1 \\ 4 & -5 & : & 7 \end{bmatrix}$$

Applying $R_2 \to R_2 - 4R_1$

$$\sim \begin{bmatrix} 1 & 3/2 & : & 1 \\ 0 & -11 & : & 3 \end{bmatrix}$$

Applying $R_2 \to -\frac{1}{11} R_2$

$$\sim \begin{bmatrix} 1 & 3/2 & : & 1 \\ 0 & 1 & : & -3/11 \end{bmatrix}$$

Applying $R_1 \to R_1 - \frac{3}{2} R_1$

$$\sim \begin{bmatrix} 1 & 0 & : & 31/22 \\ 0 & 1 & : & -3/11 \end{bmatrix}$$

Hence, the required solution is

$\frac{1}{x} = \frac{31}{22} \Rightarrow x = \frac{22}{31}$

and $\frac{1}{y} = \frac{-3}{11} \Rightarrow y = \frac{-11}{3}$

3. Use the row equivalent matrix method to solve the system of equations:

a. $x + y + z = 1$
$x + 2y + 3z = -1$
$2x - y + 2z = -4$

b. $x + 4y + z = 18$
$3x + 3y - 2z = 2$
$-4y + z = -7$

c. $9y - 5x = 3$
$x + z = 1$
$z + 2y = 2$

d. $x - y + 2z = 0$
$x - 2y + 3z = -1$
$2x - 2y + z = -3$

e. $2x - y + 4z = -3$
$x - 4z = 5$
$6x - y + 2z = 10$

f. $x + 2y - 3z = 9$
$2x - y + 2z = -8$
$3x - y - 4z = 3$

g. $3x - 2y - 3z = -3$
$2x + y + z = 6$
$x + 3y - 2z = 13$

h. $3x - 5z = -7$
$3x + 5y = 3$
$3z - 3y = 2$

Solution:

a. Here, The augmented matrix is $\left[\begin{array}{ccccc} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 3 & : & -1 \\ 2 & -1 & 2 & : & -4 \end{array}\right]$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - {}_2R_1$

$$\sim \left[\begin{array}{ccccc} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 2 & : & -2 \\ 0 & -3 & 0 & : & -6 \end{array}\right]$$

Applying $R_3 \to R_3 + 3R_2$ and $R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccccc} 1 & 0 & -1 & : & 3 \\ 0 & 1 & 2 & : & -2 \\ 0 & 0 & 6 & : & -12 \end{array}\right]$$

Applying $R_3 \to \frac{1}{6} R_3$

$$\sim \left[\begin{array}{ccccc} 1 & 0 & -1 & : & 3 \\ 0 & 1 & 2 & : & -2 \\ 0 & 0 & 1 & : & -2 \end{array}\right]$$

Applying $R_1 \to R_1 + R_3$ and $R_2 \to R_2 - 2R_3$

$$\sim \left[\begin{array}{ccccc} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & -2 \end{array}\right]$$

Hence, the required solution is $x = 1$, $y = 2$ and $z = -2$

b. Here,

The augmented matrix is

$$\left[\begin{array}{ccccc} 1 & 4 & 1 & : & 18 \\ 3 & 3 & -2 & : & 2 \\ 0 & -4 & 1 & : & -7 \end{array}\right]$$

Applying $R_2 \to R_2 - 3R_1$

$$\sim \left[\begin{array}{ccccc} 1 & 4 & 1 & : & 18 \\ 0 & -9 & -5 & : & -52 \\ 0 & -4 & 1 & : & -7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc} 1 & 4 & 1 & : & 18 \\ 0 & 1 & 5/9 & : & 52/9 \\ 0 & -4 & 1 & : & -7 \end{array}\right]$$

Applying $R_3 \to 4 R_2 + R_3$, we get

$$\sim \left[\begin{array}{ccccc} 1 & 4 & 1 & : & 18 \\ 0 & 1 & 5/9 & : & 52/9 \\ 0 & 0 & 29/9 & : & 145/9 \end{array}\right]$$

Applying $R_3 \to \frac{9}{29} \times R_3$ we get

$$\sim \left[\begin{array}{ccccc} 1 & 4 & 1 & : & 18 \\ 0 & 1 & 5/9 & : & 52/9 \\ 0 & 0 & 1 & : & 5 \end{array}\right]$$

Applying $R_2 \to R_2 - \frac{5}{9} R_3$ we get,

$$\sim \left[\begin{array}{ccc:c} 1 & 4 & 1 & 18 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Applying $R_1 \to R_1 - R_3$ we get

$$\sim \left[\begin{array}{ccc:c} 1 & 4 & 0 & 13 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Applying $R_1 \to R_1 - 4R_2$ we get

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Hence, $x = 1$, $y = 3$, $z = 5$

c. The augmented matrix is

$$\left[\begin{array}{ccc:c} -6 & 9 & 0 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 2 \end{array}\right]$$

Applying $R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ -5 & 9 & 0 & 3 \\ 0 & 2 & 1 & 2 \end{array}\right]$$

Applying $R_2 \to R_2 + 5R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 9 & 5 & 8 \\ 0 & 2 & 1 & 2 \end{array}\right]$$

Applying $R_2 \to R_2 - 4R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 2 \end{array}\right]$$

Applying $R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

Applying $R_3 \to -1R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_1 - R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Hence the solution is
$x = 3$, $y = 2$ and $z = -2$

d. The augmented matrix is

$$\left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 1 & -2 & 3 & -1 \\ 2 & -2 & 1 & -3 \end{array}\right]$$

Applying $R_{12} \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

Applying $R_2 \to -1R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

Applying $R_1 \to R_1 + R_2$

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right]$$

Applying $R_3 \to -\frac{1}{3} R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Applying $R_1 \to R_1 + R_3$ and $R_2 \to R_2 + R_3$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Hence, the required solution is
$x = 0$, $y = 2$, and $z = 1$

e. The augmented matrix is $\begin{bmatrix} 2 & -1 & 4 & : & -3 \\ 1 & 0 & -4 & : & 5 \\ 6 & -1 & 2 & : & 10 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 2 & -1 & 4 & : & -3 \\ 6 & -1 & 2 & : & 10 \end{bmatrix}$ Applying $R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & -1 & 12 & : & -13 \\ 0 & -1 & 26 & : & -20 \end{bmatrix}$ Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 6R_3$

$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & 1 & -12 & : & 13 \\ 0 & -1 & 26 & : & -20 \end{bmatrix}$ Applying $R_2 \to 1R_2$

$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & 1 & -12 & : & 13 \\ 0 & 0 & 14 & : & -7 \end{bmatrix}$ Applying $R_3 \to R_2 + R_3$

$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & 1 & -12 & : & 13 \\ 0 & 0 & 1 & : & -1/2 \end{bmatrix}$ Applying $R_3 \to \frac{1}{14} R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 7 \\ 0 & 0 & 1 & : & -1/2 \end{bmatrix}$ Applying $R_2 \to R_2 + 12R_3$ and $R_1 \to R_1 + 4R_3$

Hence, $x = 3$, $y = 7$, $z = -\frac{1}{2}$

f. The augmented matrix is $\begin{bmatrix} 1 & 2 & -3 & : & 9 \\ 2 & -1 & 2 & : & -8 \\ 3 & -1 & -4 & : & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -3 & : & 9 \\ 0 & -5 & 8 & : & -26 \\ 0 & -7 & 5 & : & -24 \end{bmatrix}$ Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & 2 & -3 & : & 9 \\ 0 & 1 & -8/5 & : & 26/5 \\ 0 & -7 & 5 & : & -24 \end{bmatrix}$ Applying $R_2 \to \frac{1}{5} R_2$

$\sim \begin{bmatrix} 1 & 0 & 1/5 & : & -7/5 \\ 0 & 1 & -8/5 & : & 26/5 \\ 0 & 0 & -31/5 & : & 62/5 \end{bmatrix}$ Applying $R_3 \to R_3 + 7R_2$ and $R_1 \to R_1 - 2R_2$

$\sim \begin{bmatrix} 1 & 0 & 1/5 & : & -7/5 \\ 0 & 1 & -8/5 & : & 26/5 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$ Applying $R_3 \to -\frac{5}{31} R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & : & -1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & -2 \end{bmatrix}$ Applying $R_1 \to R_1 - \frac{1}{5} R_3$ and $R_2 \to R_2 + \frac{1}{8} R_3$

Hence, the required solution is $x = -1$, $y = 2$ and $z = -2$

4. To control a certain crop disease, it is necessary to use 7 units of chemical A, 10 units of chemical B and 6 units of chemical C. One barrel of spray P contains 1, 4, 2 units of chemicals, one third of spray Q contains 3, 2, 2 units and one barrel of spray R contains 4, 3, 2 units of these chemicals respectively.
   a. Formulate the simultaneous linear system.
   b. Write the linear system in matrix forms as
      i. Coefficient matrix
      ii. Variable matrix
      iii. Constant matrix
   c. Express the matrix in augmented matrix form.
   d. Solve the systems by row equivalent matrix method and find the quantity of each type of spray should be used to control the disease.

Solution:

Let us tabulate the data as follows:

	Spray			
Chemical	**P**	**Q**	**R**	Requirement of chemicals (Units)
A	1	3	4	7
B	4	2	3	10
C	2	2	2	6

Let x barrels spray P, y barrels of Q and Z barrels of R be used to control the disease. Then, by questions

$x + 3y + 4z = 7$

$4x + 2y + 3z = 10$

$2x + 2y + 2z = 6$

The above equations can be written in matrix form AX = B as

$$\begin{bmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 6 \end{bmatrix}$$

AX = B where, $A = \begin{bmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 10 \\ 6 \end{bmatrix}$

Now, the augmented matrix is,

$$[A : B] = \begin{bmatrix} 1 & 3 & 4 & : & 7 \\ 4 & 2 & 3 & : & 10 \\ 2 & 2 & 2 & : & 6 \end{bmatrix}$$

Applying $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 2R_1$ we get,

$$\sim \begin{vmatrix} 1 & 3 & 4 & : & 7 \\ 0 & -10 & -13 & : & -18 \\ 0 & -4 & -6 & : & -8 \end{vmatrix}$$

Applying $R_2 \to \dfrac{R_2}{-10}$ we get,

$$\sim \begin{bmatrix} 1 & 3 & 4 & : & 7 \\ 0 & 1 & \dfrac{13}{10} & : & \dfrac{9}{5} \\ 0 & -4 & -6 & : & -8 \end{bmatrix}$$

Applying $R_1 \to R_1 - 3R_2$ and $R_3 \to R_3 + 4R_2$ we get,

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & \frac{1}{10} & \frac{8}{5} \\ 0 & 1 & \frac{13}{10} & \frac{9}{5} \\ 0 & 0 & \frac{4}{5} & \frac{4}{5} \end{array}\right]$$

Applying $R_1 \rightarrow R_1 - \frac{1}{10} \times R_3$ and $R_2 \rightarrow R_2 - \frac{13}{10} \times R_3$ we get,

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore$ $x = \frac{3}{2}$, $y = \frac{1}{2}$, $z = 1$

Hence, $\frac{3}{2}$ barrel of spray, P.$\frac{1}{2}$ barrels of spray Q and 1 barrel of spray R are used to control the disease.

EXERCISE 7.2

1. Solve the equations, by inverse matrix method and by Cramer's rule

a. $x + y = 4$
$3x - 2y = 17$

b. $2x - y = 5$
$x - 2y = 1$

c. $3x + 4y = -2$
$15x + 20y = 24$

d. $\frac{2}{3}x + y = 16$
$x + \frac{y}{4} = 14$

e. $3x + \frac{4}{y} = 10$
$-2x + \frac{3}{y} = -1$

f. $3x = 4y - 11$
$5y = -2x + 31$

Solution:

a. $x + y = 4$
$3x - 2y = 17$

Coe. of x	**Coe. of y**	**Constant**
1	1	4
3	–2	17

Now,

$D = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -2 - 3 = -5$ $\quad D_1 = \begin{vmatrix} 4 & 1 \\ 17 & -2 \end{vmatrix} = -8 - 17 = -25$

$D_2 = \begin{vmatrix} 1 & 4 \\ 3 & 17 \end{vmatrix} = 17 - 12 = 5$

The solution is $x = \frac{D_1}{D} = \frac{-25}{-5} = 5$ $\quad y = \frac{D_2}{D} = \frac{5}{-5} = -1$

b. Let,

Coe. of x	Coe. of y	Constant
2	–1	5
1	–2	1

Now,

$D = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -4 + 1 = -3$ $\qquad D_1 = \begin{vmatrix} 5 & -1 \\ 1 & -2 \end{vmatrix} = -10 + 1 = -9$

$D_2 = \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = 2 - 5 = -3$

The solution is, $x = \frac{D_1}{D} = \frac{-9}{-3} = 3$ $\qquad y = \frac{D_2}{D} = \frac{-3}{-3} = 1$

c. Let,

Coe. of x	Coe. of y	Constant
3	4	–2
15	20	24

Now,

$D = \begin{vmatrix} 3 & 4 \\ 15 & 20 \end{vmatrix} = 60 - 60 = 0$

$\therefore$ D is negative, the solution does not exist.

d. Let,

Coe. of x	Coe. of y	Constant
$\frac{2}{3}$	1	16
1	$\frac{1}{4}$	14

Now, $D = \begin{vmatrix} \frac{2}{3} & 1 \\ 1 & \frac{1}{4} \end{vmatrix} = \frac{1}{6} - 1 = \frac{-5}{6}$ $\qquad D_1 = \begin{vmatrix} 16 & 1 \\ 14 & \frac{1}{4} \end{vmatrix} = 4 - 14 = -10$

$D_2 = \begin{vmatrix} \frac{2}{3} & 16 \\ 1 & 14 \end{vmatrix} = \frac{28}{3} - 16 = \frac{-20}{3}$

The solution is $x = \frac{D_1}{D} = \frac{-10}{-\frac{5}{6}} = 12$, $y = \frac{D_2}{D} = \frac{-\frac{20}{3}}{-\frac{5}{6}} = 8$

e. Let,

Coe. of x	Coe. of y	Constant
3	4	10
–2	3	–1

$D = \begin{vmatrix} 3 & 4 \\ -2 & 3 \end{vmatrix} = 9 + 8 = 17$ $\qquad D_1 = \begin{vmatrix} 10 & 4 \\ -1 & 3 \end{vmatrix} = 30 + 4 = 34$

$D_2 = \begin{vmatrix} 3 & 10 \\ -2 & -1 \end{vmatrix} = -3 + 20 = 17$

The solution is $x = \frac{D_1}{D} = \frac{34}{17} = 2$

$\frac{1}{y} = \frac{D_2}{D} = \frac{1}{1} = 1$

or, $\frac{1}{y} = 1 \therefore y = 1$

f. Let,

Coe. of x	Coe. of y	Constant

3   –4   –11
2   5   31

$D = \begin{vmatrix} 3 & -4 \\ 2 & 5 \end{vmatrix} = 15 + 8 = 23$ $\qquad D_1 = \begin{vmatrix} -11 & -4 \\ 31 & 5 \end{vmatrix} = -55 + 124 = 69$

$D_2 = \begin{vmatrix} 3 & -11 \\ 2 & 31 \end{vmatrix} = 93 + 22 = 115$

The solution is $x = \frac{D_1}{D} = \frac{69}{23} = 3 \qquad y = \frac{D_2}{D} = \frac{115}{23} = 5$

2. Solve the following system of simultaneous linear equations by matrix inversion method.

a. $x - y + z = 4$
$x + y + z = 2$
$2x + y - 3z = 0$

b. $2x - 3y - z = 4$
$x - 2y - z = 1$
$x - y + 2z = 9$

c. $3x + 5y = 2$
$2x - 3z = -7$
$4y + 2z = 2$

d. $x - 3y - 7z = 6$
$2x + 3y + z = 9$
$4x + y = 7$

e. $x + 2y + z = 7$
$2x - y + z = 3$
$3x + y + 2z = 8$

Solution:

a. The matrix equation of given system is Ax = B

Where $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$

Now,

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & -3 \end{bmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= 1(-3 - 1) + 1(-3 - 2) + 1(1 - 2) = -4 - 5 - 1 = -10$$

$\therefore$ $|A| \neq 0$, so A^{-1} exist

Let cofactor of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$A_{11} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -3 - 1 = -4$ $\qquad A_{12} = -\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -(-3 - 2) = 5$

$A_{13} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$ $\qquad A_{21} = -\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = -(3 - 1) = -2$

$A_{22} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5$ $\qquad A_{23} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -(1 + 2) = -3$

$A_{31} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = (-1 - 1) = -2$ $\qquad A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -(1 - 1) = 0$

$A_{33} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2$

Co. factor of A = $\begin{bmatrix} -4 & 5 & -1 \\ -2 & -5 & -3 \\ -2 & 0 & 2 \end{bmatrix}$

Adj. of A = $\begin{bmatrix} -4 & 5 & -1 \\ -2 & -5 & -3 \\ -2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -2 & -2 \\ 5 & -5 & 0 \\ -1 & -3 & 2 \end{bmatrix}$

The solution given by,

$$x = A^{-1} B = \frac{1}{-10}\begin{bmatrix} -4 & -2 & -2 \\ 5 & -5 & 0 \\ -1 & -3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{-10}\begin{bmatrix} -20 \\ 10 \\ -10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ x = 2, y = –1, z = 1

b. The matrix equation of system is AX = B

Where A = $\begin{bmatrix} 2 & -3 & -1 \\ 1 & -2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, x = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, B = $\begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}$

Now,

$$|A| = \begin{bmatrix} 2 & -3 & -1 \\ 1 & -2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 2\begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} -1\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$= 2(-4 -1) +3 (2 + 1) -1 (-1 + 2) = 2x - 5 + 3 \times 3 - 1\times1 = -2$

$\therefore$ $|A| \neq 0$, A^{-1} exist

Cofactor of A = $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$A_{11} = \begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} = -4 - 1 = -5$

$A_{12} = -\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = -(2 + 1) = -3$

$A_{13} = \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = -1 + 2 = 1$

$A_{21} = \begin{vmatrix} -3 & -1 \\ -1 & 2 \end{vmatrix} = -(-6 - 1) = 7$

$A_{22} = \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} = (4 \times 1) = 5$

$A_{23} = -\begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} = -(-2 + 3) = -1$

$A_{31} = \begin{vmatrix} -3 & -1 \\ -2 & -1 \end{vmatrix} = (3 - 2) = 1$

$A_{32} = -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = -(-2 + 1) = 1$

$A_{33} = \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = -4 + 3 = -1$

Now, Co factor of A = $\begin{bmatrix} -5 & -3 & 1 \\ 7 & 5 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

Adj of A = $\begin{bmatrix} -5 & -3 & 1 \\ 7 & 5 & -1 \\ 1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -5 & 7 & 1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

The solution given by,

$$x = A^{-1} B = \frac{1}{-2}\begin{bmatrix} -5 & 7 & 1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$\therefore$ x = 2, y = –1, z = 3

c. The matrix equation of given system is Ax = B

$$\text{where } A = \begin{bmatrix} 3 & 5 & 0 \\ 2 & 0 & -3 \\ 0 & 4 & 2 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ -7 \\ 2 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 3 & 5 & 0 \\ 2 & 0 & -3 \\ 0 & 4 & 2 \end{vmatrix} = 3\begin{vmatrix} 0 & -3 \\ 4 & 2 \end{vmatrix} - 5\begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}$$

$= 3(12) - 5(4 \neq 0) + 0 = 16$

$\therefore$ $|A| \neq 0$, A^{-1} exist

$$\text{Let cofactor of } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$A_{11} = \begin{vmatrix} 0 & -3 \\ 4 & 2 \end{vmatrix} = 12$ $\qquad$ $A_{12} = -\begin{vmatrix} 2 & -3 \\ 0 & 2 \end{vmatrix} = -(4 - 0) = -4$

$A_{13} = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8$ $\qquad$ $A_{21} = \begin{vmatrix} 5 & 0 \\ 4 & 2 \end{vmatrix} = -(10) = -10$

$A_{22} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$ $\qquad$ $A_{23} = \begin{vmatrix} 3 & 5 \\ 0 & 4 \end{vmatrix} = -12$

$A_{31} = \begin{vmatrix} 5 & 0 \\ 0 & -3 \end{vmatrix} = -15$ $\qquad$ $A_{32} = \begin{vmatrix} 3 & 0 \\ 2 & -3 \end{vmatrix} = -(9) = 9$

$A_{33} = \begin{vmatrix} 3 & 5 \\ 2 & 0 \end{vmatrix} = 0 - 10 = -10$

$$\therefore \text{Cofactor of } A = \begin{bmatrix} 12 & -4 & 8 \\ -10 & 6 & -12 \\ -15 & 9 & 10 \end{bmatrix}$$

$$\text{Adj of } A = \begin{bmatrix} 12 & -4 & 8 \\ -10 & 6 & -12 \\ -15 & 9 & 10 \end{bmatrix}^T = \begin{bmatrix} 12 & -10 & -15 \\ -4 & 6 & 9 \\ 8 & -12 & 10 \end{bmatrix}$$

The solution given by,

$$X = A^{-1}B = \frac{1}{10}\begin{bmatrix} 12 & -10 & -15 \\ -4 & 6 & 9 \\ 8 & -12 & 10 \end{bmatrix}\begin{bmatrix} 2 \\ -7 \\ 2 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 64 \\ -32 \\ 120 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$$

$\therefore$ x = 4, y = –2, z = 5

d. The matrix equation of given system is AX = B

where $A = \begin{bmatrix} 1 & -3 & -7 \\ 2 & 3 & 1 \\ 4 & 1 & 0 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 9 \\ 7 \end{bmatrix}$

Now,

$$|A| = \begin{vmatrix} 1 & -3 & -7 \\ 2 & 3 & 1 \\ 4 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} - 7\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}$$

$$= -1 + 3\times -4 - 7(2 - 12) = 57$$

$\therefore$ $|A| \neq 0$, A^{-1} exist

Let cofactor of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$A_{11} = \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} = -1$ $\qquad$ $A_{12} = -\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} = 4$

$A_{13} = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = 2 - 12 = -10$ $\qquad$ $A_{21} = \begin{vmatrix} -3 & -7 \\ 1 & 0 \end{vmatrix} = -(7) = -7$

$A_{22} = \begin{vmatrix} 1 & -7 \\ 4 & 0 \end{vmatrix} = 28$ $\qquad$ $A_{23} = \begin{vmatrix} 1 & -3 \\ 4 & 1 \end{vmatrix} = -(1 + 12) - 13$

$A_{31} = \begin{vmatrix} -3 & -7 \\ 3 & 1 \end{vmatrix} = -3 + 21 = 18$ $\qquad$ $A_{32} = -\begin{vmatrix} 1 & -7 \\ 2 & 1 \end{vmatrix} = -(1 + 14) = -15$

$A_{33} = \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 3 + 6 = 9$

$\therefore$ Cofactor of $A = \begin{bmatrix} -1 & 4 & -10 \\ -7 & 28 & -13 \\ 18 & -15 & 9 \end{bmatrix}$

Adj of $A = \begin{bmatrix} -1 & 4 & -10 \\ -7 & 28 & -13 \\ 18 & -15 & 9 \end{bmatrix}^T = \begin{bmatrix} -1 & -7 & 18 \\ 4 & 28 & -15 \\ -10 & -13 & 9 \end{bmatrix}$

Now, the solution is given by,

$x = A^{-1} B$

$$= \frac{1}{57}\begin{bmatrix} -1 & -7 & 18 \\ 4 & 28 & -15 \\ -10 & -13 & 9 \end{bmatrix}\begin{bmatrix} 6 \\ 9 \\ 7 \end{bmatrix}$$

$$= \frac{1}{57}\begin{bmatrix} 57 \\ 171 \\ -114 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$\therefore$ x = 1, y = 3, z = –2

e. The matrix equation of system is AX = B.

where, $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 7 \\ 3 \\ 8 \end{bmatrix}$

Now,

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix}$$

$= (-2 - 1) - 2(4 - 3) + 1(2 + 3) = -3 - 2 + 5 = 0$

$\therefore$ $|A| = 0$, A^{-1} does not exist.

3. Solve the following system of equations by using Cramer's rule

a. $2x - 3y - z = 4$
$x - 2y - z = 1$
$x - y + 2z = 9$

b. $x + y + z = -1$
$3x + y + z = 1$
$4x - 2y + 2z = 0$

c. $6y + 6z = -1$
$8x + 6z = -1$
$4x + 9y = 8$

d. $x + 4y + z = 18$
$3x + 3y - 2z = 2$
$-4y + z = -7$

Solution:

a.

Coe. of x	Coe. of y	coe. of z	Constant
2	–3	–1	4
1	–2	–1	1
1	–1	2	9

$$D = \begin{vmatrix} 2 & -3 & -1 \\ 1 & -2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$= 2(-4 - 1) + 3(2 + 1) - 1(-1 + 2) = 2x - 5 + 3\times3 - 1 \times 1 = -2$

$$D_1 = \begin{vmatrix} 4 & -3 & -1 \\ 1 & -2 & -1 \\ 9 & -1 & 2 \end{vmatrix} = 4\begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 9 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 \\ 9 & -1 \end{vmatrix}$$

$= 4(-4 - 1) + 3(2 + 9) - 1(-1 + 18) = -4$

$$D_2 = \begin{vmatrix} 2 & 4 & -1 \\ 1 & 1 & -1 \\ 1 & 9 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ 9 & 2 \end{vmatrix} - 4\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix}$$

$= 2(2 + 9) - 4(2 + 1) - 1(9 - 1) = 2$

$$D_3 = \begin{vmatrix} 2 & -3 & 4 \\ 1 & -2 & 1 \\ 1 & -1 & 9 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ -1 & 9 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix}$$

$= 2(-18+1) + 3(9 - 1) + 4(-1 + 2) = -34 + 24 + 4 = -6$

The solution is $x = \frac{D_1}{D} = \frac{-4}{-2} = 2$ $\quad y = \frac{D_1}{D} = \frac{2}{-2} = -1$ $\quad z = \frac{D_3}{D} = \frac{-6}{-2} = 3$

b. Let,

Coe. of x	Coe. of y	coe. of z	Constant
1	1	1	–1
3	1	1	1
4	–2	2	0

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \\ 4 & -2 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix}$$

$= (2 + 2) - 1(6 - 4) + (-6 - 4) = -8$

$$D_1 = \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -2 & 2 \end{vmatrix} = -1\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} -1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix}$$

$$= -1(2 + 2) -1(2 - 0) + 1(-2 - 0) = -4 - 2 - 2 = -8$$

$$D_2 = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 1 & 1 \\ 4 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix}$$

$$= 2 + (6 - 4) + (-4) = 2 + 2 - 4 = 0$$

$$D_3 = \begin{vmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 4 & -2 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} -1\begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix} -1\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix}$$

$$= (0 + 2) -1(-4) -1(-6 - 4) = 2 + 4 + 10 = 16$$

The solution is $x = x = \frac{D_1}{D} = \frac{-8}{-8} = 1$ $\quad y = \frac{D_2}{D} = \frac{0}{-8} = 0$ $\quad z = \frac{D_3}{D} = \frac{16}{-8} = -2$

c. Let,

Coe. of x	Coe. of y	coe. of z	Constant
0	6	6	–1
8	0	6	–1
4	9	0	8

$$D = \begin{vmatrix} 0 & 6 & 6 \\ 8 & 0 & 6 \\ 4 & 9 & 0 \end{vmatrix} = -6\begin{vmatrix} 8 & 6 \\ 4 & 0 \end{vmatrix} + 6\begin{vmatrix} 8 & 0 \\ 4 & 9 \end{vmatrix} = -6(-24) + 6\times72 = 144 + 432 = 576$$

$$D_1 = \begin{vmatrix} -1 & 6 & 6 \\ -1 & 0 & 6 \\ 8 & 9 & 0 \end{vmatrix} = -1\begin{vmatrix} 0 & 6 \\ 9 & 0 \end{vmatrix} -6\begin{vmatrix} -1 & 6 \\ 8 & 0 \end{vmatrix} + 6\begin{vmatrix} -1 & 0 \\ 8 & 9 \end{vmatrix}$$

$$= -1(-54) -6(-48) + 6(-9) = 288$$

$$D_2 = \begin{vmatrix} 0 & -1 & 6 \\ 8 & -1 & 6 \\ 4 & 8 & 0 \end{vmatrix} = +1\begin{vmatrix} 8 & 6 \\ 4 & 0 \end{vmatrix} + 6\begin{vmatrix} 8 & -1 \\ 4 & 8 \end{vmatrix} = -24 + 6(64 + 4) = 384$$

$$D_3 = \begin{vmatrix} 0 & 6 & -1 \\ 8 & 0 & -1 \\ 4 & 9 & 8 \end{vmatrix} = -6\begin{vmatrix} 8 & -1 \\ 4 & 8 \end{vmatrix} -1\begin{vmatrix} 8 & 0 \\ 4 & 9 \end{vmatrix} = -6(64 + 4) -1(72 - 0) = -480$$

The solution is $x = \frac{D_1}{D} = \frac{288}{576} = \frac{1}{2}$ $\quad y = \frac{D_2}{D} = \frac{288}{576} = \frac{2}{3}$ $\quad z = \frac{D_3}{D} = \frac{-480}{570} = \frac{-5}{6}$

d. Let,

Coe. of x	Coe. of y	coe. of z	Constant
1	4	1	18
3	3	–2	2
0	–4	1	–7

Now, $D = \begin{vmatrix} 1 & 4 & 1 \\ 3 & 3 & -2 \\ 0 & -4 & 1 \end{vmatrix} = 1\begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} -4\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 3 \\ 0 & -4 \end{vmatrix}$

$$= (3 - 8) -4(3) + (-12) = -29$$

$$D_1 = \begin{vmatrix} 18 & 4 & 1 \\ 2 & 3 & -2 \\ -7 & -4 & 1 \end{vmatrix} = 18\begin{vmatrix} 3 & -2 \\ -4 & 1 \end{vmatrix} - 4\begin{vmatrix} 32 & -2 \\ -7 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ -7 & -4 \end{vmatrix}$$

$= 18(3 - 8) - 4(2 - 14) + 1(-8 + 21) = -29$

$$D_2 = \begin{vmatrix} 1 & 18 & 1 \\ 3 & 2 & -2 \\ 0 & -7 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & -2 \\ -7 & 1 \end{vmatrix} - 18\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 0 & -7 \end{vmatrix}$$

$= (2 - 14) - 18(3) + 1(-21) = -87$

$$D_3 = \begin{vmatrix} 1 & 4 & -18 \\ 3 & 3 & 2 \\ 0 & -4 & -7 \end{vmatrix} = 1\begin{vmatrix} 3 & 2 \\ -4 & -7 \end{vmatrix} - 4\begin{vmatrix} 3 & 2 \\ 0 & -7 \end{vmatrix} + 28\begin{vmatrix} 3 & 3 \\ 0 & -4 \end{vmatrix}$$

$= (-21 + 8) - 4(-21) + 18(-12) = -145$

The solution is $x = \frac{D_1}{D} = \frac{-29}{-29} = 1 \quad y = \frac{D_2}{D} = \frac{-87}{-29} = 3 \quad z = \frac{D_3}{D} = \frac{-145}{-29} = 5$

4. Rinav sells 7 shares of A and buys 9 shares of B, thus increasing his cash by Rs. 70, Arnav sells 9 shares of A and buys 14 shares f B, thus increasing his cash by Rs 80.
   a. Formulate the simultaneous linear system.
   b. Express the linear system in determinant form.
   c. Using Cramer's rule, find the price per share of A and B.

Solution:

Let Rs. x and Rs. y be the price per share of A and B respectively. Then, by question,

$7x - 9y = 70$

$9x - 14y = -80$

Coefficient of x	Coefficient of y	Constants
7	–9	70
9	–14	–80

Now, $D = \begin{vmatrix} 7 & -9 \\ 9 & -14 \end{vmatrix} = -98 + 81 = -17$

$D_1 = \begin{vmatrix} 70 & -9 \\ -80 & -14 \end{vmatrix} = -980 - 720 = -1700$

$D_2 = \begin{vmatrix} 7 & 70 \\ 9 & -80 \end{vmatrix} = -560 - 630 = -1190$

$\therefore x = \frac{D_1}{D} = \frac{-1700}{-17} = 100,\ y = \frac{D_2}{D} = \frac{-1190}{-17} = 70$

Hence, the price per share of A and B are Rs. 100 and Rs. 70 respectively.

5. A transport company has three types of trucks A, B and C which are designated to carry three different sizes of boxes, P, Q and R per load as shown below:

	Types of trucks		
Boxes	A	B	C
P	2	5	2
Q	3	2	5
R	1	9	0

   Each type of boxes should be used to carry exactly 18 boxes of size P, 18 boxes of size Q and 21 boxes of size R
   a. Formulate the simultaneous linear system.
   b. Express the linear system in determinant form.
   c. Using Cramer's rule, solve the linear system and find the number of truck.

Solution:

We have,

	Trucks of types			
Boxes	A	B	C	Total
P	2	5	2	18
Q	3	2	5	15
R	1	9	0	21

Let x, y and z be the total number of trucks of types A, B and C used respectively. Then by question,

$2x + 5y + 2z = 18$

$3x + 2y + 5z = 18$

$x + 9y + 0z = 21$

Coefficient of x	Coefficient of y	Coefficient of z	Constants
2	5	2	18
3	2	5	18
1	9	0	21

Now, $D = \begin{vmatrix} 2 & 5 & 2 \\ 3 & 2 & 5 \\ 1 & 9 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & 5 \\ 9 & 0 \end{vmatrix} - 5\begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ 1 & 9 \end{vmatrix}$

$= 2(0 - 45) - 5(0 - 5) + 2(27 - 2) = -90 + 25 + 50 = -15$

$D_1 = \begin{vmatrix} 18 & 5 & 2 \\ 18 & 2 & 5 \\ 21 & 9 & 0 \end{vmatrix} = 18\begin{vmatrix} 2 & 5 \\ 9 & 0 \end{vmatrix} - 5\begin{vmatrix} 18 & 5 \\ 21 & 0 \end{vmatrix} + 2\begin{vmatrix} 18 & 2 \\ 21 & 9 \end{vmatrix}$

$= 18(0 - 45) - 5(0 - 105) + 2(162 - 42) = -810 + 525 + 240 = -45$

$D_2 = \begin{vmatrix} 2 & 18 & 2 \\ 3 & 18 & 5 \\ 1 & 21 & 0 \end{vmatrix} = 1\begin{vmatrix} 18 & 2 \\ 18 & 5 \end{vmatrix} - 21\begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix}$

$= 1(90 - 36) - 21(10 - 6) = 54 - 84 = -30$

$D_3 = \begin{vmatrix} 2 & 5 & 18 \\ 3 & 2 & 18 \\ 1 & 9 & 21 \end{vmatrix} = 2\begin{vmatrix} 2 & 18 \\ 9 & 21 \end{vmatrix} - 5\begin{vmatrix} 3 & 18 \\ 1 & 21 \end{vmatrix} + 18\begin{vmatrix} 3 & 2 \\ 1 & 9 \end{vmatrix}$

$= 2(42 - 162) - 5(63 - 18) + 18(27 - 2) = -240 - 225 + 450 = -15$

$\therefore$ Using Cramer's rule, we get

$x = \frac{D_1}{D} = \frac{-45}{-15} = 3,\ y = \frac{D_2}{D} = \frac{-30}{-15} = 2,\ z = \frac{D_3}{D} = \frac{-15}{-15} = 1$

Hence, required number of trucks A, B and C used are 3, 2 and 1 respectively.

6. The price of commodities X, Y , and Z are respectively x, y and z rupees per unit Mr. A purchases 4 units of Z and ells 3 units of X and 5 units of Y. Mr. B purchases 3 units of Y and sells 2 units of X and 1 unit of Z. Mr. C purchases 1 unit of X and sells 4 units of Y and 6 units of Z. In the process, A, B and C earn zero profit. Rs. 5000 and Rs. 1300 profits respectively.
   a. Formulate the simultaneous linear system.
   b. Express the linear system in determinant form.
   c. Using Cramer's rule, find the prices per unit of the three commodities.

Solution:

Here the prices of commodities X, Y and Z are Rs. x, Rs. y and Rs. z per unit respectively. Then, by question,

$3x + 5y - 4z = 0$

$2x - 3y + z = 5000$

$-x + 4y + 6z = 13000$

Coefficient of x	Coefficient of y	Coefficient of z	Constants
3	5	–4	0
2	–3	1	5000
–1	4	6	13000

Now, $D = \begin{vmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & 4 & 6 \end{vmatrix} = 3\begin{vmatrix} -3 & 1 \\ 4 & 6 \end{vmatrix} - 5\begin{vmatrix} 2 & 1 \\ -1 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix}$

$= 3(-18 - 4) - 5(12 + 1) - 4(8 - 3) = -66 - 65 - 20 = -151$

$D_1 = \begin{vmatrix} 0 & 5 & -4 \\ 5000 & -3 & 1 \\ 13000 & 4 & 6 \end{vmatrix} = -5\begin{vmatrix} 5000 & 1 \\ 13000 & 6 \end{vmatrix} - 4\begin{vmatrix} 5000 & -3 \\ 13000 & 4 \end{vmatrix}$

$= -5(30000 - 13000) - 4(20000 + 39000) = -85000 - 236000 = -321000$

$D_2 = \begin{vmatrix} 3 & 0 & -4 \\ 2 & 5000 & 1 \\ -1 & 13000 & 6 \end{vmatrix} = 3\begin{vmatrix} 5000 & 1 \\ 13000 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 5000 \\ -1 & 13000 \end{vmatrix}$

$= 3(30000 - 13000) - 4(26000 + 5000) = 51000 - 214000 = -73000$

$D_3 = \begin{vmatrix} 3 & 5 & 0 \\ 2 & -3 & 5000 \\ -1 & 4 & 13000 \end{vmatrix} = 3\begin{vmatrix} -3 & 5000 \\ 4 & 13000 \end{vmatrix} - 5\begin{vmatrix} 2 & 5000 \\ -1 & 13000 \end{vmatrix}$

$= 3(-39000 - 20000) - 5(26000 + 5000) = -1747000 - 155000 = -332000$

Using Cranner's rule, we get

$x = \frac{D_1}{D} = \frac{-321000}{-151} = 2125.83$, $y = \frac{D_2}{D} = \frac{-73000}{-151} = 483.44$,

$z = \frac{D_3}{D} = \frac{-332000}{-151} = 2198.68$

Hence, required the prices of commodities X, Y and Z are Rs. 2125.83m Rs, 483.44 and Rs. 2198.68 respectively.

■

CHAPTER 8

INVERSE CIRCULAR FUNCTIONS

EXERCISE 8

1. Evaluate the following

a. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ b. $\operatorname{cosec}^{-1}(2)$ c. $\cot^{-1}\left(-\sqrt{3}\right)$

d. Arc tan $\left(\frac{2}{\sqrt{3}}\right)$ e. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution:

a. Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \theta$

Then, $\sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$

$\therefore \quad \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

b. Let $\operatorname{cosec}^{-1}(2) = \theta$

Then $\operatorname{cosec}\theta = 2$

$\frac{1}{\sin\theta} = 2$

$\sin\theta = \frac{1}{2}$

$\sin\theta = \sin\frac{\theta}{6} \Rightarrow \theta = \frac{\pi}{6}$

$\therefore \quad \operatorname{cosec}^{-1}(2) = \frac{\pi}{6}$

c. Let $\cot^{-1}(-\sqrt{3}) = \theta$

Then $\cot\theta = -\sqrt{3}$

$\therefore \quad \tan\theta = -\frac{1}{\sqrt{3}}$

$\tan\theta = \tan\frac{5\theta}{6}$

$\therefore \quad \theta = \frac{5\pi}{6}$

d. Let arc tan $\left(\frac{1}{\sqrt{3}}\right) = \theta$

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \theta$

$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\therefore \quad \theta = \frac{\pi}{6}$

e. Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \theta$

Then, $\sec\theta = \frac{2}{\sqrt{3}}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\cos\theta = \cos\frac{\pi}{6} \therefore \theta = \frac{\pi}{6}$

$\therefore \quad \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$

2. Evaluate:

a. $\cos\left(\tan^{-1}\frac{3}{4}\right)$ b. $\sin(\cot^{-1}x)$ c. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

d. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$ e. $\cos(2\cot^{-1}x)$ f. $\sin(2 \text{ Arc } \tan x)$

Solution:

a. $\cos\left(\tan^{-1}\frac{3}{4}\right)$

Let $\tan^{-1}\frac{3}{4} = \theta$

$\tan\theta = \frac{3}{4}$

$\Rightarrow$ $\text{Cos}\theta = \frac{4}{5}$

$\Rightarrow$ $\theta = \cos^{-1}\frac{4}{5}$

$\therefore$ $\text{Tan}^{-1}\frac{3}{4} = \cos^{-1}\frac{4}{5}$

Now,

$\cos\left(\tan^{-1}\left(\frac{3}{4}\right)\right) = \cos\left(\cos^{-1}\frac{4}{5}\right) = \frac{4}{5}$

b. $\sin(\cot^{-1} x)$

Let $\text{co}^{-1} x = \theta$

$\therefore$ $\cot\theta = x$

$\sin(\cot^{-1} x)$

$\sin\theta = \frac{p}{h} = \frac{1}{\sqrt{1 + x^2}}$

c. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

d. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \frac{2\pi}{3}$

e. $\cos(2\cot^{-1} x)$

Let $\cot^{-1}x = \theta$

$\therefore$ $\cot\theta = x$

Now, $\cos(2\cot^{-1} x)$

$= \cos 2\theta = \frac{\cot^2\theta - 1}{\cot^2\theta + 1} = \frac{x^2 - 1}{x^2 + 1}$

f. $\sin(2 \text{ arc } \tan x) = \sin(2\tan^{-1}x)$

Let $\tan^{-1}x = \theta$

$\tan\theta = x$

$\sin(2\tan^{-1}x) = \sin 2\theta$

$= \frac{2\tan\theta}{1 + \tan^2\theta} = \frac{2x}{1 + x^2}$

3. Find the value of the following.

a. $\cos\left[\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right]$

b. $\tan\left[\tan^{-1} x - \tan^{-1} 2y\right]$

c. $\sin^{-1}x - \cos^{-1}(-x)$

d. $\sin\left[\sin^{-1}\frac{4}{5} + \cot^{-1} 3\right]$

e. $\tan\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right]$

Solution:

a. $\cos\left[\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right]$

Let $\sin^{-1}\frac{4}{5} = 4$

$\therefore$ $\sin A = \frac{4}{5}$

$\Rightarrow$ $\cos A = \sqrt{1 - \sin^2 A} = \frac{3}{5}$ and $\tan^{-1}\frac{5}{12} = B$

$\therefore$ $\tan B = \frac{5}{12}$

Now, $\cos\left[\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right] = \cos(A + B) = \cos A.\ \cos B - \sin A\ .\ \sin B$

$$= \frac{3}{5}\cdot\frac{12}{13} - \frac{4}{5}\cdot\frac{5}{13} = \frac{36 - 20}{65} = \frac{16}{65}$$

b. $\tan\,[\tan^{-1} x - \tan^{-1}2y]$

$= \tan\left[\tan^{-1}\left(\frac{x - 2y}{1 + x.2y}\right)\right]\ \left[\because \tan^{-1}A\ \ \tan^{-1}B = \tan^{-1}\left(\frac{A - B}{1 + AB}\right)\right] = \frac{x - 2y}{1 + 2xy}$

c. $\sin^{-1}x - \cos^{-1}(-x)$

$\sin^{-1}x - (\pi - \cos^{-1}x) = \sin^{-1}x - \pi + \cos^{-1}x\ = \sin^{-1}x + \cos^{-1}x - \pi = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$

d. Let $\sin^{-1}\frac{4}{5} = A$

$\sin A = \frac{4}{5}$ $\qquad\qquad \therefore\ \cos A = \frac{3}{5}$

and $\cot^{-1}3 = B$

$\cot B = 3$

$\therefore\ \ \sin B = \frac{1}{\sqrt{10}}\ \cos B = \frac{3}{\sqrt{10}}$

Now, $\sin\left(\sin^{-1}\frac{4}{5} + \cot^{-1}3\right)$

$\sin(A + B)\ = \sin A\ .\ \cos B + \cos A\ .\ \sin B = \frac{4}{5}\cdot\frac{3}{\sqrt{10}} + \frac{3}{5}\cdot\frac{1}{\sqrt{10}}\ = \frac{15}{5\sqrt{10}} = \frac{3}{\sqrt{10}}$

e. Let $\cos^{-1}\frac{4}{5} = A$ $\qquad$ and $\tan^{-1}\frac{2}{3} = B$

$\therefore\ \ \cos A = \frac{4}{5}$ $\qquad\qquad \tan B = \frac{2}{3}$

$\sin A = \frac{3}{5}$ $\qquad\qquad \sin B = \frac{2}{\sqrt{13}}$ and $\cos B = \frac{3}{\sqrt{13}}$

Now, $\tan\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right]$

$$= \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\ .\ \tan B} = \frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4}\cdot\frac{2}{3}} = \frac{\frac{9 + 8}{12}}{\frac{12 - 6}{12}} = \frac{17}{6}$$

4. Prove that:

a. $\sin^{-1}(3x - 4x^3) = 3\sin^{-1}x$

b. $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1} x$

c. $\tan^{-1}2 - \tan^{-1}1 = \tan^{-1}\frac{1}{3}$

d. $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$

e. $\tan^{-1}x = \frac{1}{2}\ \sin^{-1}\frac{2x}{1+x^2}$

f. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

g. $\cot^{-1}3 + \operatorname{cosec}^{-1}\sqrt{5} = \frac{\pi}{4}$

h. $\tan^{-1}\frac{m}{n} - \tan^{-1}\frac{m - n}{m + n} = \frac{\pi}{4}$

i. $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$

j. $\tan^{-1}\frac{1}{3} + \sec^{-1}\frac{\sqrt{5}}{2} = \frac{\pi}{4}$

Solution:

a. Prove that $\sin^{-1}(3x - 4x^3) = 3\sin^{-1}x$

Let $x = \sin\theta$ then $\sin^{-1}x = \theta$

LHS $\sin^{-1}(3x - 4x^3) = \sin^{-1}(3\sin\theta - 4\sin^3\theta) = \sin(\sin 3\theta) = 3\theta = 3\theta = 3\sin^{-1}x$ RHS

b. $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x$

Let $x = \cos\theta$

$\therefore \quad \cos^{-1}x = \theta$

Taking LHS:

$\cos^{-1}(4x^3 - 3x) = \cos{-1}(4\cos^3\theta - 3\cos\theta) = \cos^{-1}(\cos^3\theta) = 3\theta = 3\cos^{-1}x$ RHS

c. $\tan^{-1}2 - \tan^{-1}1 = \tan^{-1}\left(\frac{1}{3}\right)$

LHS $\tan^{-1}2 = \tan^{-1}1 = \tan^{-1}\left(\frac{(2-1)}{1+2.1}\right)\left[\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$

$= \tan^{-1}\left(\frac{1}{3}\right)$ RHS

d. $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$ +-+

Let $\sec^{-1}x = \theta$ then $x = \sec\theta$

$x = \text{cosec}\left(\frac{\pi}{2} - \theta\right)$

$\text{cosec}^{-1}x = \frac{\pi}{2} - \theta \Rightarrow \theta + \text{cosec}^{-1}x = \frac{\pi}{2}$

$\therefore \quad \sec^{-1}x + \text{cosce}^{-1}x = \frac{\pi}{2}$

e. $\text{Tan}^{-1}x = \frac{1}{2}\sin^{-1}x\frac{2x}{1+x^2}$

Let $x = \tan\theta$

$\therefore \quad \tan^{-1}x = \theta$

Now, $\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \frac{1}{2}\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \frac{1}{2}\sin^{-1}(\sin^2\theta) = \frac{1}{2}.2\theta = \theta = \tan^{-1}x$

f. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

LHS $\left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5}\right) + \left(\tan+\frac{1}{7} + \tan^{-1}\frac{1}{8}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{15}}\right) + \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{56}}\right) = \tan^{-1}\left(\frac{8}{14}\right) + \tan^{-1}\left(\frac{15}{55}\right)$

$= \tan^{-1}\left(\frac{4}{7}\right) + \tan^{-1}\left(\frac{3}{11}\right) = \tan^{-1}\left(\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{12}{77}}\right) \Rightarrow \tan^{-1}\left(\frac{65}{65}\right) = \tan^{-1}1 = \frac{\pi}{4}$

g. $\cot^{-1}3 + \text{cosec}^{-1}\sqrt{4} = \tan^{-1}\left(\frac{1}{3}\right) + \text{cosec}^{-1}\sqrt{(5)}$ ... ... (i)

Let $\text{cosec}^{-1}\sqrt{5} = \theta$ then, $\text{cosec}\theta = \sqrt{5}$

$\therefore \quad \frac{h}{p} = \frac{\sqrt{5}}{1}$

$\therefore$ h = $\sqrt{5}$, p = 1 then b = 2

from fig.

$\tan\theta = \frac{1}{2} \therefore \theta = \tan^{-1}\frac{1}{2}$

$\operatorname{cosec}^{-1}\sqrt{5} = \tan^{-1}\left(\frac{1}{2}\right)$

from (i),

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{6}}\right) = \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}1 = \frac{\pi}{4}$$

h. $\tan^{-1}\frac{m}{n} - \tan^{-1}\left(\frac{m-n}{m+n}\right)$

$$\tan^{-1}\left(\frac{\frac{m}{n}-\frac{m-n}{m+n}}{1+\frac{m}{n}\cdot\left(\frac{m-n}{m+n}\right)}\right) = \tan^{-1}\left(\frac{\frac{m^2+mn-mn+n^2}{n(m+n)}}{\frac{mn+n^2+m^2-mn}{n(m+n)}}\right) = \tan^{-1}1 = \frac{\pi}{4}$$

i. Let $\sin^{-1}\frac{12}{13} = \theta$ and $\cos^{-1}\frac{4}{5} = \beta$

$\therefore$ $\sin\theta = \frac{12}{13}$ then $\cos\beta = \frac{4}{5}$

$\therefore$ $\tan\theta = \frac{12}{5}$ $\therefore$ $\tan\beta = \frac{3}{4}$

$\theta = \tan^{-1}\left(\frac{12}{5}\right)$ $\therefore$ $\beta = \tan^{-1}\left(\frac{3}{4}\right)$

$\therefore$ $\sin^{-1}\left(\frac{12}{13}\right) = \tan^{-1}\left(\frac{12}{5}\right)$ $\cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

Now, $\sin^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$$\tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{63}{16}\right) = \tan^{-1}\left(\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{36}{20}}\right) + \tan^{-1}\left(\frac{63}{16}\right)$$

$$= \tan^{-1}\left(\frac{63}{-16}\right) + \tan^{-1}\left(\frac{63}{16}\right) = \tan^{-1}\left(\frac{-63}{16}\right) + \tan^{-1}\left(\frac{63}{16}\right)$$

$$= -\tan^{-1}\left(\frac{63}{16}\right) + \tan^{-1}\left(\frac{63}{16}\right) \quad [\because \tan^{-1}(-x) = -\tan^{-1}(x)] = 0$$

j. $\tan^{-1}\left(\frac{1}{3}\right) + \sec^{-1}\frac{\sqrt{5}}{2}$

Let $\sec^{-1}\left(\frac{\sqrt{5}}{2}\right) = \theta$

$\therefore$ $\sec\theta = \frac{\sqrt{5}}{2}$

$\therefore$ $\frac{h}{b} = \frac{\sqrt{5}}{2} \therefore p = 1$

$\tan\theta = \frac{p}{b} = \frac{1}{2}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right)$

or, $\sec^{-1}\left(\frac{\sqrt{5}}{2}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Therefore, $\tan^{-1}\left(\frac{1}{3}\right) + \sec^{-1}\left(\frac{\sqrt{5}}{2}\right) = \tan^{-1}\left(\frac{1}{3}\right) + \sec^{-1}\left(\frac{\sqrt{5}}{2}\right)$

$= \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left(\dfrac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{6}}\right)$

$= \tan^{-1}1 = \frac{\pi}{4}$

5. Solve.

a. $\cos^{-1}x - \sin^{-1}x = 0$

b. $\sin^{-1}\frac{1}{2}x = \cos^{-1}x$

c. $\cos^{-1}x = \cos^{-1}\frac{1}{2x}$

d. $\tan^{-1}x - \cot^{-1}x = 0$

e. $\tan^{-1}\left[\frac{x-1}{x-2}\right] + \tan^{-1}\left[\frac{x+1}{x+2}\right] = \tan^{-1}1$

f. $\sin^{-1}2x - \sin^{-1}\sqrt{3}\,x = \sin^{-1}x \ (x > 0)$

g. $3\tan^{-1}\frac{1}{2+\sqrt{3}} - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}$

h. $\sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2} = 2\tan^{-1}x$

i. $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$

Solution:

a. $\cos^{-1}x = \sin^{-1}x = 0$

$\cos^{-1}x = \sin^{-1}x$

or, $\sin^{-1}\sqrt{1-x^2} = \sin^{-1}x$

or, $\sqrt{1-x^2} = x$

Squaring both sides

$1 - x^2 = x^2$

$1 = 2x^2$

$\therefore \ x = \pm\frac{1}{\sqrt{2}}$

b. $\sin^{-1}\frac{x}{2} = \cos^{-1}x$

or, $\sin^{-1}\frac{x}{2} = \sin^{-1}\sqrt{1-x^2})$

$\therefore \ \frac{x}{2} = \sqrt{1-x^2}$

Squaring both sides

or, $\frac{x^2}{4} = 1 - x^2$

or, $5x^2 = 4$

$\therefore \ x = \pm\frac{2}{\sqrt{5}}$

c. $\cos^{-1}x = \cos^{-1}\frac{1}{2x}$

$\therefore \ x = \frac{1}{2x}$

$x^2 = \frac{1}{2}$

$\therefore \ x = \pm\frac{1}{\sqrt{2}}$

d. $\tan^{-1}x - \cot^{-1}x = 0$

$\tan^{-1}x = \cot^{-1}x$

or, $\tan^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)$

$\therefore \ x = \frac{1}{x}$

$\therefore \ x = \pm 1$

e. $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \tan^{-1}1$

or, $\tan^{-1}\left\{\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right\} = \tan^{-1} 1$

or, $\tan^{-1}\left(\frac{x^2 + 2x - x - 2 + x^2 + x - 2x - 2}{x^2 - 4 - x^2 + 1}\right) = \tan^{-1}1$

$\tan{-1}\left(\frac{2x^2 - 4}{-3}\right) = \tan^{-1}1$

or, $\frac{2x^2 - 4}{-3} = 1$

or, $2x^2 - 4 = -3$

or, $2x^2 = 1$

$\therefore$ $x = \pm\frac{1}{\sqrt{2}}$

f. $\sin^{-1}2x - \sin^{-1}\sqrt{3}x = \sin^{-1}x$

or, $\sin^{-1} 2x - \sin^{-1}x = \sin^{-1}x\sqrt{3}x$

$\sin^{-1}\left\{2x \,.\, \sqrt{1-x^2} - x \,.\, \sqrt{1-4x^2}\right\} = \sin^{-1}\sqrt{3}x$

$2x\sqrt{1-x^2} - x\sqrt{1-4x^2} = \sqrt{3}\,x$

$x\left(2\sqrt{1-x^2} - \sqrt{1-4x^2}\right) = \sqrt{3}\,x$

$\therefore$ $x\left(2 - \sqrt{1-x^2} - \sqrt{1-4x^2} - \sqrt{3}\right) = 0$

Either $x = 0$ Or, $2\sqrt{1-x^2} - \sqrt{1-4x^2} - \sqrt{3} = 0$

$2\sqrt{1-x^2} - \sqrt{3} = \sqrt{1-4x^2}$

Squaring both sides, we get, $x = \frac{1}{2}$

Hence, $x = 0, \frac{1}{2}$.

g. The given equation is

$3\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right) - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}$

or, $3\tan^{-1}(2-\sqrt{3}) - \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{3}$

or, $\tan^{-1}\left\{\frac{3(2-\sqrt{3})\ \ (2-\sqrt{3}^3)}{1-3(2-\sqrt{3})^2}\right\} - \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{x}$

$\therefore$ $3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

or, $\tan^{-1}\left(\frac{12\sqrt{3} - 20}{12\sqrt{3} - 20}\right) - \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{x}$

or, $\tan^{-1}1 - \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{x}$

or, $\tan^{-1}\left(\frac{1-\frac{1}{3}}{1+1\,\frac{1}{3}}\right) = \tan^{-1}\frac{1}{x}$

or, $\tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\frac{1}{x}$

$\therefore\ \frac{1}{2} = \frac{1}{x}\ \therefore\ x = 2$

h. $\sin^{-1}\left(\frac{2a}{1+a^2}\right) - \cos^{-1}\left(\frac{1-b^2}{1+b^2}\right) = 2\tan^{-1}x$

or, $2tn^{-1}a - 2\tan^{-1}b = 2\tan^{-1}x$

or, $\tan^{-1}a - \tan^{-1}b = \tan^{-1}x$

or, $\tan^{-1}\left(\frac{a-b}{1+ab}\right) = \tan^{-1}x$

$\therefore\ x = \frac{a-b}{1+ab}$

i. $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$

or, $\tan^{-1}\left\{\frac{x+1+x-1}{1-(x+1)(x-1)}\right\} = \tan^{-1}\left(\frac{8}{31}\right)$

$\tan^{-1}\left(\frac{2x}{1-x^2+1}\right) = \tan^{-1}\left(\frac{8}{31}\right)$

$\therefore\ \frac{2x}{2-x^2} = \frac{8}{31}$

or, $8 - 4x^2 = 31x$

or, $4x^2 + 31x - 8 = 0$

or, $4x^2 + 32x - x - 8 = 0$

or, $4x(x+8) - 1(x+8) = 0$

$\therefore\ x = -8$ or $\frac{1}{4}$

6. Prove the following.

a. $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$

b. $\tan(2\tan^{-1}x) = \tan\left(\tan^{-1}\frac{2x}{1-x^2}\right) = \frac{2x}{1-x^2}$

c. $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi = 2\left(\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right)$

d. $\sin^{-1}\sqrt{\frac{x-b}{a-b}} = \cos^{-1}\sqrt{\frac{a-x}{a-b}} = \tan^{-1}\sqrt{\frac{x-b}{a-x}}$

Solution:

a. Let $x = \tan\theta$ then $2\tan^{-1}x = 2\tan^{-1}\tan\theta = 2\theta$

$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin^2\theta) = 2\theta$

$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos^2\theta) = 2\theta$

$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}\tan^2\theta) = 2\theta$

Combining the above results, we get the required result.

Hence, $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

b. $\tan(2\tan^{-1}x) = \tan\left(\tan^{-1}\frac{2x}{1-x^2}\right) = \frac{2x}{1-x^2}$

$$2\tan\left(\tan^{-1}x + \tan^{-1}x^3\right) = 2\tan\tan^{-1}\left(\frac{x + x^3}{1 - x^4}\right) = 2\,\frac{x(1 + x^2)}{(1 - x^2)\,(1 + x^2)} = \frac{2x}{1 - x^2}$$

Hence, $\text{tn}(2\tan^{-1} x) = 2\tan\left(\tan^{-1}x + \tan^{-1} x^3\right)$

c. LHS $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \tan^{-1}1 + \tan^{-1}\left(\frac{2+3}{1-6}\right) = \tan^{-1}1 + \tan^{-1}(-1) = \frac{\pi}{4} + \frac{3\pi}{4} = \pi$

RHS, $= 2\left(\tan^{-1} 1 + \tan^{-1} \text{eq}\,\frac{1}{2} + \tan^{-1}\frac{1}{3}\right) = 2\left\{\tan^{-1}1 + \tan{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}\right)\right\}$

$= 2\left\{\tan^{-1}1 + \tan^{-1}1\right\} = 2\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \pi$

Hence, LHS = RHS

d. Put $\sin^{-1}\sqrt{\frac{x-a}{a-b}} = A$ ... (i)

$\Rightarrow \sqrt{\frac{x-a}{a-b}} = \sin A$

We have,

$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{x-b}{a-b}} = \sqrt{\frac{a-b-x+b}{a-b}} = \sqrt{\frac{a-x}{a-b}}$

$\Rightarrow$ $A = \cos^{-1}\sqrt{\frac{a-x}{a-b}}$ ... (ii)

And, $\tan A = \frac{\sin A}{\cos A} = \frac{\sqrt{\frac{x-a}{a-b}}}{\sqrt{\frac{a-x}{a-b}}} = \sqrt{\frac{x-b}{a-x}}$

$\Rightarrow$ $A = \tan^{-1}\sqrt{\frac{x-b}{a-x}}$ ... (iii)

From (i), (ii) and (iii), we have,

$\sin^{-1}\sqrt{\frac{x-b}{a-b}} = \cos^{-1}\sqrt{\frac{a-x}{a-b}} = \tan^{-1}\sqrt{\frac{x-b}{a-x}}$

7. If $\sec^{-1} x = \text{cosec}^{-1} y$, prove that $\frac{1}{x^2} + \frac{1}{y^2} = 1$.

Solution:

$\text{cosec}^{-1} = \sec^{-1} y$

$\Rightarrow$ $\sin^{-1}\frac{1}{x} = \cos^{-1}\frac{1}{y}$

$\Rightarrow$ $\sin^{-1}\frac{1}{x} = \sin^{-1}\sqrt{1 - \left(\frac{1}{y}\right)^2}$

$\Rightarrow$ $\frac{1}{x} = \sqrt{1 - \left(\frac{1}{y}\right)^2}$

$\Rightarrow$ $\left(\frac{1}{x}\right)^2 = 1 - \left(\frac{1}{y}\right)^2$

$\Rightarrow \quad \frac{1}{x^2} = 1 - \frac{1}{y^2}$

$\Rightarrow \quad \frac{1}{x^2} + \frac{1}{y^2} = 1$

8. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{\pi}{2}$. Prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

Solution:

$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{\pi}{2}$

Put $\sin^{-1} x = A \Rightarrow x = \sin A \Rightarrow \cos A = \sqrt{1 - x^2}$

$\sin^{-1} y = B \Rightarrow y = \sin B \Rightarrow \cos B = \sqrt{1 - y^2}$

$\sin^{-1} z = C \Rightarrow z = \sin C$

Now,

$A + B + C = \frac{\pi}{2}$

$A + B = \frac{\pi}{2} - C$

$\Rightarrow \quad \cos(A + B) = \cos\left(\frac{\pi}{2} - C\right)$

$\Rightarrow \quad \cos A. \cos B - \sin A. \sin B = \sin C$

$\Rightarrow \quad \sqrt{1 - x^2}\ \sqrt{1 - y^2} - xy = z$

$\Rightarrow \quad \sqrt{1 - x^2}\ \sqrt{1 - y^2} = xy + z$

Squaring on both sides, we get

$(1 - x^2)(1 - y^2) = (xy + z)^2$

$\Rightarrow \quad 1 - x^2 - y^2 + x^2y^2 = x^2y^2 + 2xyz + z^2$

$\Rightarrow \quad 1 = x^2 + y^2 + z^2 + 2xyz$

$\therefore \quad x^2 + y^2 + z^2 + 2xyz = 1$

9. If $\cot^{-1} x + \cot^{-1} y + \cot^{-1} z = \pi$, show that $xy + yz + zx = 1$

Solution:

We have,

$\cot^{-1} x + \cot^{-1} y + \cot^{-1} z = \pi$

$\cot^{-1} x + \cot^{-1} y = \pi - \cot^{-1} z$

$\cot^{-1}\left(\frac{xy - 1}{x + y}\right) = \pi - \cot^{-1} z$

or, $\frac{xy - 1}{x + y} = \cot(\pi - \cot^{-1} z)$

or, $\frac{xy - 1}{x + y} = -\cot \cot^{-1} z$

or, $\frac{xy - 1}{x + y} = -z$

$xy - 1 = -xz - yz$

$\therefore \quad xy + yz + zx = 1$

10. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, show that $x + y + z = xyz$

Solution:

Given,

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$$

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \pi - \tan^{-1}z$$

or, $\dfrac{x+y}{1-xy} = \tan(\pi - \tan^{-1}z)$

or, $\dfrac{x+y}{1-xy} = -\tan\tan^{-1}z$ $\quad$ $(\because \tan(\pi-\theta) = -\tan\theta)$

$$\frac{x+y}{1-xy} = -z$$

$$x + y = -z + xyz$$

$\therefore$ $x+y+z = xyz$

11. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, prove that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$
$= 2xyz$.

Solution:

Let $\sin^{-1}x = A \Rightarrow \sin A = x$ $\quad$ $\therefore \cos A = \sqrt{1-x^2}$

$\sin^{-1}y = B \Rightarrow \sin B = y$ $\quad$ $\therefore \cos B = \sqrt{1-y^2}$

$\sin^{-1}z = C \Rightarrow \sin C = z$ $\quad$ $\therefore \cos C = \sqrt{1-z^2}$

Since,

$\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$

i.e. $A + B + C = \pi$

$\therefore$ $A + B = \pi - C$

$\therefore$ $\text{Sin}(A + B) = \sin(\pi - c) = \sin C$

$\cos(A + B) = \cos(\pi - c) = -\cos C$

Now,

Taking, LHS, $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$

$= \sin A.\cos A + \sin B.\cos B + \sin C.\cos C = \dfrac{1}{2}(\sin 2A + \sin 2B) + \sin C.\cos C$

$= \dfrac{1}{2}\, 2\sin\dfrac{2A+2B}{2}.\cos\dfrac{2A-2B}{2} + \sin C.\cos C$

$= \sin(A + B).\cos(A - B) + \sin C.\cos C$

$= \sin C.\cos(A - B) + \sin C.\cos C = \sin C\{\cos(A - B) + \cos C\}$

$= \sin C\{\cos(A - B) - \cos(A + B)\}$ $\quad$ $[\because \cos(A+B) = -\cos C]$

$= \sin C.2\sin A.\sin B = 2\sin A.\sin B.\sin C = 2\,x.y.z = 2xyz$

Hence, $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$

■

CHAPTER 9

TRIGONOMETRIC EQUATIONS AND GENERAL VALUES

EXERCISE 9

1. Find the principal solution of

a. $2\cos x - 1 = 0$ b. $\sqrt{3}\sec x = 2$ c. $\tan x = \frac{-1}{\sqrt{3}}$ d. $\sin x = \frac{1}{\sqrt{2}}$

Solution:

a. $2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$\cos x = \cos\frac{\pi}{3}$

$\therefore\ x = \frac{\pi}{3}, \frac{5\pi}{3}$

b. $\sqrt{3}\sec x = 2$

$\sec x = \frac{1}{\sqrt{3}}$

$\cos x = \frac{\sqrt{3}}{2}$

$\therefore\ \cos x = \cos\frac{\pi}{6}, \cos\left(2\pi - \frac{\pi}{6}\right)$

$\therefore\ x = \frac{\pi}{6}, \frac{11\pi}{6}$

c. $\tan x = -\frac{1}{\sqrt{3}}$

$\tan x = \tan\left(\pi - \frac{\pi}{6}\right), \tan\left(2\pi - \frac{\pi}{6}\right)$

$\tan x = \tan\frac{5\pi}{6}, \tan\frac{11\pi}{6}\ \therefore\ x = \frac{5\pi}{6}, \frac{11\pi}{6}$

d. $\sin x = \frac{1}{\sqrt{2}}$

$\sin x = \sin\frac{\pi}{4}, \sin\left(\pi - \frac{\pi}{4}\right)$

$\sin x = \sin\frac{\pi}{4}, \sin\frac{3\pi}{4}\ \therefore\ x = \frac{\pi}{4}, \frac{3\pi}{4}$

2. Find the general solution of

a. $\cos^2 x = \frac{1}{2}$ b. $\cos 3x = \frac{-1}{\sqrt{2}}$

c. $\cos 3x = \sin 2x$ d. $\tan^2 x = \frac{1}{3}$

Solution:

a. $\cos^2 x = \frac{1}{2}$

$\cos^2 x = \cos^2\frac{\pi}{4}$

$\therefore\ x = n\pi \pm \frac{\pi}{4}$

(Since $\cos^2\theta = \cos^2\propto \Rightarrow \theta = n\pi \pm \propto$)

b. $\cos 3x = -\frac{1}{\sqrt{2}}$

$\cos 3x = \cos\frac{3\pi}{4}$

$\therefore$ The general solution is

$3x = 2n\pi \pm \frac{3\pi}{4} \Rightarrow x = \frac{2n\pi}{3} \pm \frac{\pi}{4}$

c. $\cos 3x = \sin 2x$

$\cos 3x = \cos\left(\frac{\pi}{2} - 2x\right)$

$\therefore\ 3x = 2n\pi \pm \left(\frac{\pi}{2} - 2x\right)$ ($\because \cos\theta = \cos\propto \Rightarrow \theta = 2n\pi \pm \propto$)

$3x = 2n\pi + \frac{\pi}{2} - 2x$ or, $3x = 2n\pi - \frac{\pi}{2} + 2x$

$5x = 2x\pi + \frac{\pi}{2}$ $x = 2n\pi - \frac{\pi}{2}$

$5x = (4n + 1)\frac{\pi}{2}$ $\therefore$ $x = (4n - 1)\frac{\pi}{2}$

$5x = (4n + 1)\frac{\pi}{2}$

$\therefore$ $x = (4x + 1)\frac{\pi}{10}$

Hence, $x = (4n + 1)\frac{\pi}{10}, (4n - 1)\frac{\pi}{2}$

d. $\tan^2x = \frac{1}{3}$

$\tan^2x = \left(\frac{1}{\sqrt{3}}\right)^2$

or, $\tan^2x = \left(\tan\frac{\pi}{6}\right)^2$

$\therefore$ $x = n\pi \pm \frac{\pi}{6}$ ($\because \tan^2\theta = \tan^2\propto \Rightarrow \theta = n\pi \pm \propto$)

3. Find the general solution of the following

a. $\sin 2x + \cos x = 0$

b. $\tan^3 x - 3\tan x = 0$

c. $\sin ax + \cos bx = 0$

d. $\tan x + \cot x = 2$

Solution:

a. $\sin2x + \cos x = 0$

or, $2\sin x.\cos x + \cos x = 0$

or, $\cos x(2\sin x + 1) = 0$

Either $\cos x = 0$

or, $\sin x = -\frac{1}{2}$

$\therefore$ $x = (2x + 1)\frac{\pi}{2}$

$\sin x = \sin\left(-\frac{\pi}{6}\right)$

$\therefore$ $x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$

$\therefore$ $x = (2n + 1)\frac{\pi}{2}, n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$

b. $\tan^3x = 3\tan x = 0$

or, $\tan x(\tan^2x - 3) = 0$

Either $\tan x = 0$ or, $\tan^2x - 3 = 0$

$\therefore$ $x = n\pi$ $\tan^2x = (\sqrt{3})^2$

$\tan^2x = \tan^2\frac{\pi}{3}$

$\therefore$ $x = n\pi \pm \frac{\pi}{3}$

$\therefore$ $x = n\pi, n\pi \pm \frac{\pi}{3}$

c. $\text{Sin}ax + \cos bx = 0$

or, $-\sin ax = \cos bx$

$\cos bx = \cos\left(\frac{\pi}{2} + ax\right)$

$\therefore$ $bx = 2ns\pi \pm \left(\frac{\pi}{2} + ax\right)$ ($\because \cos\theta = \cos\propto \Rightarrow \theta = 2n\pi \pm \propto \forall n\in z$)

Taking positive sign Taking negative sign

$bx = 2n\pi + \frac{\pi}{2} + ax$

$(b–a)\ x = 2n\pi + \frac{\pi}{2}$

$\therefore \quad x = 1(4n + 1)\frac{\pi}{2}$

$bx = 2n\pi - \frac{\pi}{2} - ax$

$(b + a)x = 2n\pi - \frac{\pi}{2}$

$x = \frac{1}{(b+a)}(4n - 1)\frac{\pi}{2}$

Hence, $x = \frac{(4n + 1)}{b - a}\frac{\pi}{2}, \frac{(4n - 1)}{b+a}\frac{\pi}{2}$

d. $\tan x + \cot x = 2$

or, $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = 2$

or, $\sin^2 x + \cos^2 x = 2\sin x \cdot \cos x$

$1 = \sin 2x$

$\therefore \quad \sin 2x = \sin\frac{\pi}{2}$

$\therefore \quad 2x = n\pi \pm (-1)^n\frac{\pi}{2}$ $(\because \sin\theta = \sin\propto\ \theta = n\pi \pm (-1)^n \propto, \forall n)$

$x = \frac{n\pi}{2} + (-1)^n\frac{\pi}{4}$

4. Solve the following equations for general solution.

a. $4\cos^2 x + 6\sin^2 x = 5$

b. $\cos^2 x - \sin^2 x + \cos x = 0$

c. $3\cos^2 x + 5\sin^2 x = 4$

d. $4\sin^2 x - 8\cos x + 1 = 0$

Solution:

a. $4\cos^2 x = 6\sin^2 x = 5$

$4 - 4\sin^2 x + 6\sin^2 x = 5$

$2\sin^2 x = 1$

$\sin^2 x = \left(\frac{1}{\sqrt{2}}\right)^2$

$\sin^2 x = \sin^2\frac{\pi}{4}$

$\therefore \quad x = n\pi \pm \frac{\pi}{4}$

$(\because \sin^2\theta = \sin^2\propto \Rightarrow \theta = n\pi \pm \propto\ \forall n \in z)$

b. $\cos^2 x - \sin^2 x + \cos x = 0$

$\cos^2 x - 1 + \cos^2 x + \cos x = 0$

or, $2\cos^2 x + \cos x - 1 = 0$

or, $2\cos^2 x + 2\cos x - \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

either $2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$\therefore \quad \cos x = \cos\frac{\pi}{3}$ $\therefore\ x = 2n\pi \pm \frac{\pi}{3}$

or, $\cos x + 1 = 0$

$\cos x = -1$

$\cos x = \cos\pi$

$\therefore \quad x = 2n\pi \pm \pi$

c. $3\cos^2 x + 5\sin^2 x = 4$

or, $3 – 3\sin^2 x + \sin^2 x = 4$

$2\sin^2 x = 1$

$\therefore \quad \sin^2 x = \left(\frac{1}{\sqrt{2}}\right)^2$

$\sin^2 x = \sin^2\frac{\pi}{4}$

$\therefore \quad x = n\pi \pm \frac{\pi}{4}$

d. $4\sin^2 x - 8\cos x + 1 = 0$

$4 - 4\cos^2 x - 8\cos x + 1 = 0$

or, $4\cos^2 x + 8\cos x - 5 = 0$

or, $4\cos^2 x + 10\cos x - 2\cos x - 5 = 0$

or, $2\cos x(2\cos x + 5) - 1(2\cos x + 5) = 0$

$(2\cos x - 1)(2\cos x + 5) = 0$

Either $2\cos x - 1 = 0$ or, $\cos x = -\frac{5}{2}$

$\cos x = \frac{1}{2}$

$\cos x = \cos\frac{\pi}{3}$ $\therefore\ x = 2n\pi \pm \frac{\pi}{3}$

5. Solve:

a. $\cos x + \cos 2x + \cos 3x = 0$ b. $\sin 3x + \sin x = \sin 2x$

c. $\cos 3x + \cos x = \cos 2x$ d. $2 \tan x - \cot x = -1$

Solution:

a. $\cos x + \cos 2x + \cos 3x = 0$

$(\cos x + \cos 3x) + \cos 2x = 0$

or, $2\cos\left(\frac{x + 3x}{2}\right) . \cos\left(\frac{3x - x}{2}\right) + \cos 2x = 0$

$2\cos 2x . \cos x + \cos 2x = 0$

or, $\cos 2x\,(2\cos x + 1) = 0$

Either $\cos 2x = 0$ or, $2\cos x+1 = 0$

$\cos 2x = \cos\frac{\pi}{2}$ $\cos x = -\frac{1}{2}$

$\therefore\ 2x = (2n + 1)\frac{\pi}{2}$ $\cos x = \cos\frac{2\pi}{3}$

$\therefore\ x = (2n+1)\frac{\pi}{4}$ $\therefore\ x = 2n\pi \pm \frac{2\pi}{3} = (6n \pm 2)\frac{\pi}{3}$

b. $\text{Sin}3x + \sin x = \sin^2 x$

$2\sin\left(\frac{3x + x}{2}\right) . \cos\left(\frac{3x - 3}{2}\right) = \sin^2 x$

or, $2 \sin 2x . \cos x - \sin^2 x = 0$

$\sin 2x\,(2\cos x - 1) = 0$

Either or, $2 \cos x - 1 = 0$

or, $\sin 2x = 0$ or, $\cos x = \frac{1}{2}$

or, $2x = n\pi$ or, $\text{cox} = \cos\frac{\pi}{3}$

$\therefore\ x = \frac{n\pi}{3}$ $\therefore\ x = 2n\pi \pm \frac{\pi}{3} = (6n \pm 1)\frac{\pi}{3}$

$\therefore\ x = \frac{n\pi}{2}, 2n\pi \pm \frac{\pi}{3}$

c. $\cos 3x + \cos x = \cos 2x$

or, $2\cos 2x . \cos x = \cos 2x$

$\cos 2x\,(2 \cos x - 1) = 0$

Either $\cos 3x = 0$ or, $2\cos x - 1 = 0$

$\therefore\ 2x = (2n + 1)\frac{\pi}{2}$ or, $\cos x = \frac{1}{2}$

$\therefore\ x = (2n + 1)\frac{\pi}{4}$ or, $\cos x = \cos\frac{\pi}{3}$ $\therefore\ x = 2n\pi \pm \frac{\pi}{3}$

Hence, the general solution $x = (2n + 1)\frac{\pi}{4}, (6n \pm 1)\frac{\pi}{3}$

d. $2\tan x - \cot x = -1$

$2\tan x - \frac{1}{\tan x} = -1$

or, $2\tan^2 x - 1 = -\tan x$

or, $2\tan^2 x + \tan x - 1 = 0$

$2\tan^2 x + 2\tan x - \tan x - 1 = 0$

or, $2\tan x\,(\tan x + 1) - 1(\tan x + 1) = 0$

$(\tan x + 1)\,(2\tan x - 1) = 0$

Either tanx + 1 = 0

$\tan x = -1$

$\tan x = \tan\frac{3\pi}{4}$ or, $\tan\left(-\frac{\pi}{4}\right)$

$\therefore$ $x = n\pi + \frac{3\pi}{4}$ or, $n\pi - \frac{\pi}{4}$

or, $2\tan x - 1 = 0$

or, $\tan x = \frac{1}{2}$

or, $x = \tan^{-1}\frac{1}{2}$ $\therefore$ $x = n\pi + \tan^{-1}\frac{1}{2}$

Hence, the general solution are

$x = n\pi - \frac{\pi}{4}, n\pi + \tan^{-1}\frac{1}{2}$

6. Solve:

a. $\sqrt{3}\sin x - \cos x = \sqrt{2}$

b. $\tan x + \sec x = \sqrt{3}$

c. $\sin x + \sqrt{3}\cos x = \sqrt{2}$

d. $\sqrt{2}\sec x + \tan x = 1$

e. $\sin x + \cos x = \frac{-1}{\sqrt{2}}$

Solution:

a. $\sqrt{3}\sin x - \cos x = \sqrt{2}$

Dividing each term by 2

$\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = \frac{1}{\sqrt{2}}$

$\sin\frac{\pi}{3}\sin x - \cos\frac{\pi}{3}\cos x = \frac{1}{\sqrt{2}}$

$-\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$

or, $\cos\left(x + \frac{\pi}{3}\right) = -\frac{1}{\sqrt{2}}$

or, $\cos\left(x + \frac{\pi}{3}\right) = \cos\left(\frac{3\pi}{4}\right)$

or, $x + \frac{\pi}{3} = 2n\pi \pm \frac{3\pi}{4}$ $\therefore$ $x = 2n\pi - \frac{\pi}{3} \pm \frac{3\pi}{4}$

or, $\cos\frac{\pi}{6}\sin x - \sin\frac{\pi}{6}\cos x = \frac{1}{\sqrt{2}}$

or, $\sin\left(x - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$

or, $\sin\left(x - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}$

$\therefore$ $x - \frac{\pi}{6} = n\pi \pm (-1)^n\frac{\pi}{4}$

$\therefore$ $x = n\pi + \frac{\pi}{6} \pm (-1)^n\frac{\pi}{4}$

b. $\tan x + \sec x = \sqrt{3}$

$\frac{\sin x}{\cos x} + \frac{1}{\cos x} = \sqrt{3}$

or, $\sin x + 1 = \sqrt{3}\cos x$

or, $\sqrt{3}\cos x - \sin x = 1$ ... ... (i)

Dividing (i) by $\sqrt{\sqrt{3}^2 + (-1)^2} = 2$

$\therefore$ $\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$

or, $\cos\left(x + \frac{\pi}{6}\right) = \cos\left(2x\ \pi \pm \frac{\pi}{3}\right)$

$\Rightarrow$ $x = 2n\pi + \frac{\pi}{6}, 2n\pi - \frac{\pi}{2}$

$\sin x = \sin \frac{3\pi}{2} \therefore x = n\pi \pm (-1)^n$

$\frac{3\pi}{2}$

or, $2\sin x = 1$

or, $\sin x = \sin\frac{\pi}{6} \;\therefore x = n\pi \pm (-1)^n \frac{\pi}{6}$

c. $\text{Sinx} + \sqrt{3}\cos x = \sqrt{2}$

Dividing both sides by

$\sqrt{(\text{coeff. of sim})^2 + (\text{coeff. of cosx})^2}$

$= \sqrt{1 + + (\sqrt{3})^2 = 2}$

$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{1}{\sqrt{2}}$

or, $\sin x.\ \sin\frac{\pi}{6} + \cos x\ .\ \cos\frac{\pi}{6} = \cos\frac{\pi}{4}$

$\cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}$

$\therefore\ x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}$

$\therefore\ x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$

d. $\sqrt{2}\sec x + \tan x = 1$

$\sqrt{2} + \sin x = \cos x$

or, $\cos x - \sin x = \sqrt{2}$

Dividing both sides by $\sqrt{2}$

$\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = 1$

$\cos\frac{\pi}{4}\ .\ \cos x - \sin\frac{\pi}{4}\ .\ \sin x = \cos 0$

$\therefore\ \cos\left(x + \frac{\pi}{4}\right) = \cos 0^o$

$\therefore\ x + \frac{\pi}{4} = 2n\pi \pm 0$

$x = 2n\pi - \frac{\pi}{4}$

e. $\text{Sinx} + \cos x = -\frac{1}{\sqrt{2}}$

Dividing both sides by $\sqrt{2}$

$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = -\frac{1}{2}$

$\cos\left(x - \frac{\pi}{4}\right) = \cos\left(\frac{2\pi}{3}\right)$

or, $x - \frac{\pi}{4} = 2n\pi \pm \frac{2\pi}{3} \;\therefore x = 2n\pi \pm \frac{2\pi}{3} + \frac{\pi}{4}$

7. Solve:

a. $\sin 2x + \sin 4x + \sin 6x = 0$

b. $\sin x + \sin 3x + \sin 5x = 0$

c. $\cos 3x + \cos x - \cos 2x = 0$

d. $\cos x + \sin x = \cos 2x + \sin 2x$

e. $\tan x + \tan 2x = 1 - \tan x.\ \tan 2x$

f. $\tan x + \tan 2x + \sqrt{3}\tan x.\ \tan 2x = \sqrt{3}$

g. $\tan 3x + \tan 4x + \tan 7x = \tan 3x.\ \tan 4x.\ \tan 7x$

Solution:

a. $\sin 2x + \sin 4x + \sin 6x = 0$

or, $(\sin 2x + \sin 6x) + \sin 6x = 0$

or, $2\sin 4x\ .\ \cos 2x + \sin 4x = 0$

$\sin 4x\,(2\cos 2x + 1) = 0$

Either, $\sin 4x = 0$

$\therefore\ 4x = n\pi$

$\therefore\ x = \frac{n\pi}{4}$

or, $2\cos 2x + 1 = 0$

or, $\cos 2x = -\frac{1}{2}$

or, $\cos 2x = \cos\left(\frac{2\pi}{3}\right)$

or, $2x = 2n\pi \pm \frac{2\pi}{3} \;\therefore x = n\pi \pm \frac{\pi}{3}$

$\therefore \quad x = \frac{n\pi}{4}, n\pi \pm \frac{\pi}{3}$

b. (sinx + sin5x) + sin3x = 0

or, 1sin3x . cos2x + sin3x = 0

or, sin3x (2cos2x + 1) = 0

Either

or, sin3x = 0

$\therefore \quad 3x = n\pi$

$\therefore \quad x = \frac{n\pi}{3}$

or, 2cos2x + 1 = 0

or, $\cos 2x = -\frac{1}{2}$

or, $\cos 2x = -\frac{1}{2}$

or, $\cos 2x = \cos\left(\frac{2\pi}{3}\right)$

or, $2x = 2n\pi \pm \frac{2\pi}{3} \quad \therefore x = n\pi \pm \frac{\pi}{3}$

Hence, $x = \frac{n\pi}{3}, n\pi \pm \frac{\pi}{3}$

c. cos3x + cosx – cos2x = 0

or, 2cos2x . cosx – cos2x = 0

or, cos2x (2cosx – 1) = 0

Either cos2x = 0

$\therefore \quad 2x = (2n + 1)\frac{\pi}{2}$

$\therefore \quad x = (2n + 1)\frac{\pi}{4}$

or, 2cosx – 1 = 0

or, $\cos x = \frac{1}{2}$

or, $\cos x = \cos\frac{\pi}{3} \quad \therefore x = 2n\pi \pm \frac{\pi}{3}$

Hence, $x = (2n + 1)\frac{\pi}{4}, 2n\pi \pm \frac{\pi}{3}$

d. cosx + sinx = cos2x + sin2x

or, $\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = \frac{1}{\sqrt{2}}\cos 2x + \frac{1}{\sqrt{2}}\sin 2x$

or, $\cos\frac{\pi}{4} . \cos x + \sin\frac{\pi}{4}\sin x = \cos\frac{\pi}{4}\cos 2x + \sin\frac{\pi}{4} . \sin 2x$

or, $\cos\left(x - \frac{\pi}{4}\right) = \cos\left(2x - \frac{\pi}{4}\right)$

$\therefore \quad 2x - \frac{\pi}{4} = 2n\pi \pm \left(x - \frac{\pi}{4}\right)$

$$2x - \frac{\pi}{4} = \begin{cases} 2n\pi + x - \frac{\pi}{4} \\ 2n\pi - x + \frac{\pi}{4} \end{cases}$$

Either $x = 2n\pi$

or, $3x = 2n\pi + \frac{\pi}{2}$

i.e. $x = \frac{2n\pi}{3} + \frac{\pi}{6} = (4n + 1)\frac{\pi}{6}$

Hence, $x = 2n\pi, (4n + 1)\frac{\pi}{6}$

e. tanx + tan2x = 1 – tanx . tan2x

or, $\frac{\tan 2x + \tan x}{1 - \tan 2x . \tan x} = 1$

or, tan(2x + x) = 1

f. $\tan x + \tan 2x + \sqrt{3}\tan x . \tan 2x = \sqrt{3}$

or, $\tan x + \tan 2x = \sqrt{3}(1 - \tan x . \tan 2x)$

or, $\tan3x = \tan\frac{\pi}{4}$

$\therefore$ $3x = n\pi + \frac{\pi}{4}$

($\because \tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$)

$\therefore$ $x = \frac{n\pi}{3} + \frac{\pi}{12}$

or, $\frac{\tan2x + \tan x}{1 - \tan2x.\ \tan x} = \sqrt{3}$

or, $\tan(2x + x) = \sqrt{3}$

or, $\tan3x = \tan\left(\frac{\pi}{3}\right)$

or, $3x = n\pi + \frac{\pi}{3}$

$\therefore$ $x = \frac{n\pi}{3} + \frac{\pi}{9}$

g. We have

Tan3x + tan4x + tan7x

= tan3x . tan4x . tan7x

$\Rightarrow$ Tan3x + tan4x = –tan7x + tan3x tan4x . tan7x

$\Rightarrow$ tan3x + tan4x = –tan7x (1–tan3x.tan4x)

$\Rightarrow$ $\frac{\tan3x + \tan4x}{1 - \tan3x\ .\ \tan4x - \tan7x}$

$\Rightarrow$ tan(3x + 4x) = –tan7x . tan7x + tan7x = 0

$\Rightarrow$ 2tan7x = 0

$\Rightarrow$ tan7x = 0 = tan0

$\Rightarrow$ $7x = n\pi + 0$

$\therefore$ $x = \frac{n\pi}{7}$

8. Solve:

a. $\tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right) = 4$

b. $2\sin^2 x + \sin^2 2x = 2$

c. $\tan px = \cot qx$

Solution:

a. $\tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right) = 4$

or, $\dfrac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\ .\ \tan\theta} + \dfrac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\ .\ \tan\theta} = 4$

or, $\frac{1 + \tan\theta}{1 - \tan\theta} + \frac{1 - \tan\theta}{1 + \tan\theta} = 4$

or, $(1 + \tan\theta)^2 + (1 - \tan\theta)^2 = 4(1 + \tan\theta)(1 - \tan\theta)$

or, $1 + 2\tan\theta + \tan^2\theta + 1 - 2\tan\theta + \tan^2\theta = 4 - 4\tan^2\theta$

or, $6\tan^2\theta = 2$

or, $\tan^2\theta = \left(\frac{1}{\sqrt{3}}\right)^2$

$\therefore$ $\tan^2\theta = \tan2\left(\frac{\pi}{6}\right)$ $\therefore$ $\theta = n\pi \pm \frac{\pi}{6}$, $n \in z$

b. $2\sin^2x + \sin^2 2x = 2$

or, $2\sin^2x + 4\sin^2x \cdot \cos^2x - 2 = 0$

or, $2\sin^2x + 4\sin^2x(1-\sin^2x) - 2 = 0$

or, $2\sin^2x + 4\sin^2x\ 4\sin^4x - 2 = 0$

or, $-4\sin^4x - 6\sin^2x - 2 = 0$

or, $2\sin^4x - 3\sin^2x + 1 = 0$

$(\sin^2x - 1)(2\sin^2x - 1) = 0$

Either

$\sin^2x - 1 = 0$ or, $2\sin^2x - 1 = 0$

$\sin^2x = 1$ $\sin^2x = \left(\frac{1}{\sqrt{2}}\right)^2$

$\sin^2x = \sin^2\frac{\pi}{2}$ $\sin^2x = \sin^2\left(\frac{\pi}{4}\right)$

$\therefore$ $x = n\pi \pm \frac{\pi}{2}$ $\therefore$ $x = n\pi \pm \frac{\pi}{4}$

Hence, $x = n\pi \pm \frac{\pi}{2}, n\pi \pm \frac{\pi}{4}$

c. $\tan px = \cot qx$

$\frac{\sin px}{\cos px} = \frac{\cos qx}{\sin qx}$

or, $\cos px. \cos qx - \sin px . \sin qx = 0$

$\cos(px + qx) = 0$

or, $\cos(p + q)x = 0$

$\therefore$ $(p + q)x = (2n + 1)\frac{\pi}{2}$

$\therefore$ $x = \frac{(2n + 1)}{p + q} \cdot \frac{\pi}{2}$

9. Find the solutions of

a. $\tan 2x = \tan x$ $(-\pi \le x \le \pi)$

b. $\tan x - 3 \cot x = 2 \tan 3x$ for $0 \le x \le 2\pi$

Solution:

a. $\tan^2x = \tan x$ $(-\pi \le x \le \pi)$

or, $\frac{2\tan x}{1 - \tan^2x} = \tan x$

or, $2\tan x - \tan x(1 - \tan^2x) = 0$

or, $\tan x(2 - 1 + \tan^2x) = 0$

or, $\tan x(1 + \tan^2x) = 0$

Either $\tan x = 0$

$\tan x = \tan 0^o, \tan\pi, \tan(-\pi)$

$\therefore$ $x = 0^o, \pi, -\pi$

b. $\tan x - 3\cot x = 2\tan^3x$ $(0 \le x \le 2\pi)$

$\tan x - \frac{3}{\tan x} = 2\left(\frac{3\tan x - \tan^3x}{1 - 3\tan^2x}\right)$

or, $\frac{\tan^2x - 3}{\tan x} = \frac{6\tan x - 2\tan^3x}{1 - 3\tan^2x}$

or, $\tan^2x - 3\tan^4x - 3 + 9\tan^2x = 6\tan^2x - 2\tan^4x$

or, $\tan^3x - 4\tan^2x + 3 = 0$

or, $\tan^4x - 3\tan^2x - \tan^2x + 3 = 0$

or, $\tan^2x(\tan^2x - 3) - 1(\tan^2x - 3) = 0$

or, $(\tan^2x - 1)(\tan^2x - 3) = 0$

Either, $\tan^2x - 1 = 0$

$\Rightarrow$ $\tan^2x = 1$

or, $\tan x = \pm 1$

or, $\tan x = \tan\frac{\pi}{4}, \tan\frac{3\pi}{4}, \tan\frac{5\pi}{4}, \tan\frac{7\pi}{4}$

$\therefore$ $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

or, $\tan^2x = 3$

$\therefore$ $\tan x = \pm\sqrt{3}$

tanx = tan $\frac{\pi}{3}$, tan$\frac{2\pi}{3}$, tan$\frac{4\pi}{4}$, tan$\frac{5\pi}{3}$

$\therefore$ x = $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$

Hence, x = $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, $\frac{5\pi}{3}$

10. Find the solutions of the equation (general solution not required)
cot x + cot y = 2 and 2 sin x. sin y = 1

Solution:

Given equations

cotx + coty = 2 ... ... (i) and 2sinx. siny = 1 ... ... (ii)

$\Rightarrow$ $\frac{\text{cosx . siny + sinx cosy}}{\text{sinx . siny}} = 2$

or, sinx cosy + cosx. siny = 2sinx . siny

or, sin(x + y) = 1 using (ii)

or, sin (x + y) = sin 90°

$\therefore$ x + y = 90° ... ... (iii)

Also, 2sinx . siny = 1

or, cos(x – y) = cos (x + y) = 1

or, cos (x – y) = cos 90° = 1

or, cos (x – y) = 0 = 1

or, cos (x – y) = 1

or, cos (x – y) = cos0°

$\therefore$ x – y = 0° ... ... (iv)

Solving (iii) and (iv) we get

x = 45° = $\frac{\pi}{4}$ and y = 45° = $\frac{\pi}{4}$

$\therefore$ x = $\frac{\pi}{4}$, y = $\frac{\pi}{4}$

■

CHAPTER 10

CONIC SECTION

EXERCISE 10.1

1. Find the eccentricity, co-ordinates of the vertices and foci, the length of the latus rectum, major axis and minor axis of the following ellipses.

a. $\frac{x^2}{16} + \frac{y^2}{4} = 1$ b. $\frac{x^2}{9} + \frac{y^2}{25} = 1$

c. $3x^2 + 4y^2 = 36$ d. $5x^2 + 9y^2 = 45$

e. $5x^2 + 4y^2 = 1$

Solution:

a. $\frac{x^2}{16} + \frac{y^2}{4} = 1$ … … … (1)

Comparing (1) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 16$, $b^2 = 4$

$\therefore$ a = 4, b = 2

Now, eccentricity (e) = $\sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{16}} = \frac{\sqrt{3}}{2}$

Co-ordinate of vertices = $(\pm a, 0) = (\pm 4, 0)$

Co-ordinate of foci = $(\pm ae, 0) = \left(14 . \sqrt{32}, 0\right) = (\pm 2\sqrt{3}, 0)$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2 \times 4}{4} = 2$

Major axis = 2a = 2×4 = 8

Minor axis = 2b = 2×2 = 4

b. $\frac{x^2}{9} + \frac{y^2}{25} = 1$ … … … (1)

Compare (1) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 9$, $b^2 = 25$

$\therefore$ a = 3, b = 5

Now, eccentricity (e) = $\sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

Co–ordinate of vertices = $(0, \pm b) = (0, \pm 5)$

Co-ordinate of foci = $(0, \pm be) = \left(0, \pm \times \frac{4}{5}\right) = (0, \pm 4)$

Length of latus rectum = $\frac{2a^2}{b} = \frac{2 \times 9}{5} = \frac{18}{5}$

Major axis = 2b = 2×5 = 10

Minor axis = 2a = 2×3 = 6

c. $3x^2 + 4y^2 = 36$

or, $\frac{3x^2}{36} + \frac{4y^2}{36} = 1$

or, $\frac{x^2}{12} + \frac{y^2}{9} = 1$ … … … (1)

Comparing (1) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 12, b^2 = 9$

$\therefore \quad a = 2\sqrt{3}, b = 3$

Now, eccentricity (e) $= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{12}} = \frac{\sqrt{3}}{\sqrt{12}} = \frac{1}{2}$

Co–ordinate of vertices = $(\pm a, 0) = (\pm 2\sqrt{3}, 0)$

Co–ordinate of foci = $(\pm ae, 0) = (\pm 2\sqrt{3}\,\frac{1}{2}, 0) = (\pm\sqrt{3}, 0)$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2 \times 9}{2\sqrt{3}} = 3\sqrt{3}$

Major axis $= 2a = 2\times2\sqrt{3} = 4\sqrt{3}$

Minor axis $= 2b = 2\times3 = 6$

d. $5x^2 + 9y^2 = 45$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{5} = 1$

Here, $a^2 = 9, b^2 = 5$

Now,

Eccentricity (e) $= \sqrt{1 - b^2/a^2} = \sqrt{1 - 5/9} = \frac{2}{3}$

Vertices $= (\pm 3, 0)$

Foci $= (\pm 3 \times \frac{2}{3}, 0)$

$= (\pm 2, 0)$

Lotus rectum $= \frac{2b^2}{a} = \frac{2\times5}{3} = \frac{10}{3}$

Major axis $= 2a = 2\times3 = 6$

Minor axis $= 2b = 2\times\sqrt{5} = 2\sqrt{5}$

e. We have,

$5x^2 + 4y^2 = 1$

$\Rightarrow \frac{x^2}{(1/5)} + \frac{y^2}{(1/4)} = 1$

Here, $a^2 = \frac{1}{5}$ and $b^2 = \frac{1}{4}$

$\therefore \quad b > a > 0$

Eccentricity (e) $= \sqrt{1 - a^2/b^2}$

$= \sqrt{1 - \frac{(1/5)}{(1/4)}}$

$= \sqrt{1 - \frac{4}{5}} = \frac{1}{\sqrt{5}}$

Vertices $= \left(0, \pm\frac{1}{2}\right)$

Foci $= (0, \pm be)$

$= (0, \pm \frac{1}{2} \times \frac{1}{\sqrt{5}}) = (0, \pm \frac{1}{2\sqrt{5}})$

Latus rectum

$= \frac{2a^2}{b} = \frac{2 \times \frac{1}{5}}{\frac{1}{2}} = \frac{4}{5}$

Major axis = 2a = $2 \times \frac{1}{\sqrt{5}} = \frac{2}{\sqrt{5}}$

Minor axis = 2b = $2 \times \frac{1}{2} = 1$

2. Deduce the equation of ellipse in standard position with the following data.

a. A focus at (-2, 0) and a vertex at (5, 0)

b. Vertex at (0, 10) and eccentricity = $\frac{4}{5}$

c. Foci at (±2, 0) and eccentricity = $\frac{1}{2}$

d. A vertex at (0, 8) and passing through $(3, \frac{32}{5})$

e. Passing through the points (1, 4) and (-3, 2).

Solution:

a. Here, a = 5, ae = 2

$\therefore \quad 5e = 2 \Rightarrow e = \frac{2}{5}$

Now, using $b^2 = a^2 (1 - e^2) = 25 \left(1 - \frac{4}{25}\right) = 21$

So, the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or, $\frac{x^2}{25} + \frac{y^2}{21} = 1$

b. Here, major axis is along the y–axis.

So, $b = 10 \Rightarrow b^2 = 100$ and $e = \frac{4}{5}$

Now, using $a^2 = b^2 (1 – e^2) = 100 \left(1 - \frac{16}{25}\right) = 36$

So, the equation of ellipse is $\frac{x^2}{36} + \frac{y^2}{100} = 1$

c. Here, foci = $(\pm 2, 0) = (\pm ae, 0) \Rightarrow ae = 2$ and $e = \frac{1}{2} \Rightarrow a = \frac{2}{1/2} = 4$

and $e^2 = 1 - \frac{b^2}{16} \Rightarrow \frac{1}{4} = \frac{16 - b^2}{16} \Rightarrow 4 = 16 - b^2 \Rightarrow b^2 = 12$

Using equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$

or, $3x^2 + 4y^2 = 48$

d. Here, major axis is along the y–axis.

So, b = 8

The equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{64} = 1$

which passes through $\left(3, \frac{32}{5}\right)$, so

or, $\frac{9}{a^2} + \frac{(32/5)^5}{64} = 1$

or, $\frac{9}{a^2} + \frac{1025}{25 \times 64} = 1$

or, $\frac{9}{a^2} + \frac{16}{25} = 1$

or, $\frac{9}{a^2} = \frac{9}{25} \Rightarrow a^2 = 25$

$\therefore$ The equation of ellipse is $\frac{x^2}{25} + \frac{y^2}{64} = 1$

e. Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ … … … (1)

Since, (1) passes through (1, 4) and (–3, 2), so

$\frac{1}{a^2} + \frac{16}{b^2} = 1$ and $\frac{9}{a^2} + \frac{4}{b^2} = 1$

Solving these two equations, we get

$a^2 = \frac{140}{12} = \frac{35}{3}$ and $b^2 = \frac{140}{8} = \frac{35}{2}$

From equation (1), equation of ellipse is $\frac{x^2}{35/3} + \frac{y^2}{35/2} = 1$

or, $\frac{3x^2}{35} + \frac{2y^2}{35} = 1$

or, $3x^2 + 2y^2 = 35$

3. Find the eccentricity, the co-ordinates of the centre and the foci of the following ellipse.

a. $\frac{(x+2)^2}{16} + \frac{(y-5)^2}{9} = 1$

b. $\frac{(x-3)^2}{9} + \frac{(y-5)^2}{25} = 1$

c. $x^2 + 4y^2 - 4x + 24y + 24 = 0$

d. $9x^2 + 5y^2 - 30y = 0$

e. $9x^2 + 4y^2 + 40y + 18x + 73 = 0$

Solution:

a. Comparing (i) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

We get, $h = -2$, $k = 5$, $a^2 = 16$, $b^2 = 9$

$\therefore$ $a = 4$, $b = 3$

Now, $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$

The co–ordinate of vertices = $(h \pm a, k) = (-2 \pm 4, 5) = (-6, 5)$ and $(2, 5)$

So, the co–ordinate of centre = $\left(-\frac{6+2}{2}, \frac{5+5}{2}\right) = (-2, 5)$

And co–ordinate of foci = $(h \pm ae, k) = \left(-2 \pm 4 \cdot \frac{\sqrt{7}}{4}, 5\right) = (-2 \pm \sqrt{7}, 5)$

b. We have,

$\frac{(x-3)^2}{9} + \frac{(y-5)^2}{25} = 1$ which is in the form of $\frac{(x-h)^2}{9} + \frac{(y-k)^2}{25} = 1$

where $h = 3$, $k = 5$, $a^2 = 9$ and $b^2 = 25$

$\therefore$ $a = 3$ and $b = 5$

Since (b) (a) 0. So, the ellipse in along y–axis.

eccentricity (e) = $\sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

Co–ordinate of the center = (h, k) = (3, 5)

Foci of the ellipse = $(h, k \pm be) = \left(3, 5 \pm 5 \times \frac{4}{5}\right) = (3, 1)$ and $(3, 9)$

c. $x^2 + 4y^2 - 4x + 24y + 24 = 0$

or, $(x-2)^2 + 4(y+3)^2 = 4 + 36 - 24 = 16$

or, $\dfrac{(x-2)^2}{16} + \dfrac{(y+3)^2}{4} = 1$ … … … (1)

Comparing (i) with $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, we get

$a^2 = 16$, $b^2 = 4$, $h = 2$, $k = -3$

Now, $e = \sqrt{1 - \dfrac{b^2}{a^2}} = \sqrt{1 - \dfrac{4}{16}} = \dfrac{\sqrt{3}}{2}$

Foci $= (h \pm ae, k) = \left(2 \pm 4.\dfrac{\sqrt{3}}{2}, -3\right) = \left(2 \pm 2\sqrt{3}, -3\right)$ and centre $(h, k) = (2, -3)$

d. We have,

$9x^2 + 5y^2 - 30y = 0$

$\Rightarrow 9x^2 + 5(y^2 - 6y) = 0$

$\Rightarrow 9x^2 + 5(y^2 - 2.y.3 + 3^2 - 3^2) = 0$

$\Rightarrow 9x^2 + 5[(y-3)^2 - 9] = 0$

$\Rightarrow 9x^2 + 5(y-3)^2 = 45$

Dividing by 45 on both sides, we get

$\dfrac{x^2}{5} + \dfrac{(y-3)^2}{9} = 1$ which is in the form of $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$; where $h = 0, k = 3$,

$a^2 = 5$ and $b^2 = 9$

Since $b > a > 0$. So, the ellipse is along y–axis.

Hence, Eccentricity (e) $= \sqrt{1 - a^2/b^2} = \sqrt{1 - \dfrac{5}{9}} = \dfrac{2}{3}$

Co–ordinate of the center $= (h, k) = (0, 3)$

Foci of the ellipse $= (h, k \pm be) = \left(0, 3 \pm 3 \times \dfrac{2}{3}\right) = (0,5)$ and $(0, 1)$

e. We have,

$9x^2 + 4y^2 + 40y + 18x + 73 = 0$

$\Rightarrow (9x^2 + 18x) + (4y^2 + 40y) + 73 = 0$

$\Rightarrow 9[x^2 + 2.x.1 + 1^2 - 1^2] + 4[y^2 + 2.5.y + 5^2 - 5^2] + 73 = 0$

$\Rightarrow 9[(x+1)^2 - 1] + 4[(y+5)^2 - 25] + 73 = 0$

$\Rightarrow 9(x+1)^2 - 9 + 4(y+5)^2 - 100 + 73 = 0$

$\Rightarrow 9(x+1)^2 + 4(y+5)^2 = 36$

$\Rightarrow \dfrac{(x+1)^2}{4} + \dfrac{(y+5)^2}{9} = 1$; which is in the form of $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$; where $h = -1, k = -5, a^2 = 4$ and $b^2 = 9$

$\therefore$ $a = 2$ and $b = 3$

Since $b > a > 0$. So, the ellipse is along y–axis

Hence, eccentricity (e) $= \sqrt{1 - \dfrac{a^2}{b^2}} = \sqrt{1 - \dfrac{4}{9}} = \dfrac{\sqrt{5}}{3}$

Co–ordinate of the center $(h, k) = (-1, -5)$

Foci of the ellipse $= (h, k \pm be) = \left(-1, -5 \pm 3 \times \dfrac{\sqrt{5}}{3}\right) = (-1, -5 \pm \sqrt{5})$

4. Find the equation of ellipse whose

   a. Major axis is twice its minor axis and which passes through the point (0,1).

   b. Latus rectum 3 and eccentricity is $\dfrac{1}{\sqrt{2}}$.

   c. Distance between the two foci is 8 and the semi latus rectum is 6.

d. Latus rectum is equal to the half its major axis and which passes through the point (4, 3).

e. Foci are at (±2, 0) and length of latus rectum is 6.

Solution:

a. The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ … … … (1)

It is given that a = 2b and ellipse passes through (0, 1)

So, $\frac{0}{a^2} + \frac{y^2}{b^2} = 1$

or, $b^2 = 1$ $\quad \therefore b = 1$

and a = 2b = 2

from (1), $\frac{x^2}{4} + \frac{y^2}{1} = 1$

$x^2 + 4y^2 = 4$ is the required equation of an ellipse.

b. Here, equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

It is given, length of latus rectum = 3

or, $\frac{2b^2}{a} = 3$ $\quad \therefore b^2 = \frac{3a}{2}$

Using $e^2 = 1 - \frac{b^2}{a^2}$ $\quad$ or, $\frac{1}{2} = 1 - \frac{3a}{2a^2}$

or, $a = 2a - 3$ $\quad$ or, $a = 3$

and $b^2 = \frac{3.3}{2} = \frac{9}{2}$

So, the equation of ellipse is $\frac{x^2}{a} + \frac{y^2}{9/2} = 1$

or, $x^2 + 2y^2 = 9$

c. Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let a > b distance between foci = 8

i.e. $2ae = 8 \therefore ae = 4$

and semi latus rectum = $\frac{b^2}{a} = 6$

or, $b^2 = 6a$

Using $b^2 = a^2 (1 - e^2)$

or, $6a = a^2\left(1 - \frac{16}{a^2}\right)$ $\quad (\therefore e = 4/a)$

or, $6 = a\left(\frac{a^2 - 16}{a^2}\right)$

or, $a^2 - 6a - 16 = 0$

or, $a = 8, -2$ (but $a \neq -2$)

So, $b^2 = 6 \times 8 = 48$

$\therefore$ The equation of ellipse is $\frac{x^2}{64} + \frac{y^3}{48} = 1$

or, $3x^2 + 4y^2 = 192$

d. Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ … … … (1)

which passes through (4, 3)

So, $\frac{16}{a^2} + \frac{9}{b^2} = 1$ … … … (ii)

Also, $\frac{2b^2}{a} = \frac{1}{2} 2a$

or, $ab^2 = a^2$

Put $a^2 = 2b^2$ in (ii), then

or, $\frac{16}{2b^2} + \frac{9}{b^2} = 1$

or, $8 + 9 = b^2 \quad \therefore b^2 = 17$

and, $a^2 = 2\times b^2 = 34$

So, from (1), equation of ellipse is $\frac{x^2}{34} + \frac{y^2}{17} = 1$

or, $x^2 + 2y^2 = 34$

e. Here, foci = $(\pm ae, 0) = (\pm 2, 0)$

$\Rightarrow ae = 2 \qquad \therefore e = \frac{2}{a}$

and length of latus rectum $\frac{2b^2}{a} = 6$

or, $b^2 = 3a$

Also, $e = \sqrt{1 - \frac{b^2}{a^2}}$

or, $\frac{2}{a} = \sqrt{1 - \frac{3a}{a^2}}$

or, $\frac{4}{a^2} = 1 - \frac{3}{a}$

or, $\frac{4}{a} = a - 3$

or, $4 = a^2 - 3a$

or, $a^2 - 3a - 4 = 0$

or, $a^2 - 4a + a - 4 = 0$

or, $a(a - 4) + 1(a - 4) = 0$

$\therefore$ $a = -1, 4$ (but $a \neq -1$)

and $b^2 = 3\times 4 = 12$

Hence, the equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{12} = 1$

EXERCISE 10.2

1. Find the eccentricity, co-ordinates of the vertices and foci, length of latus rectum, length of transverse axis and conjugate axis of the hyperbola.

a. $\frac{x^2}{25} - \frac{y^2}{16} = 1$ b. $\frac{x^2}{9} - \frac{y^2}{25} = -1$ c. $3x^2 - 4y^2 = 36$

Solution:

a. $\frac{x^2}{25} - \frac{y^2}{16} = 1$ ... ... (1)

Compare (1) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 25, b^2 = 16 \qquad \therefore a = 5, b = 4$

Now, eccentricity (e) = $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{16}{25}} = \frac{\sqrt{41}}{5}$

Co–ordinate of vertices $(\pm a, 0) = (\pm 5, 0)$

Co–ordinate of foci $(\pm ae, 0) = (\pm 5.\frac{\sqrt{41}}{5}, 0) = \left(\pm\sqrt{41}, 0\right)$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2\times 16}{5} = \frac{32}{5}$

Length of transverse axis = $2a = 2\times 5 = 10$ Length of conjugate axis = $2b = 2\times 4 = 8$

b. $\frac{x^2}{9} - \frac{y^2}{25} = 1$

We have,

$\frac{x^2}{9} - \frac{y^2}{25} = 1$ which is in the form of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; where $a^2 = 9$ and $b^2 = 25$

$\therefore$ a = 3 and b = 5

Since the hyperbola is along y–axis

Hence, eccentricity (e) = $\sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{9}{25}} = \frac{\sqrt{34}}{5}$

Co–ordinate of the vertices = $(0, \pm b) = (0, \pm 5)$

Foci of the hyperbola = $(0, \pm be) = \left(0, \pm \times \frac{\sqrt{34}}{5}\right) = \left(0, \pm\sqrt{34}\right)$

Lenth of the latus rectum = $\frac{2a^2}{b} = \frac{2 \times 9}{5} = \frac{18}{5}$

Length of transverse axis = $2b = 2\times b = 2\times 5 = 10$

Length of conjugate axis = $2a = 2\times 3 = 6$

c. $3x^2 - 4y^2 = 36$

or, $\frac{x^2}{12} - \frac{y^2}{9} = 1$ which is in the form of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; where $a^2 = 12$, $b^2 = 9$

$\therefore$ $a = 2\sqrt{3}$ and b = 3

Since the hyperbola is along x–axis

Hence, eccentricity (e) = $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{12}} = \frac{\sqrt{21}}{2\sqrt{3}} = \frac{\sqrt{7}}{2}$

Co–ordinate of the vertices = $(\pm a, 0) = (\pm 2\sqrt{3}, 0)$

Foci of the hyperbola = $(\pm ae, 0) = \left(\pm 2\sqrt{3} . \frac{\sqrt{7}}{2}, 0\right) = (\pm\sqrt{21}, 0)$

Length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \times 9}{2\sqrt{3}} = 3\sqrt{3}$

Length of the transverse axis = $2a = 2\times 2\sqrt{3} = 4\sqrt{3}$

Length of the conjugate axis = $2b = 2\times 3 = 6$

2. Find the eccentricity, co-ordinates of vertices and foci, length of latus rectum, length of transverse axis and conjugate axis of the hyperbola.

a. $\frac{(x+1)^2}{144} - \frac{(y-1)^2}{25} = 1$ b. $5x^2 - 20y^2 - 20x = 0$

c. $16x^2 - 9y^2 + 96x - 72y + 144 = 0$

Solution:

a. $\frac{(x+1)^2}{144} - \frac{(y-1)^2}{25} = 1$ … … … (1)

Compare (1) with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get

$h = -1$, $k = 1$, $a^2 = 144$, $b^2 = 25$

Now, eccentricity (e) = $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{25}{144}} = \frac{13}{12}$

Cor–ordinate of vertices = $(h \pm a, k)$

$= (-1 \pm 12, 1) = (-13, 1)$ and $(11, 1)$

Co–ordinate of foci = $(h \pm ae, k) = (-1 \pm 12 \times \frac{13}{12}, 1) = (-14, 1)$ and $(12, 1)$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2 \times 25}{12} = \frac{25}{6}$

Length of conjugate axis = 2b = 2 × 5 = 10

Length of transverse axis = 2a = 2 × 12 = 24

b. $5x^2 - 20y^2 - 20x = 0$

or, $x^2 - 4y^2 - 4x = 0$

or, $(x - 2)^2 - 4y^2 = 4$

or, $\frac{(x-2)^2}{4} - \frac{y^2}{1} = 1$ … … … (1)

Compare (1) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, then (h, k) = (2, 0), a = 2, b = 1

Now, eccentricity (e) = $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$

Co–ordinate of vertices = (h ± a, k) = (2 ± 2, 0) = (4, 0)| and (0, 0)

Co–ordinate of foci = (h ± ae, k) = $\left(2 \pm 2.\frac{\sqrt{5}}{2}, 0\right) = \left(2 \pm \sqrt{5}, 0\right)$

Length of latus rectum = $\frac{2b^2}{a} = 2 . \frac{1}{2} = 1$

Length of transverse axis = 2a = 2.2 = 4

Length of conjugate axis = 2b = 2.1 = 2

c. $16x^2 - 9y^2 + 96x - 72y + 144 = 0$

or, $16(x^2 + 6x) - 9(y^2 + 8y) + 144 = 0$

or, $16(x + 3)^2 - 9(y + 4)^2 + 144 - 144 + 144 = 0$

or, $16(x + 3)^2 - 9(y + 4)^2 = -144$

or, $\frac{(x+3)^2}{9} - \frac{(y+4)^2}{16} = -1$ … … … (1)

Compare (1) with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$, we get

(h, k) = (–3, –4), $a^2 = 9$, $b^2 = 16$. (b > a)

Now, eccentricity (e) = $\sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$

Co–ordinate of vertices = (h, k ± b) = (–3, –4 ± 4) = (–3, 0) and (–3, –8)

Co–ordinate of foci = (h, k ± be) = $(-3, -4 \pm 4.\frac{5}{4})$ = (–3, 1) and (–3, –9)

Length of latus rectum = $\frac{2a^2}{b} = 2 . \frac{9}{4} = \frac{9}{2}$

Length of transverse axis = 2b = 2×4 = 8

Length of conjugate axis = 2a = 2×3 = 6

3. Find the equation of hyperbola in standard position satisfying the given conditions.

   a. Transverse and conjugate axis are respectively 4 and 5.
   b. Foci at (±3,0) and eccentricity $\frac{3}{2}$.
   c. Latus rectum is 4 and eccentricity is 3.
   d. Vertex at (0, 8) and passing through $(4, 8\sqrt{2})$
   e. Vertices at (0, ±7), e = $\frac{4}{3}$
   f. Focus at (6, 0) and a vertex at (4, 0)

Solution:

a. Let the equation of hyperbola be

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ … … … (1)

Where, 2a = 4 and 2b = 5 $\Rightarrow$ a = 2, b $= \frac{5}{2}$

$\therefore$ from (1), equation of hyperbola is

$\frac{x^2}{4} - \frac{y^2}{25/4} = 1$

or, $\frac{x^2}{4} - \frac{4y^2}{25} = 1$

b. Foci = ($\pm$ 3, 0), eccentricity (e) = $\frac{3}{2}$

Here, e = 3 and ae = 3

$\Rightarrow$ a = $\frac{3x^2}{3}$ = 2

Using, $b^2 = a^2(e^2 - 1)$

or, $b^2 = 4\left(\frac{9}{4} - 1\right) = 5$

$\therefore$ The equation of hyperbola is

$\frac{x^2}{4} - \frac{y^2}{5} = 1$

c. Here, e = 3 and $\frac{2b^2}{a} = 4$

Now, $b^2 = \frac{4a}{2}$

Using $b^2 = a^2(e^2 - 1)$

$\frac{4a}{2} = a^2(9 - 1)$

or, 2 = 8a $\Rightarrow$ a = $\frac{1}{4}$ $\therefore$ $a^2 = \frac{1}{16}$

and $b^2 = \frac{4a}{2} = \frac{4\frac{1}{4}}{2} = \frac{1}{2}$

So, the equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

or, $\frac{x^2}{1/16} - \frac{y^2}{1/2} = 1$

or, $16x^2 - 2y^2 = 1$

d. Here, vertex = (0, $\pm$ b) = (0, 8)

$\Rightarrow$ b = 8

Let the equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ … … … (1)

Which passes through (4, $8\sqrt{2}$), then

$\frac{4^2}{a^2} - \frac{(8\sqrt{2})^2}{64} = -1$

or, $\frac{16}{a^2} - \frac{128}{64} = -1$

or, $\frac{16}{a^2} = -1 + 2 \Rightarrow a^2 = 16$

Hence, from (1), $\frac{x^2}{16} - \frac{y^2}{64} = -1$

e. Here, b = 7, e = $\frac{4}{3}$

Using $a^2 = b^2(e^2 - 1) = 49\left(\frac{16}{9} - 1\right) = 49 \times \frac{7}{9} = \frac{343}{9}$

Hence, the equation of hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

or, $\frac{x^2}{343/9} - \frac{y^2}{49} = -1$

or, $9x^2 - 7y^2 = -343$

or, $9x^2 - 7y^2 + 343 = 0$

f. Here, ae = 6 and a = 4

Then, $e = \frac{6}{4} = \frac{3}{2}$

Using $b^2 = a^2(e^2 - 1)$

$b^2 = 16\left(\frac{9}{4} - 1\right) = 16 \times \frac{5}{4} = 20$

Now, the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or, $\frac{x^2}{16} - \frac{y^2}{20} = 1$

■

CHAPTER 11

CO-ORDINATE IN SPACE

EXERCISE 11.1

1. Find the distance between the points:
   a. (–2, 1, 0) and (3, 5, –2) b. (–4, 7, –7) and (–2, 1, –10)

Solution:

a. A(–2, 1, 0) and B(3, 5, –2)
Here,
$x_1 = -2 \quad x_2 = 3$
$y_1 = 1 \quad y_2 = 5$
$z_1 = 0 \quad z_2 = -2$
Using distance formulae,
$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(3 - (-2))^2 + (5 - 1)^2 + (-2 - 0)^2}$$
$$= \sqrt{5^2 + 4^2 + (-2)^2} = \sqrt{25 + 16 + 4}$$
$$= \sqrt{45} = 3\sqrt{5} \text{ units}$$

b. P(–4, 7, –7) and Q(–2, 1, –10)
Here,
$x_1 = -4 \quad x_2 = -2$
$y_1 = 7 \quad y_2 = 1$
$z_1 = -7 \quad z_2 = -10$
Using distance formula,
$$PQ = \sqrt{(x_2 - x_1)_2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(-2 - (-4))^2 + (1 - 7)^2 + (-10 - (-7))^2}$$
$$= \sqrt{(-2 + 4)^2 + (-6)^2 + (-10 + 7)^2}$$
$$= \sqrt{2^2 + (-6)^2 + (-3)^2} = \sqrt{4 + 36 + 9}$$
$$= \sqrt{49} = 7 \text{ units}$$

2. Show that the following points are collinear.
   a. (3, –2, 4), (1, 1, 1) and (–1, 4, –2) b. (1, –2, 3) , (2, 3, –4) and (0, –7, 10)
   c. (1, 2, 3), (4, 0 ,4) and (–2, 4, 2)

Solution:

a. Using distance formula
$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(1 - 3)^2 + (1 + 2)^2 + (1 - 4)^2}$$
$$= \sqrt{(-2)^2 + 3^2 + (-3)^2} = \sqrt{4 + 9 + 9} = \sqrt{22} \text{ units}$$
Again,
$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(-1 - 1)^2 + (4 - 1)^2 + (-2 - 1)^2}$$
$$= \sqrt{(-2)^2 + (3)^2 + (-3)^2} = \sqrt{4 + 9 + 9} = \sqrt{22} \text{ units}$$
Finally,
$$AC = \sqrt{(-1-3)^2 + (4 + 2)^2 + (-2-4)^2} = \sqrt{(-4)^2 + (6)^2 + (-6)^2}$$
$$= \sqrt{16 + 36 + 36} = \sqrt{88} = 2\sqrt{22} \text{ units}$$
Now, $AB + BC = 2\sqrt{22}$
Since, AB + BC = AC, the given points are collinear.

b. Using distance formula,
$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(2 - 1)^2 + (3 + 2)^2 + (-4 - 3)^2}$$

$= \sqrt{1^2 + 5^2 + (-7)^2} = \sqrt{1 + 25 + 49} = \sqrt{75} = 5\sqrt{3}$ units

Again,

$QR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(0-2)^2 + (-7-3)^2 + (10+4)^2}$

$= \sqrt{(-2)^2 + (-10)^2 + (14)^2} = \sqrt{4 + 100 + 196} = \sqrt{300} = 10\sqrt{3}$ units

Finally,

$PR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(0 - 1)^2 + (-7 + 2)^2 + (10 - 3)^2}$

$= \sqrt{(-1)^2 + (-5)^2 + 7^2} = \sqrt{1 + 25 + 49} = \sqrt{75} = 5\sqrt{3}$ units

Now, $PQ + PR = 10\sqrt{3}$

Since PQ + PR = QR, the given points are collinear.

c. Using distance formula,

$xy = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(4 - 1)^2 + (0 - 2)^2 + (4 - 3)^2}$

$= \sqrt{(4 - 1)^2 + (0 - 2)^2 + (4 - 3)^2}$

$= \sqrt{3^2 + (-2)^2 + 1^2}$

$= \sqrt{9 + 4 + 1}$

$= \sqrt{14}$ units

Again,

$yz = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2\ (z_2 - z_1)^2} = \sqrt{(-2 - 4)^2 + (4 - 0)^2 + (z - 4)^2}$

$= \sqrt{(-6)^2 + 4^2 + (-2)^2} = \sqrt{36 + 16 + 4} = \sqrt{56} = 2\sqrt{14}$ units

Finally,

$xz = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2\ (z_2 - z_1)^2} = \sqrt{(-2-1)^2 + (4 -2)^2 + (2 - 3)^2}$

$= \sqrt{(-3)^2 + 2^2 + (-1)^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$ units

Since, yz = xz + xy

So, the points are collinear.

3. Find the Co - ordinates of the mid - points of the line joining the points.

a. (–2, 6, –4) and (4, 0, 8)　　b. (–1, –2, –1) and (4, 7, 6)

Solution:

a. (–2, 6, –4) and (4, 0, 8)

Here,

$x_1 = -2$　$x_2 = 4$

$y_1 = 6$　$y_2 = 0$

$z_1 = -4$　$z_2 = 8$

Now,

Using mid-point formula,

M(x, y, z)

$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

$= \left(\frac{-2 + 4}{2}, \frac{6 + 0}{2}, \frac{-4 + 8}{2}\right)$

$= \left(\frac{2}{2}, \frac{6}{2}, \frac{4}{2}\right)$

$\therefore$ Mid–points = (1, 3, 2)

b. (–1, –2, –1) and (4, 7, 6)

Here,

$x_1 = -1$　$x_2 = 4$

$y_1 = -2$　$y_2 = 7$

$z_1 = -1$　$z_2 = 6$

Now,

Using mid-point formula,

M(x, y, z)

$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

$= \left(\frac{-1 + 4}{2}, \frac{-2 + 7}{2}, \frac{-1 + 6}{2}\right)$

$= \left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$

$\therefore$ Mid–point $= \left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$

4. Find the Co- ordinate of the point which divides the join of the points (3, 3, 1) and (3, –6, 4) internally in the ratio 2: 1.

Solution:

Here,

Let the points of line be A(3, 3, 1) and B(3, –6, 4). And the ratio that divides the line is 2:1. So,

$x_1 = 3$ $\quad$ $x_2 = 3$

$y_1 = 3$ $\quad$ $y_2 = -6$

$z_1 = 1$ $\quad$ $z_2 = 4$

Also, m : n = 2 : 1

Using section formula we get,

$$P(x, y, z) = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

$$= \left(\frac{2\times3 + 1\times3}{2 + 1}, \frac{2\times(-6) + 1 \times 3}{2 + 1}, \frac{2\times4 + 1\times1}{2 + 1}\right) = \left(\frac{9}{3}, \frac{-9}{3}, \frac{9}{3}\right)$$

$\therefore$ P(x, y, z) = (3, – 3, 3)

5. Find the Co- ordinates of the point which divides the Join of the points (3, 4, –5) and (1, 3, –2) externally in the ratio 5:4.

Solution:

Here,

Let the point of the line be M(3, 4, –5) and N(1, 3, –2) and the ratio that divides the time is 5 : 4. So,

$x_1 = 3$ $\quad$ $x_2 = 1$

$y_1 = 4$ $\quad$ $y_2 = 3$ $\quad$ m : n = 5 : 4

$z_1 = -5$ $\quad$ $z_2 = -2$

Using section formula we get

$$P(x, y, z) = \left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right)$$

$$= \left(\frac{5 \times 1 - 4 \times 3}{5 - 4}, \frac{5 \times 3 - 4 \times 4}{5 - 4}, \frac{5 \times (-2) - 4 \times (-5)}{5 - 4}\right)$$

$$= (5 - 12, 15 - 16, -10 + 20)$$

$\therefore$ P(x, y, z) = (–7, –1, 10)

6. If the internal section of two points P(2, –4, 3) and Q(x, y, z) in the ratio 2: 1 is (–2, 2, –3), then find Q.

Solution:

Here,

The internal section of two points P(2, –4, 3) and Q(x, y, z) in the ratio 2 : 1 is A(–2, 2, –3). So,

$x_1 = 2$ $\quad$ $x_2 = x$ $\quad$ and internal point m : n

$y_1 = -4$ $\quad$ $y_2 = y$ $\quad$ p = –2 = 2 : 1

$z_1 = 3$ $\quad$ $z_2 = z$ $\quad$ q = 2

r = –3

Now,

Using section formula we get,

$$p = \frac{-mx_2 + nx_1}{m + n}$$

or, $-2 = \dfrac{2 \times x + 1 \times 2}{2 + 1}$

or, $-2 \times 3 = 2x + 2$

or, $-6 = 2x + 2$

or, $2x = -8$

$\therefore$ x = –4

Similarly, $q = \dfrac{my_2 + ny_1}{m + n}$

or, $2 = \dfrac{2 \times y + 1 \times (-4)}{2 + 1}$

or, 6 = 2y – 4

or, 2y = 6 + 4

or, $y = \dfrac{10}{2}$ $\therefore$ y = 5

Again,

$r = \dfrac{mz_2 + nz_1}{m + n}$

or, $-3 = \dfrac{2 \times z + 1 \times 3}{2 + 1}$

or, –3 × 3 = 2z + 3

or, –9 = 2z + 3

or, 2z = –9 – 3

or, $z = \dfrac{-12}{2}$ $\therefore$ z = –6

Hence, Q(x, y, z) = (–4, 5, –6)

7. a. Given three Collinear Points A(3, 2, –4), B(5, 4, –6) and C(9, 8, –10). Find the ratio in which B divides AC.

b. Find the ratio in which the xy-plane divides the join of the points (–2, 4, 7) and (3, –5, 8).

c. Find the Co- ordinates of a point where the line through the point A (1, 2, 3) and B(4, –4, 9) meets the zx - plane.

Solution:

a. Given, $A(x_1, y_1, z_1) = (3, 2, -4)$

B(x, y, z) = (5, 4, –6)

and $C(x_2, y_2, z_2) = (9, 8, -10)$ be three collinear points.

So, let B divide AC in the ratio x : 1

So, $x = \dfrac{kx_2 + x_1}{k + 1}$

or, $5 = \dfrac{k \times 9 + 3}{k + 1}$

or, 5k + 5 = 9k + 3

or, 5k – 9k = 3 – 5

or, –4k = –2

$\therefore$ $k = \dfrac{1}{2}$ i.e. k : 1 = 1 : 2

$\therefore$ B divides AC in the ratio of 1 : 2

b. Let, xy – plane divides the line joining the points (–2, 4, 7) and (3, –5, –8) in the ratio of k : 1.

At the xy – plane, z = 0

Now, Using the section formula,

or, $z = \dfrac{mz_2 + nz_1}{m + n}$

or, $z = \dfrac{kz_2 + z_1}{k + 1}$

or, $0 = \dfrac{k(-8) + 7}{k + 1}$

or, k8 + 7 = 0

$\therefore$ $k = \dfrac{7}{8}$ or, k : 1 = 7 : 8

$\therefore$ Required ratio is 7:8

c. Let xz plane divides the line jointing the points A(1, 2, 3) and B(4, –4, 9) in the ratio k : 1.

At the xz – plane, y = 0

Now, Using the section formula,

or, $y = \dfrac{my_2 + ny_1}{m + n}$

or, $y = \dfrac{k\,x_2 + y_1}{k + 1}$

or, $0 = \dfrac{k(-4) + 2}{k + 1}$

or, $-4k + 2 = 0$

or, $4k = 2$

$\therefore$ $k = \dfrac{1}{2}$ or, $k : 1 = 1 : 2$

$\therefore$ Required ratio is 1:2

Then using $x = \dfrac{mx_2 + nx_1}{m + n} = 2$ and $z = \dfrac{mz_2 + nz_1}{m + n} = 5$

$\therefore$ The required point is (2, 0, 5).

8. a. Find the locus of a points which are equidistance from the two fixed points (1, 2, 1) and (3, –4, 2)

   b. Find the locus of a point P(x, y, z) satisfying the conditions $PA^2 + PB^2 = 6$, where A(–1, 2, –1) and B(0, 3, –2) are two fixed points.

Solution:

a. Let P(x, y, z) be any point on the locus. Let
Let A(1, 2, 1) and B(3, –4, 2) be two points.
By the given condition

$PA = PB$

or, $PA^2 = PB^2$

or, $(x - 1)^2 + (y - 2)^2 + (2 - 1)^2 = (x - 3)^2 + (y + 4)^2 + (z - 2)^2$

or, $x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 2z + 1 = x^2 - 6x + 9 + y^2 + 8y + 16 + z^2 - 4z + 4$

or, $x^2 - x^2 + y^2 - y^2 + z^2 - z^2 - 2x + 6x - 4y - 8y - 2z + 4z + 1 + 4 + 1 - 9 - 16 - 4 = 0$

or, $4x - 12y + 2z - 23 = 0$ is the required equation of locus.

b. Here, $PA^2 = (x + 1)^2 + (y - 2)^2 + (z - 1)^2 = x^2 + y^2 + z^2 + 2x - 4y + 2z + 6$

and $PB^2 = x^2 + (y - 3)^2 + (z + 2)^2 = x^2 + y^2 + z^2 - 6y + 4z + 113$

Since $PA^2 + PB^2 = 6$

So, we have

$x^2 + y^2 + z^2 + 2x - 4y + 2z + 6 + x^2 + y^2 + z^2 - 6y + 4z + 13 = 6$

$\Rightarrow$ $2x^2 + 2y^2 + 2z^2 + 2x - 10y + 6z + 13 = 0$

which is the required equation of the locus of a point.

9. Find the point where the line Joining the points (2, –3, 1) and (3, –4, –5) cuts the plane $2x + y + z = 7$.

Solution:

Let (2, –3, 1) and (3, –4, 5) divides the plane $2x + y + z = 7$ in the ratio k : 1.
So,

$$(x, y, z) = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

$$= \left(\frac{k.3 + 2}{k + 1}, \frac{-4k - 3}{k + 1}, \frac{-5k + 1}{k + 1}\right) \quad \text{... (i)}$$

Also, (x, y, z) satisfies the plane $2x + y + z = 7$

So, $2\left(\dfrac{3k + 2}{k + 1}\right) + \left(\dfrac{-4k - 3}{k + 1}\right) + \left(\dfrac{-5k + 1}{k + 1}\right) = 7$

or, $6k + 4 - 4k - 3 - 5k + 1 = 7k + 7$

or, $2k - 5k - 7k = 7 - 4 + 3 - 1$ $\therefore$ $k = \dfrac{-1}{2}$

Put the value of $k = \dfrac{-1}{2}$ in (i), we get the required point (x, y, z) = (1, –2, 7).

10. Prove that the centre of the sphere that passes through (3, 2, 2), (–1, 1, 3), (2, 1, 2) and (1, 0, 4) is at (1, 3, 4). Find also the radius of sphere.

Solution:

Let P(x, y, z) be the centre of the sphere passing through the points A(3, 2, 2), B(–1, 1, 3), C(2, 1, 2) and D(1, 0, 4).

Here, According to the question,

P(x, y, z) = (1, 3, 4)

Now, Using distance formula,

$D^2 = (x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2$ we have,

or, $AP^2 = (x – 3)^2 + (y–2)^2 + (z–2)^2 = x^2–6x + 9 + y^2–4y + 4 + z^2 – 4z + 4$ ....(i)

or, $BP^2 = (x + 1)^2 + (y – 1)^2 + (z – 3)^2$

$= x^2 + 2x + 1 + y^2 – 2y + 1 + z^2 – 6z + 9$ .... (ii)

or, $CP^2 = (x – 2)^2 + (y–1)^2 + (z–2)^2 = x^2 – 4x + 4 + y^2 – 2y +1+ z^2 – 4z + 4$ .. (iii)

$DP^2 = (x – 1)^2 + (y – 0)^2 + (z – 4)^2 = x^2 – 2x + 1 + y^2 + z^2 – 8z + 16$ ...... (iv)

$\therefore$ AP = BP

$\Rightarrow AP^2 = BP^2$

$\Rightarrow x^2 + y^2 + z^2 – 6x – 4y – 4z + 17 = x^2 + y^2 + z^2 + 2x – 2y – 6z + 11$

$\Rightarrow 2x + 6x – 2y + 4y – 6z + 4z = 17 – 11$

$\Rightarrow 8x + 2y – 2z = 6$

$\Rightarrow 4x + y – z = 3$

Similarly making (i) equal to (iii) and (i) equal to (iv)

We have,

$x^2 – 6x + 17 + y^2 – 4y + z2 – 4z = x^2 + y^2 + z^2 – 4x – 2y – 4z + 9$

or, $–2x – 2y = –8$

or, $x + y = 4$ .... (B)

Also, solving equation (iii) and (iv) we get

$–x – y + 2z = 4$ .... (C)

Hence, solving equation (A), (B) and (C) we get

P(x, y, z) = (1, 3, 4)

Hence proved.

Also, radius (r) = AP $= \sqrt{x^2 + y^2 + z^2 – 6x – 4y – 4z + 17}$

$= \sqrt{1 + 9 + 16 – 6 – 12 – 16 + 17} = \sqrt{9} = 3$ units

11. a. Prove that the points (–4, 9, 6), (0, 7, 10) and (–1, 6, 6) are the vertices of a right angled isosceles triangle.

b. Show that the points (2, 0 ,–4), (4, 2, 4) and (10, 2, –2) are the vertices of an equilateral triangle.

Solution:

Let A(0, 7, 10), B(–1, 6, 6) and C(–4, 9, 6) be three points.

For AB

$AB^2 = (x_2 – x_1)^2 + (y_2 – y_1)^2 + (z^2 – z_1)^2$

$= (–1 – 0)^2 + (6 – 7)^2 + (6 – 70)^2 = 1+1+16 = 18$

$AB = \sqrt{18} = 3\sqrt{2}$ units

For BC

$BC^2 = (x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2$

$= (–4 + 1)^2 + (9 – 6)^2 + (6 – 6)^2 = 9 + 9 + 0 = 18$

$BC = \sqrt{18} = 3\sqrt{2}$ units

For AC

$CA^2 = (x_2 – x_1)^2 + (y_2 – y_1)^2 + (z_2 – z_1)^2$

$= (0 + 4)2 + (7 – 9)^2 + (10 – 6)^2 = 16+4+16 = 36$

$CA = \sqrt{36} = 6$ units

Now, By Pythagoras theorem
We have,
$CA^2 = AB^2 + BC^2$
Also AB = AC
$\therefore$ A, B and C are the vertices of right angled isosceles triangle.

b. Let A(2, 0, –4) B(4, 2, 4) and C(10, 2, –2) be three points.
$AB^2 = 14 - 2)^2 + (2 - 0)^2 + (4 + 4)^2 = 4 + 4 + 64 = 72$
Again, $BC^2 = (10 - 4)^2 + (2 - 2)^2 + (-2 - 4)^2 = 36 + 0 + 36 = 72$
Similarly, $CA^2 = (2 - 10)^2 + (0 - 2)^2 + (-4 + 2)^2 = 64 + 4 + 4 = 72$
So, AB = BC = CA
$\therefore$ A, B and C are the vertices of an equilateral triangle.

12. a. Show that the points (1, 2, 3), (–1, –2, –1), (2, 3, 2) and (4, 7, 6) are the verticels of a parallelnram. Also show that this is not a rectangle.

b. Three consecutive vertices of a parallelnram ABCD are A(–5, 5, 2), B(–9, –1, 2) an C(–3, –3, 0) .Find the Co-ordinates of the fourth vertex.

Solution

a. Let A(1, 2, 3), B(–1, –2, –1), C(2, 3, 2)and D(4, 7, 6) be four points
$AB^2 = (-1 - 1)^2 + (-2 - 2)^2 + (-1 - 3)^2 = 36$
$\therefore$ AB = 6
$BC^2 = (2 + 1)^2 + (3 + 2)^2 + (2 + 1)^2 = 43$
$\therefore$ BC $= \sqrt{43}$
$CD^2 = (4 - 2)^2 + (7 - 3)^2 + (6 - 2)^2 = 36$
$\therefore$ CD = 6
$DA^2 = (1 - 4)^2 + (2 - 7)^2 + (3 - 6)^2 = 43$
$\therefore$ DA $= \sqrt{43}$
Hence, AB = CD and BC = DA so, A, B, C and D are the vertices of a parallelnram.
Here, AC $= \sqrt{(2 - 1)^2 + (3 - 2)^2 + (2 - 3)^2} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ units
Again, BD $= \sqrt{(4 + 2)^2 + (7 + 2)^2 + (6 + 1)^2} = \sqrt{36 + 81 + 49} = \sqrt{166}$ units
Since, the two diagonals AC and BD are not equal.
$\therefore$ The points A, B, C and D do not represent a rectangle.

b. Let, $D(\overline{x}, \overline{y}, \overline{z})$ be the point of intersection of the diagonals AC and BD.
For AC : A (–5, 5, 2) and C(–3, –3, 0)
The coordinates of the mid-point AC

$$= \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}, \frac{z_2 + z_1}{2}\right) = \left(\frac{-3 - 5}{2}, \frac{5 - 3}{2}, \frac{2 + 0}{2}\right) = (-4, 1, 1)$$

For BD : B(–9, –1, 2) and $D(\overline{x}, \overline{y}, \overline{z})$
Since mid point of AC = midpoint of BD

or, $x = \dfrac{x_1 + x_2}{2}$

or, $-4 = \dfrac{-9 + x}{2}$ or, $-8 = -9 + \overline{x}$ $\therefore \overline{x} = -8 + 9 = 1$

or, $y = \dfrac{y_1 + y_2}{2}$

or, $1 = \dfrac{-1 + \overline{y}}{2}$ or, $2 = -1 + \overline{y}$ $\therefore \overline{y} = 2 + 1 = 3$

or, $z = \dfrac{z_1 + z_2}{2}$

or, $1 = \frac{2 + z}{2}$ or, $2 = 2 + \bar{z}$ $\therefore \bar{z} = 2 - 2 = 0$

$\therefore$ The coordinates of D = $(\bar{x}, \bar{y}, \bar{z}) = (1, 3, 0)$

13. Find the third vertex of the triangle whose centroid is (7, –2, 5) and other two vertices are (2, 6, –4) and (15, –10, 16).

Solution:

Let $(x_1, y_1, z_1) = (2, 6, -4)$
$(x_2, y_2, z_2) = (15, -10, 16)$, $(x, y, z) = (7, -2, 5)$ and $(x_3, y_3, z_3) = ?$
By the centroid formula,

or, $x = \frac{x_1 + x_2 + x_3}{3}$

or, $7 = \frac{2 + 15 + x_3}{3}$

or, $21 = 17 + x_3$

$\therefore$ $x_3 = 21 - 17 = 4$

$y = \frac{y_1 + y_2 + y_3}{3}$

or, $-6 = -4 + y_3$

$\therefore$ $y_3 = -6 + 4 = -2$

or, $z = \frac{z_1 + z_2 + z_3}{3}$

or, $5 = \frac{-4 + 16 + z_3}{3}$

or, $15 = 12 + z_3$

or, $z_3 = 15 - 12$

$\therefore$ $z_3 = 3$

$\therefore$ $(x_3, y_3, z_3) = (4, -2, 3)$

EXERCISE 11.2

1. If a line makes an angle $\frac{\pi}{4}$ with each of the x- axis and y- axis, what angle does it makes with z- axis?

Solution:

$\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{4}$

But $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or, $\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} + \cos^2\gamma = 1$

or, $\frac{1}{2} + \frac{1}{2} + \cos^2\gamma = 1$

or, $1 + \cos^2\gamma = 1$

or, $\cos^2\gamma = 0$

or, $\cos\gamma = 0$

$\therefore$ $\gamma = \cos^{-1}(0)$ $\therefore$ $\gamma = \frac{\pi}{2}$

$\therefore$ The angle is $\frac{\pi}{2}$.

2. Find the direction cosines of a line which is equally inclined with the axes of co-ordinates.

Solution:

Let the angle made by a line with 3 axes be α, α, α.
Now, we know
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
or, $\cos^2\alpha + \cos^2\alpha \cos^2\alpha = 1$
or, $3\cos^2\alpha = 1$
or, $\cos\alpha = \pm \dfrac{1}{\sqrt{3}}$
Similarly, for $\cos^2\alpha$ and $\cos^2\alpha$
$\cos\alpha = \pm \dfrac{1}{\sqrt{3}}$ and $\cos\alpha = \pm \dfrac{1}{\sqrt{3}}$
$\therefore$ The direction cosines $= \left(\pm \dfrac{1}{\sqrt{3}}, \pm \dfrac{1}{\sqrt{3}}, \pm \dfrac{1}{\sqrt{3}}\right)$

3. If α, β and γ are the direction angles of a line prove that $\cos 2\alpha + \text{Cos } 2\beta + \text{Cos } 2\gamma + 1 = 0$.

Solution:

Given,f α, β and γ be the angles made by the line with the co-ordinates axis.
$\text{Cos}^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
Multiplying by 2 on both sides we get,
or, $2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma = 2$
or, $1 + \cos^2\alpha + 1 + \cos^2\beta + 1 + \cos^2\gamma = 2$
or, $3 + \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 2$
$\therefore$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 1 = 0$

4. Find the direction cosines of each of the lines whose direction ratios are.

a. 6, 2, –3 b. –1, –2, –3

Solution:

a. Here, a = 6, b = 2 and c = –3
The direction cosines are

$$l = \frac{1}{\sqrt{a^2 + b^2 + c^2}} = \frac{6}{\sqrt{6^2 + 2^2 + (-3)^2}} = \frac{6}{\sqrt{49}} = \frac{6}{7}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{2}{\sqrt{49}} = \frac{2}{7} \text{ and, } n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{-3}{\sqrt{49}} = \frac{-3}{7}$$

$\therefore$ The direction cosines (l, m, n) $= \left(\dfrac{6}{7}, \dfrac{2}{7}, \dfrac{-3}{7}\right)$

b. Here, a = –1, b = –2, and c = –3
Now, The direction cosines are:

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{-1}{\sqrt{(-1)^2 + (-2)^2 + (-3)^2}} = \frac{-2}{\sqrt{(-1)^2 + (-2)^2 + (-3)^2}} = \frac{-3}{\sqrt{(-1)^2 + (-2)^2 + (-3)^2}}$$

$$= \frac{-1}{\sqrt{14}} = \frac{-2}{\sqrt{14}} = \frac{-3}{\sqrt{14}}$$

$\therefore$ The direction cosines are (l, m, n) $= \left(\dfrac{-1}{\sqrt{14}}, \dfrac{-2}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}\right)$

5. Find the direction cosines of each of the following lines joining the points.
   a. (–2, 1, –8) and (4, 3 –5)
   b. (5, 2 , 8) and (7, –1, 9)

Solution:

a. $(x_1, y_1, z_1) = (-2, 1, -8)$ and $(x_2, y_2, z_2) = (4, 3, -5)$

$$PQ = r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(4 + 2)^2 + (3 - 1)^2\ (-5 + 8)^2}$$
$$= \sqrt{6^2 + 2^2 + 3^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7$$

$\therefore$ The direction cosines of PQ are

$$l = \frac{x_2 - x_1}{r} = \frac{4 + 2}{7} = \frac{6}{7},\quad m = \frac{y_2 - y_1}{r} = \frac{3 - 1}{7} = \frac{2}{7},\quad n = \frac{z_2 - z_1}{r} = \frac{-5 + 8}{7} = \frac{3}{7}$$

$\therefore$ The direction cosines are $(l, m, n) = \left(\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$

b. $(x_1, y_1, z_1) = (5, 2, 8)$ and $(x_2, y_2, z_2) = (7, -1, 9)$

$$AB = r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(7 - 5)^2 + (-1 - 2)^2 + (9 - 8)^2}$$
$$= \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$\therefore$ The direction cosines of AB are:

$$l = \frac{x_2 - x_1}{r},\ m = \frac{y_2 - y_1}{r},\ n = \frac{z_2 - z_1}{r} = \frac{7 - 5}{\sqrt{14}} = \frac{-1 - 3}{\sqrt{14}} = \frac{9 - 8}{\sqrt{14}} = \frac{2}{\sqrt{14}} = \frac{-3}{\sqrt{14}} = \frac{1}{\sqrt{14}}$$

$\therefore$ The direction cosines $(l, m, n) = \left(\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$

6. a. Find the angle between the line whose direction cosines are proportional to 1, 2, 2 and 2, 3, 6.
   b. If A, B, C are the points (1, 4, 2), (–2, 1, 2) and (2, –3, 4) respectively, find the angles of triangle ABC.

Solution:

a. $a_1 = 1 \qquad b_1 = 2, \qquad c_1 = 2$
$a_2 = 2, \qquad b_2 = 3, \qquad c_2 = 6$

We have,

$$\text{Cos}\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{1\times2 + 2\times3 + 2\times6}{\sqrt{1^2 + 2^2 + 2^2}\sqrt{2^2 + 3^2 + 6^2}} = \frac{20}{3\times7}$$

or, $\theta = \cos^{-1}\left(\frac{20}{21}\right)$

$\therefore\ \theta = \cos^{-1}\left(\frac{20}{21}\right)$

b. For AB :

$$AB = \sqrt{(1 + 2)^2 + (4 - 1)^2 + (2 - 2)^2} = \sqrt{9 + 9 + 0} = 3\sqrt{2} \text{ units}$$

and $(l_1, m_1, n_1) = \left(\frac{-2 - 1}{3\sqrt{2}}, \frac{1 - 4}{3\sqrt{2}}, \frac{2 - 2}{3\sqrt{2}}\right) = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

For BC

$$BC = \sqrt{(2 + 2)^2 + (-3 - 1)^2 + (4 - 2)^2} = \sqrt{16 + 16 + 4} = \sqrt{36} = 6 \text{ units}$$

and $(l_2, m_2, n_2) = \left(\frac{2 + 2}{6}, \frac{-3 - 1}{6}, \frac{4 - 2}{6}\right) = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$

We know,

$$\cos B = l, l_2 + m_1m_2 + n_1n_2 = \frac{-1}{\sqrt{2}} \times \frac{2}{3} + -\frac{1}{\sqrt{2}} \times \frac{-2}{3} + 0 \times \frac{1}{3} = \frac{-\sqrt{2}}{3} + \frac{\sqrt{2}}{3} + 0 = 0$$

or, $\cos B = 0$

or, $B = \frac{\pi}{2}$ $\therefore$ $B = \frac{\pi}{2} = 90^o$

Hence, the lines are perpendicular.

For AC

$AC = \sqrt{(2-1)^2 + (-3-4)^2 + (4-2)^2} = \sqrt{1 + 49 + 4} = \sqrt{54} = 3\sqrt{6}$ units.

Again, $(l_2, m_2, n_2) = \frac{2-1}{3\sqrt{6}}, \frac{-3-4}{3\sqrt{6}}, \frac{4-2}{3\sqrt{6}} = \left(\frac{1}{3\sqrt{6}}, \frac{-7}{3\sqrt{6}}, \frac{2}{3\sqrt{6}}\right)$

Similarly,

$\text{Cos } A = l_1 l_2 + m_1m_2 + n_1n_2$

$$= \frac{-1}{\sqrt{2}} \times \frac{1}{3\sqrt{6}} + \frac{-1}{\sqrt{2}} \times \frac{-7}{3\sqrt{6}} + 0 \times \frac{2}{3\sqrt{6}} = \frac{-\sqrt{3}}{18} + \frac{7\sqrt{3}}{18} + 0 = \frac{1}{\sqrt{3}}$$

$\Rightarrow \text{Cos}A = \frac{1}{\sqrt{3}}$

$\therefore \quad A = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

7. a. Show that the line Joining the points (1, 2, 3) and (–1, –2, –3) is parallel, to the line Joining points (2, 3, 4) and (5, 9, 13).

   b. Prove that the line joining the points (0, 4, 1) and (2, 6, 2) is perpendicular to the line Joining the points (4, 5, 0) and (2, 6, 2).

Solution:

a. For the live joining the points (1, 2, 3) and (–1, –2, –3) $(x_1, y_1, z_1) = (1, 2, 3)$ and $(x_2, y_2, z_2) = (-1, -2, -3)$

$a_1 = x_2 - x_1 = -1 - 1 = -2$ $\quad b_1 = y_2 - y_1 = -2 - 2 = -4$ $\quad c_1 = z_2 - z_1 = -3 - 3 = -6$

For the live joining the points (2, 3, 4) and (5, 9, 13) $(x_1, y_1, z_1) = (2, 3, 4)$ and $(x_2, y_2, z_2) = (5, 9, 13)$

$a_2 = x_2 - x_1 = 5 - 2 = 3$

$b_2 = y_2 - y_1 = 9 - 3 = 6$

$c_2 = z_2 - z_1 = 13 - 4 = 9$

Now,

$\frac{a_1}{a_2} = \frac{-2}{3}, \frac{b_1}{b_2} = \frac{-4}{6} = \frac{-2}{3}, \frac{c_1}{c_2} = \frac{-6}{9} = \frac{-2}{3} \therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Hence, the two lines are parallel.

b. For the line joining the points (0, 4, 1) and (2, 6, 2) $(x_1, y_1, z_1) = (0, 4, 1)$ and $(x_2, y_2, z_2) = (2, 6, 2)$

$a_1 = x_2 - x_1 = 2 - 0 = 2$ $\quad b_1 = y_2 - y = 6 - 4 = 2$ $\quad c_1 = 2_2 - 2_1 = 2 - 1 = 1$

For the line joining the points (4, 5, 0) and (2, 6, 2) $(x_1, y_1, z_1) = (4, 5, 0)$ and $(x_2, y_2, z_2) = (2, 6, 2)$

$a_2 = x_2 - x_1 = 2 - 4 = -2$ $\quad b_2 = y_2 - y_1 = 6 - 5 = 1$ $\quad c_2 = z_2 - z_1 = 2 - 0 = 2$

Now, $a_1\, a_2 = -4$ $\quad b_1\, b_2 = 2$

$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$

or, $-4 + 2 + 2 = 0$

or, $-4 + 4 = 0$

$\therefore \quad 0 = 0$

8. a. For what value of k is the line joining the points (1, 2, 3) and (4, 5, k) parallel to the line joining points (–4, 3, –6) and (2, 9 , 2)?

   b. For what value of k is the line joining the points (k, 2, 3) and (–1, –2, –3) is perpendicular to the line Joining the points (–2, 1, 5) and (3, 3, 2)?

Solution:

a. Here,

The two points of a line (1, 2, 3) and (4, 5, k) is parallel to two points of a line (–4, 3, –6) and (2, 9, 2).

So,

$a_1 = x_2 - x_1 = 4 - 1 = 3$, $b_1 = y_2 - y_1 = 5 - 2 = 3$, $c_1 = k - 3 = k - 3$

Again, $a_2 = x_2 - x = 2 + 4 = 6_1$, $b_2 = 9 - 3 = 6$, $c_2 = z_2 - z_1 = 2 + 6 = 8$

Since the two lines are parallel to each other we know,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

or, $\frac{3}{6} = \frac{3}{6} = \frac{k-3}{8}$

or, $\frac{3}{6} = \frac{k-3}{8}$

or, $24 = 64 - 18$

or, $24 + 18 = 64$

or, $k = \frac{42}{6}$ $\therefore$ $k = 7$

b. Here, the line joining the points (k, 2, 3) and (–1, –2, –3) is perpendicular to the line joining the points (–2, 1, 5) and (3, 3, 2)

Now, The direction ratios are:

$a_1 = x_2 - x_1 = -1 - k$, $b_1 = y_2 - y_1 = -2 - 2 = -4$, $c_1 = z_2 - z_1 = -3 - 3 = -6$

Again,

$a_2 = x_2 - x_1 = 3 + 2 = 5$, $b_2 = y_2 - y_1 = 3 - 1 = 2$, $c_2 = z_2 - z_1 = 2 - 5 = -3$

Since the two lines are perpendicular to each other. We know,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

or, $(-1 - k) \times 5 + (-4) \times 2 + (-6) \times (-3) = 0$

or, $18 - 13 = 5k + 0$

or, $5 - 0 = 5k$ $\therefore$ $k = 1$

9. Find the direction cosines of the line which is perpendicular to the lines with directions cosines proportional to 1, 2, 3 and –1, 3, 5.

Solution:

Here,

Let l, m and n be the direction cosines of the line perpendicular to the given line.

$l + 2m + 3n = 0$ .... (i)

$-l + 3m + 5n = 0$ .... (ii)

By cross = multiplication method

$$\frac{l}{10-9} = \frac{m}{-5-3} = \frac{n}{3+2}$$

or, $\frac{l}{1} = \frac{m}{-8} = \frac{n}{5} = \frac{1}{\sqrt{1^2 + 8^2 + 5^2}} = \frac{1}{\sqrt{90}}$

$\therefore$ $l = \frac{1}{\sqrt{90}}$, $m = \frac{-8}{\sqrt{90}}$, $n = \frac{5}{\sqrt{90}}$

10. The projection of a line on the axes are 1, 2, 2. Find the length of the line and it's direction cosines.

Solution

Suppose l, m, n are the direction cosines of a line and its length

Then, $lr = 1$

mr = 2
nr = 2
Squaring and adding there relations
$r^2(l^2 + m^2 + n^2) = 12 + 22 + 22 = 9$
or, $r^2 = 9 \therefore r = 3$ units
So length of the line = 3

Hence, $l = \frac{1}{r} = \frac{1}{3}$

$m = \frac{2}{r} = \frac{2}{3}$, $n = \frac{2}{r} = \frac{2}{3}$

11. Find the Projection of the Join of pair of points (3, –1, 2) and (5, –7, 4) on the following lines.
    a. On a line whose direction cosines are proportional to 1, –1, 2.
    b. On a line joining the points (0, 1, 0) and (1, 3, 7)

Solution

$(x_1, y_1, z_1) = (3, -1, 2)$ and $(x_2, y_2, z_2) = (5, -7, 4)$
Here,

a. a = 1, b = –1, c = 2

$$\therefore \quad l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{1 + 1 + 4}} = \frac{1}{\sqrt{6}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{-1}{\sqrt{1 + 1 + 4}} = \frac{-1}{\sqrt{6}} \qquad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}} = \frac{2}{\sqrt{1 + 1 = 4}} = \frac{2}{\sqrt{6}}$$

The required projection on the line
$= (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)n$

$$= (5 - 3) \times \frac{1}{\sqrt{6}} + (-7 + 1) \times -\frac{1}{\sqrt{6}} + (4-2) \times \frac{2}{\sqrt{6}} = 2\sqrt{6}$$

b. Here,

$$r = \sqrt{(1 - 0)^2 + (3-1)^2 + (7-0)^2} = \sqrt{54} = 3\sqrt{6}$$

Now,

$$l = \frac{x_2 - x}{r} = \frac{1 - 0}{3\sqrt{6}} = \frac{1}{3\sqrt{6}},$$

$$m = \frac{y_2 - y_1}{r} = \frac{3 - 1}{3\sqrt{6}} = \frac{2}{3\sqrt{6}} \text{ and } n = \frac{z_2 - z_1}{r} = \frac{7 - 0}{3\sqrt{6}} = \frac{7}{3\sqrt{6}}$$

The projection
$= (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$

$$= (5-3) \times \frac{1}{3\sqrt{6}} + (-7 + 1) \times \frac{2}{3\sqrt{6}} + (4-2) \times \frac{7}{3\sqrt{6}} = \frac{4}{3\sqrt{6}}$$

12. Find the direction Cosines l, m, n of two lines which satisfy the equation
    a. $4l + 3m - 2n = 0$ and $lm - mn + nl = 0$
    b. $2l + 2m - n = 0$ and $mn + nl + lm = 0$

Solution

a. Here, $4l + 3m - 2n = 0$ .... (i)
$lm + mn + nl + 0$ .... (ii)

From the equation (i), $n = \frac{4l + 3m}{2}$

From the equation (ii)

$$lm - m\,\frac{(4l + 3m)}{2} + l\frac{(4l + 3m)}{2} = 0$$

or, $lm - \frac{4ml}{2} - \frac{3m^2}{2} + \frac{4l^2}{2} + \frac{3ml}{2} = 0$

or, $lm - 2ml - \frac{3}{2}m^2 + 2l^2 + \frac{3ml}{2} = 0$

or, $2l^2 + \frac{ml}{2} - \frac{3}{2}m^2 = 0$

or, $4l^2 + ml - 3m^2 = 0$

or, $4l^2 + 4ml - 3ml - 3m^2 = 0$

or, $4l\,(4l + m) - 3m\,(l + m) = 0$

or, $(l + m)\,(4l - 3m) = 0$

$\therefore$ $l + m = 0$ .... (iii)

$4l - 3m = 0$ .... (iv)

From equation (i) and (iii)

$4l + 3m - 2n = 0$ and $l + m + 0n = 0$

$\therefore \ \frac{l}{0+2} = \frac{m}{-2-0} = \frac{n}{4-3}$

or, $\frac{l}{2} = \frac{m}{-2} = \frac{n}{2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{4 + k + 1}} = \frac{1}{3}$

$\therefore \ l = \frac{2}{3}, \ m = -\frac{2}{3}, \ n = \frac{1}{3}$

From equation (i) and (iv)

$4l + 3m - 2n = 0$ and $4l - 3m + 0n = 0$

$\therefore \ \frac{l}{0-6} = \frac{m}{-8-0} = \frac{n}{-12-12}$

or, $\frac{l}{6} = \frac{m}{8} = \frac{n}{2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{36 + 64 + 576}} = \frac{1}{\sqrt{676}} = \frac{1}{26}$

$\therefore \ l = \frac{6}{26} = \frac{3}{13}$

$m = \frac{8}{26} = \frac{4}{13}, \ n = \frac{24}{26} = \frac{12}{13}$

b. Here,

$2l + 2m - n = 0$ .... (i)

$mn + nl + lm = 0$ .... (ii)

Using equation (i) in equation (ii) we have

$m(2l + 2m) + l(2l + 2m) + lm = 0$

or, $2lm + 2m^2 + 2l^2 + 2lm + lm = 0$

or, $2m^2 + 5lm + 2l^2 = 0$

or, $2m^2 + (4 + 1)\,lm + 2l^2 = 0$

or, $2m^2 + 4lm + lm + 2l^2 = 0$

or, $2m(m + 2l) + l(m + 2l) = 0$

or, $(m + 2l)\,(2m + l) = 0$

$\therefore$ $m + 2l = 0$ .... (iii)

$2m + l = 0$ .... (iv)

from (i) and (iii)

$2l + 2m - n = 0$ and $2l + m + 0.n = 0$

$\therefore \ \frac{l}{0+1} = \frac{m}{0+2} = \frac{n}{2-4}$

or, $\dfrac{l}{1} = \dfrac{m}{2} = \dfrac{n}{-2} = \dfrac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{1 + k + 4}} = \dfrac{1}{3}$

$\therefore\ l = \dfrac{1}{3}$, $m = \dfrac{2}{3}$, $n = -\dfrac{2}{3}$

from (i) and (iv)

$2l + 2m - n = 0$ and $l + 2m + 0.n = 0$

$\therefore\ \dfrac{l}{0 + 2} = \dfrac{m}{0 + 1} = \dfrac{n}{4 - 2}$

or, $\dfrac{l}{2} = \dfrac{m}{1} = \dfrac{n}{2} = \dfrac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{4 + 1 + 4}} = \dfrac{1}{3}$

$\therefore\ l = \dfrac{2}{3}$, $m = \dfrac{1}{3}$, $n = \dfrac{2}{3}$

13. Find the angle between two lines whose direction cosines are given by $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$

Solution

Here, $l + m + n = 0$ .... (i)

or, $n = -1 - m$

Putting the value of n in

$l^2 + m^2 - n^2 = 0$ .... (ii)

or, $l^2 + m^2 - (-1 - m)^2 = 0$

or, $2lm = 0$

or, $lm = 0$

$\therefore\ l = 0$ .... (iii)

and $m = 0$ .... (iv)

from (i) and (iii)

$l + m + n = 0$ and $l + 0.m + 0.n = 0$

or, $\dfrac{l}{0 - 0} = \dfrac{m}{1 - 0} = \dfrac{n}{0 - 1}$

or, $\dfrac{l}{0} = \dfrac{m}{1} = \dfrac{n}{-1} = \dfrac{1}{\sqrt{0 + 1 + 1}} = \dfrac{1}{\sqrt{2}}$

$\therefore\ l = 0$, $m = \dfrac{1}{\sqrt{2}}$, $n = -\dfrac{1}{\sqrt{2}}$

from (i) and (iv)

$l + m + n = 0$ and $0.l + m + 0.n = 0$

$\therefore\ \dfrac{l}{0 - 1} = \dfrac{m}{0 - 0} = \dfrac{n}{1 - 0}$

or, $\dfrac{l}{-1} = \dfrac{m}{0} = \dfrac{n}{1} = \dfrac{1}{\sqrt{1 + 0 + 1}} = \dfrac{1}{\sqrt{2}}$

$\therefore\ l = -\dfrac{1}{\sqrt{2}}$, $m = 0$, $n = \dfrac{1}{\sqrt{2}}$

$\therefore\ \cos\theta = 0\left(\dfrac{1}{\sqrt{2}}\right) + \left(\dfrac{1}{\sqrt{2}}\right).0 + \left(\dfrac{-1}{\sqrt{2}}\right)\left(\dfrac{1}{\sqrt{2}}\right) = -\dfrac{1}{2}$

$\therefore\ \theta = \dfrac{2\pi}{3} = 120^o$

EXERCISE 11.3

1. a. Find the intercepts made by plane $2x + 3y + 4z = 24$ on co- ordinates axes.
   b. Reduce the equation $2x - y + 2z = 4$ to normal form also determine the direction cosines of the normal and length of perpendicular to it from the origin.

Solution:

a. The equation of the plane is $2x + 3y + 4z = 24$

Dividing both sides by 24, $\frac{2x}{24} + \frac{3y}{24} + \frac{4z}{24} = \frac{24}{24}$

or, $\frac{x}{12} + \frac{4}{8} + \frac{z}{6} = 1$

The intercepts on the x–axis, y–axis and z–axis are 12, 8 and 6 respectively.

b. To reduce the equation of the plane $2x - y + 2z = 4$ into normal form,

Divide each term by $\sqrt{2^2 + (-1)^2 + 2^2} = 3$

$\therefore$ $\frac{2x}{3} - \frac{y}{3} + \frac{2z}{3} = \frac{4}{3}$ is in normal form where length of perpendicular from origin is $\frac{4}{3}$ units.

The dc's are

$$\frac{2}{\sqrt{2^2 + (-1)^2 + 2^2}}, \frac{-1}{\sqrt{2^2 + (-1)^2 + 2^2}}, \frac{2}{\sqrt{2^2 + (-1)^2 + 2^2}}$$

i.e. $\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$

2. a. Find the equation of plane which makes the intercepts 2, 3, 4 on x - axis, y-axis and z - axis respectively.
   b. Find the equation of plane which makes equal intercepts on the axes and passes through the point (2, 3, 4).

Solution:

a. The equation of plane which cuts intercepts 2, 3, 4 on the coordinate axes is

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

or, $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$

$\therefore$ $6x + 4y + 3z = 12$

b. Here, $a = b = c$

The equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$x + y + z = a$ .... (i)

If the plane (i) passes through (2, 3, 4) then $2 + 3 + 4 = a$

$\therefore$ $a = 9$

From equation (i) $x + y + z = 9$ which is the equation of the plane.

3. a. Find the equation of the plane passing through the points
      i. (2, 3, –3), (1, 1, –2) and (–1, 1, 2)
      ii. (2, 2, 2), (3, 1, 1) and (6, –4, –6)
   b. Show that the four points (1, 3, –1), (3, 5, 1), (0, 2, –2) and (2, 1, –2) are coplanar.

Solution:

a.

i. The equation of the plane through (2, 3, –3) a(x – 2) + b(y –3) + c(z + 3) = 0 ... (i)
If the plane passes through (1, 1, –2) and (–1, 1, 2) then,
a(1 – 2) + b(1 – 3) + c(–2 + 3) = 0
or, –a –2b + c = 0 .... (ii)
Again,
a(–1 – 2) + b(1 – 3) + c(2 + 3) = 0
or, –3a – 2b + 5c = 0 .... (iii)
From (ii) and (iii) and cross multiplication gives

$$\frac{a}{-10 + 2} = \frac{b}{-3 + 5} = \frac{c}{2 - 6}$$

$$\frac{a}{-8} = \frac{b}{2} = \frac{c}{-4} = k \text{ (say)}$$

$\therefore$ a = –8k, b = 2k c = –4k
Substituting the values of a, b and c in equation (i) we get,
–8h (x – 2) + 2k(y – 3) –4k (2 + 3) = 0
or, –8x + 16 + 2y – 6 – 4 – 12 = 0
or, –8x + 2y – 4z – 2 = 0
or, –4x + y + 4z – 2 = 0
or, 4x – y – 4z + 2 = 0

ii. The equation of the plane through (2, 2, 2) is
a(x – 2) + b(y – 2) + c(z – 2) = 0 .... (i)
If the plane passes through (3, 1, 1) and (6, –4, –6) then,
a(3 – 2) + b(1 – 2) + c(1 – 2) = 0
or, a – b – c = 0 .... (ii)
Again,
a(6 – 2) + b(–4 – 2) + c(–6 – 2) = 0
or, 4a – 6b – 8c = 0 .... (iii)
From (ii) and (iii) and cross multiplication gives

$$\frac{a}{8 - 6} = \frac{b}{-4 + 8} = \frac{c}{-6 + 4}$$

$$\frac{a}{2} = \frac{b}{4} = \frac{c}{-2} = k \text{ (say)}$$

$\therefore$ a = 2k, b = 4k, c = –2k
Substituting the values of a, b and c in equation (i)
2k(x – 2) + (4k) y–2) + (–2) (z– 2) = 0
or, 2k – 4 + 4y – 8 – 2z + 4 = 0
or, 2x + 4y – 2z = 8
or, 2x + 4y – 2z = 8
or, x + 2y – z = 4

b. The points (1, 3, –1), (3, 5, 1), (0, 2, –2) and (2, 1, –2) are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3 - 1 & 5 - 3 & 1 + 1 \\ 0 - 1 & 2 - 3 & -2 + 1 \\ 2 - 1 & 1 - 3 & -2 + 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 1 & -2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow -2\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} = 0$$

$\Rightarrow -2 \times 0 = 0 \Rightarrow 0 = 0$ (true)

Hence, the given four points are coplanar.

4. Show that the equation of the plane through (α, β, γ) and parallel to the plane $ax + by + cz = 0$ is $ax + by + cz = a\alpha + b\beta + c\gamma$.

Solution:

The equation of the plane parallel to the plane $ax + by + cz = 0$ is

$ax + by = cz + k = 0$ .... (i)

If the plane (i) passes through (α, β, γ) then

$a\alpha = b\beta = c\gamma + k = 0$

$\therefore\ k = -a\alpha - b\beta - c\gamma$

Substituting the value of k in (i) we have

$ax + by + cz = a\alpha + b\beta + c\gamma$

which is the required equation of the plane.

5. Find the angle between the planes:

a. $x + 2y + 3z = 6$ and $3x - 3y + z = 1$ b. $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$

Solution:

a. $x + 2y + 3z = 6$ and $3x - 3y + z = 1$

Here,

$x + 2y + 3z - 6 = 0$ and $3x - 3y + 2 - 1 = 0$

$a_1 = 1$	$a_2 = 3$	$b_1 = 2$
$b_2 = -3$	$c_1 = 3$	$c_2 = 1$

Now, Angle between the two planes

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{1 \times 3 + 2 \times (-3) + 3 \times 1}{\sqrt{1^2 + 2^2 + 3^2}\sqrt{3^2 + (-3)^2 + 1^2}} = \frac{0}{\sqrt{1 + 4 + 9}\sqrt{9 + 9 + 1}}$$

$\therefore\ \cos\theta = 0\ \therefore\ \theta = \frac{\pi}{2}$

b. $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$

Here, Repeating the same procedure as No. 5a

$\therefore\ \theta = \frac{\pi}{2}$

6. For what value of k will the following pair of the planes be perpendicular?

a. $x - 6y + 8z = 4$, and $4x + ky + z = 7$b. $2x - ky + 5z = 9$ and $2x + 3y + 4z = 7$

Solution:

a. The equation of the two planes is given:

$x - 6y + 8z = 4$ .... (i)

$4x + ky + z = 7$ .... (ii)

Now,

$a_1 = 1$	$a_2 = 4$	$b_1 = -6$
$b_2 = k$	$c_1 = 8$	$c_2 = 1$

We know, the equation for two planes being perpendicular is given by,

$a_1a_2 = b_1b_2 + c_1c_2 = 0$

or, $1.4 + (-6).\ k + 8.1 = 0$

or, $4 - 6k + 8 = 0$

or, $-6k = -12$

$\therefore$ $k = 2$

b. The two equations of the plane is given

$2x - ky + 5z = 9$ .... (i)

$2x + 3y + 4z = 7$ .... (ii)

We know,

The equation of two planes being perpendicular is given by

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

or, $2 \times 2 + (-4) \times 3 + 5 \times 4 = 0$

or, $-3x = -24$

$\therefore$ $k = 8$

7. Show that the plane $2x + 3y - 4z = 3$ is parallel to the plane $10x + 15y - 20z = 12$ and perpendicular to the plane $3x + 2y + 3z = 5$.

Solution:

For parallel

Two planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Here,

$a_1 = 2,\ b_1 = 3\ c_1 = -4$

$a_2 = 10,\ b_2 = 15,\ c_2 = -20$

$$\frac{a_1}{a_2} = \frac{2}{10} = \frac{1}{5},\ \frac{b_1}{b_2} = \frac{3}{15} = \frac{1}{5},\ \frac{c_1}{c_2} = \frac{-4}{-20} = \frac{-1}{-5} = \frac{1}{5}$$

$$\therefore\ \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$\therefore$ The planes $2x + 3y - 4z = 3$ and $10x + 15y - 202 = 12$ are parallel.

For perpendicular

Two planes $a_1x + b_1y + c_1z + d_1 = 0$

and $a_2x + b_2y + c_2z + d_2 = 0$

are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Here, $a_1 = 2,\ b_1 = 3,\ c_1 = -4$ $\quad$ $a_2 = 3,\ b_2 = 2,\ c_2 = 3$

And $a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + 3 \times 2 - 4 \times 3 = 6 + 6 - 12 = 12 - 12 = 0$

$\therefore$ The two planes $2x + 3y - 4z = 3$ and $3x + 2y + 3z = 5$ are perpendicular.

8. a. Find the equation of plane through $(-2, 3, 4)$ and perpendicular to the planes
$2x + 3y + 4z = 6$ and $3x + 2y + 2z = 8$.

b. Find the equation of plane through the point $(1, 2, 3)$ and normal to the planes
$x - y - z = 5$ and $2x - 5y - 3z = 7$.

Solution:

a. An equation of a plane passing through $(-2, 3, 4)$ so, the equation is

$a(x + 2) + b(y - 3) + c(z - 4) = 0$ .... (i)

Now, It is perpendicular to the equation $2x + 3y + 4z = 6$ then,

$a \times 2 + b \times 3 + c \times 4 = 0$

or, $2a + 3b + 4c = 0$ .... (ii)

Again,
It is perpendicular to the equation 3x + 2y + 2 = 9 then,
$a \times 3 + 2 \times b + 2 \times c = 0$
or, $3a + 2b + 2c = 0$ .... (iii)
By cross multiplication
$2a + 3b + 4c = 0$
$3a + 2b + 2c = 0$
$\therefore \frac{a}{6-8} = \frac{b}{12-4} = \frac{c}{4-9} = k$ (say)
$\frac{a}{-2} = \frac{b}{8} = \frac{c}{-5} = k$ (say)
$\Rightarrow$ $a = -2k, b = 8k, c = -5k$
Substituting the values of a, b, c in equal (i) we have,
$-2k(x + 2) + 8k(y - 3) + (-5k)(z - 4) = 0$
or, $-2x - 4 + 8y - 24 - 5z + 20 = 0$
or, $-2x + 8y - 5z - 28 + 20 = 0$
or, $-2x + 8y - 5z - 8 = 0$
or, $2x - 8y + 5z + 8 = 0$ is the required equation of the plane.

b. Here,
Repeating the same procedure as in No. 8a
So, the required equation of the plane is $2x - y + 3z = 9$

9. a. Find the equation of plane through P (a, b, c) and perpendicular to OP.
b. Find the equation of a plane through (3, 2, 1) and is perpendicular to the line joining the points (–5, 3, 7) and (2, –4, 5).

Solution:

a. The equation of the plane through P(a, b, c) is
$A(x - a) + B(y - b) + C(z - c) = 0$ .... (i)
The direction cosines of OP are proportional to a – 0, b – 0, c – 0
i.e., a, b, c
Since the plane (i) is perpendicular to OP,
$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = k$ (let)
$A = ak, B = bk, C = ck$
Substituting the values of A, B, C in (i)
$ak(x - a) + bk(y - b) + ck(z - c) = 0$
or, $a(x - a) + b(y - b) + c(z - c) = 0$
$\therefore$ $ax + by + cz = a^2 + b^2 + c^2$

b. The equation of the plane through P(3, 2, 1) is
$a(x - 3) + b(y - 2) + c(z - 1) = 0$ .... (i)
The direction cosines of MN are proportional to 2 + 5, –4 – 3, 5 – 7
i.e, 7, –7, –2
Since the plane (i) is perpendicular to MN,
$\frac{a}{7} = \frac{b}{-7} = \frac{c}{-2} = k$ (say)
$\therefore$ $a = 7k, b = -7k, c = -2k$
Now,
Substituting the value of a, b, c in equation (i) we have,
$7k(x - 3) - 7k(y - 2) - 2k(z - 1) = 0$
or, $7x - 21 - 7y + 14 - 2z + 2 = 0$
or, $7x - 7y - 2z - 5 = 0$ is the required equation of the plane.

10. a. Find the equation of plane which passes through the points (–1, 1, 2) and (1, –1, 1) and is perpendicular to the plane $x + 2y + 2z = 5$.

b. Find the equation of plane passing through the intersection of the plane $x + y + z = 5$ and $2x + 3y + 4z - 5 = 0$ and passing through the origin.

c. Find the equation of plane through the line of intersection of the planes $x + 2y + 3z + 4 = 0$ and $4x + 3y + 4z + 1 = 0$ and passing through the point (1, –3, – 1).

Solution:

a. Any plane passing through (–1, 1, 2) is

$a(x + 1) + b(y - 1) + c(z - 2) = 0$ .... (i)

But, it passes through (1, –1, 1) so

or, $a(1 + 1) + b(-1 - 1) + c(1 - 2) = 0$

or, $2a - 2b - c = 0$ .... (ii)

The plane (i) is perpendicular to the given plane $x + 2y + 2z = 5$.

i.e. if $a + 2b + 2c = 0$ ... ... (iii)

From equation (ii) and (iii) we have

or, $\dfrac{a}{-4+2} = \dfrac{b}{-1-4} = \dfrac{c}{4+2}$

$\therefore \dfrac{a}{-2} = \dfrac{b}{-5} = \dfrac{c}{6} = k$ (say)

$\Rightarrow a = -2k, b = -5k, c = 6k$

From equation (i) we get

$-2k(x + 1) - 5k(y - 1) + 6k(z - 2) = 0$

or, $-2x - 2 - 5y + 5 + 6z - 12 = 0$

or, $-2x - 5y + 6z - 9 = 0$

or, $2x + 5y - 6z + 9 = 0$

b. Here, two planes are

$x + y + z = 5$ .... (i)

and $2x + 3y + 4z - 5 = 0$ .... (ii)

Then the equation of plane through intersection of (i) and (ii) is

$x + y + z - 5 + \lambda (2x + 3y + 4z - 5) = 0$

or, $x + y + z - 5 + 2\lambda x + 3\lambda y + 4\lambda z - 5\lambda = 0$

or, $(1 + z\lambda), c + (1 + 3\lambda) y + (1 + 4\lambda) z - 5 - 5\lambda = 0$ .... (iii)

and the plane (iii) passes through (0, 0, 0) so

$(1 + 2\lambda) 0 + (1 + 3\lambda) 0 + (1 + 4\lambda) 0 - 5 - 5\lambda = 0$

or, $-5 - 5\lambda = 0$

or, $-5 = 5\lambda$

$\Rightarrow \lambda = -1$

So, the required equation of plane is

$x + y + z - 5 - 1(2x + 3y + 4z - 5) = 0$

or, $x + y + z - 5 - 2x - 3y - 4z + 5 = 0$

or, $-x - 2y - 3z = 0$

$\therefore x + 2y + 3z = 0$

c. Here, two planes are:

$x + 2y + 3z + 4 = 0$ ... ... (i)

$4x + 3y + 4z + 1 = 0$ ... ... (ii)

The equation of the plane through the intersection is given by,

$x + 2y + 3z + 4 + \lambda(4x + 3y + 4z + 1) = 0$

or, $x + 2y + 3z + 4 + 4\lambda x + 3\lambda y + 4\lambda z + \lambda = 0$

or, $x(1 + 4\lambda) + y(2 + 3\lambda) + z(3 + 4\lambda) + 4 + \lambda = 0$ ... ... (ii)

And, the plane (iii) passes through the point (1, –3, –1)

$1(1 + 4\lambda) + (-3)(2 + 3\lambda) + (-1)(3 + 4\lambda) + 4 + \lambda = 0$

or, $1 + 4\lambda - 6 - 9\lambda - 3 - 4\lambda + 4 + \lambda = 0$

or, $1 - 6 - 3 + 4 - 9\lambda + \lambda = 0$

or, $-9 + 5 - 8\lambda = 0$

or, $-4 = 8\lambda$

$\therefore \quad \lambda = -\frac{4}{8} = -\frac{1}{2}$

So, the required equation of the plane is

$x + 2y + 32 + 4 + \left(-\frac{1}{2}\right)(4x + 3y + 4z + 1) = 0$

or, $x + 2y + 3z + 4 - \frac{4x}{2} - \frac{3y}{2} - \frac{4z}{2} - \frac{1}{2} = 0$

or, $x + 2y + 3z + 4 - 2x - \frac{3y}{2} - 2z = \frac{1}{2}$

or, $-x - \frac{3y}{2} + 2y + z + 4 - \frac{1}{2} = 0$

or, $-x - \frac{3y + 4y}{2} + z + \frac{8 - 7}{2} = 0$

or, $-2x + y + 2z + 7 = 0$

or, $2x - y - 2z = 7$ is the required equation of the plane.

11. Find the equation of planes through the intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z = 0$ and perpendicular to the plane $5x + 3y + 6z + 8 = 0$

Solution:

The equation of the plane through the intersection of the given planes is

$x + 2y + 3z - 4 + \lambda(2x + y - z) = 0$ .... (i)

or, $x + 2y + 3z - 4 + 2\lambda x + \lambda y - \lambda z = 0$

or, $(1 + 2\lambda)x + (2 + \lambda)y + (3 - \lambda)z - 4 = 0$

Since the plane (i) is perpendicular to the plane: $5x + 3y + 6z + 8 = 0$

So, $(1 + 2\lambda)5 + (2 + \lambda)3 + (3 - \lambda).6 = 0$ $\quad [\because a_1a_2 + b_1b_2 + c_1c_2 = 0]$

or, $5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0$

or, $29 + 7\lambda = 0$

or, $\lambda = -\frac{29}{7}$

Substituting the value of λ in equation (i) we get

or, $x + 2y + 3z - 4 - \frac{29}{7}(2x + y - z) = 0$

or, $x + 2y + 3z - 4 - \frac{58x}{7} - \frac{29y}{7} + \frac{29z}{7} = 0$

or, $7x - 58x + 14y - 29y + 21z + 29z - 28 = 0$

or, $-51x - 15y + 50z - 28 = 0$

$\therefore \quad 51x + 15y - 50z + 28 = 0$ is the required equation of the planes.

12. a. Find the distance of the point

i. $(3, 4, -5)$ from the plane $2x - 3y + 3z + 27 = 0$

ii. From the origin on the plane $3x - 2y + 6z = 17$

b. Prove that the points$(1, -1, 3)$ and $(3, 3, 3)$ are equidistance from the plane $5x + 2y - 7z + 9 = 0$

c. Show that the distance between the parallel planes $3x + 3y - 6z + 1 = 0$ and $6x + 4y - 12z + 9 = 0$ is $\frac{1}{2}$.

Solution:

a. i. The given plane is,

$2x - 3y + 3z + 27 = 0$

The distance from the point (3, 4, –5) to the plane $2x - 3y + 3z + 27 = 0$ is,

$$\pm \frac{2 \times 3 - 3 \times 4 + 3 \times -5 + 27}{\sqrt{2^2 + (-3)^2 + 3^2}} = \pm \frac{(6 - 12 - 15 + 27)}{\sqrt{4 + 9 + 9}}$$

$$= \pm \frac{6}{\sqrt{22}} = \frac{6}{\sqrt{22}} \text{ (in magnitude)}$$

Solution:

The distance fromf the point (1, –1, 3) to the plane $5x + 2y - 7z + 9 = 0$ is

$$= \pm \frac{5\times1 + 2\times(-1) - 7\times3+9}{\sqrt{5^2 + 2^2 + (-7)^2}} = \pm \frac{5 - 2 - 21 + 9}{\sqrt{78}} = \pm \frac{9}{\sqrt{78}} = \frac{9}{\sqrt{78}}$$

Again, the distance from the point (3, 3, 3) to the plane $5x + 2y - 7z + 9 = 0$ is

$$\pm \frac{5\times3 + 2\times(3) - 7\times3+9}{\sqrt{5^2 + 2^2 + (-7)^2}}$$

$$= \pm \frac{15 + 6 - 21 + 9}{\sqrt{78}}$$

$$= \pm \frac{9}{\sqrt{78}} = \frac{9}{\sqrt{78}}$$

Hence, the given two points are equidistance from the given plane.

b. Here, the two points (1, –1, 3) and (3, 3, 3) are equidistance from the equation of the plane $5x + 2y - 7z + 9 = 0$.

Firstly, $x_1 = 1$, $y_1 = -1$, $z_1 = 3$

and the equation of the plane is given by $ax_1 + by_1 + cz_1 + d = 0$

So,

$$\text{Distance} = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{5 \times 1 + 2 \times (-1) + (-7) \times 3 + 9}{\sqrt{5^2 + 2^2 + (-7)^2}}\right|$$

$$= \left|\frac{5 - 2 - 21 + 9}{\sqrt{25 + 4 + 49}}\right| = \left|-\frac{9}{\sqrt{78}}\right|$$

$$\text{Distance} = \frac{9}{\sqrt{78}} \text{ units}$$

Similarly, $x_2 = 3$, $y_2 = 3$, $z_2 = 3$

$$\text{Distance} = \left|\frac{ax_2 + by_2 + cz_2 + d}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{5 \times 3 + 2 \times 3 + (-7) \times 3 + 9}{\sqrt{5^2 + 2^2 + (-7)^2}}\right|$$

$$= \left|\frac{15 + 6 - 21 + 9}{\sqrt{25 + 4 + 49}}\right| = \left|\frac{9}{\sqrt{78}}\right|$$

$$\therefore \ \text{Distance} = \frac{9}{\sqrt{78}} \text{ units.}$$

Since the two point (1, –1, 3) and (3, 3, 3) are in at the same distance from the given plane.

$\therefore$ They are at equidistance from the plane.

c. The equation of two parallel planes is given by,

$3x + 3y - 6z + 1 = 0$ .... (i)

$6x + 4y - 12z + 9 = 0$ .... (ii)

Now, In equation (i) let $y = z = 0$ then, $x = -\frac{1}{3}$

i.e, $\left(-\frac{1}{3}, 0, 0\right)$ is the point which lies in equation (i) plane.

i.e, $(x_1, y_1, z_1) = \left(-\frac{1}{3}, 0, 0\right)$

$$\text{Distance} = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{6 \times \left(-\frac{1}{3}\right) + 4 \times 0 - 12 \times 0 + 9}{\sqrt{6^2 + 4^2 + (-12)^2}}\right|$$

$$= \left|\frac{7}{\sqrt{36 + 16 = 144}}\right| = \left|\frac{7}{\sqrt{196}}\right| = \left|\frac{7}{14}\right|$$

$\therefore$ Distance $= \frac{1}{2}$

Hence, according to the qn, the distance between the two planes is $\frac{1}{2}$ units

13. A variable plane is at a constant distance 3p from the origin and meets the axes in the points A, B and C. Prove that the locus of the centroid of triangle ABC is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{P^2}$

Solution:

Let the plane (ΔABC) whose vertices are
A(a, 0, 0), B (0, b, 0), and (0, 0, c) at a distance of 3p units from the origin.

So, the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

or, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$ .... (i)

Since, 1^v distance of the equation (i) from (0, 0, 0) is 3p,

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{\frac{1}{a} . 0 + \frac{1}{b} . 0 + \frac{1}{c} . 0 - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}\right|$$

or, $3p = \dfrac{1}{\sqrt{\frac{1}{q^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$

or, $\frac{1}{(3p)^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ .... (ii)

Let (α, β, α) be the centroid of ΔABC

So, $\alpha = \frac{a + 0 + 0}{3} \therefore \alpha = \frac{a}{3} \Rightarrow a = 3\alpha$

Similarly, b = 3β and c = 3ν

Now, From equation (ii)

$\frac{1}{9p^2} = \frac{1}{(3\alpha)^2} + \frac{1}{(3\beta)^2} + \frac{1}{(3\nu)^2}$

or, $\frac{1}{9p^2} = \frac{1}{9\alpha^2} + \frac{1}{9\beta^2} + \frac{1}{9\nu^2}$

or, $\frac{1}{p^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\nu^2} \quad \therefore \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\nu^2} = \frac{1}{p^2}$ proved.

Hence, the locus of the centroid of ΔABC is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$

■

CHAPTER 12

PRODUCT OF VECTORS

EXERCISE 12.1

1. Find $\vec{a} \times \vec{b}$ if

   a. $\vec{a} = 2\vec{i} - 3\vec{k}, \vec{b} = 2\vec{j} + 4\vec{k}$ b. $\vec{a} = 2\vec{i} + 4\vec{k}, \vec{b} = 3\vec{j} - 2\vec{k}$

   c. $\vec{a} = 2\vec{i} + 3\vec{j} + \vec{k}, \vec{b} = -\vec{i} - 2\vec{j} + 3\vec{k}$

Solution

a. Given, $\vec{a} = 2\vec{i} - 3\vec{k} = (2, 0, -3)$

$\vec{b} = 2\vec{j} + 4\vec{k} = (0, 2, 4)$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -3 \\ 0 & 2 & 4 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & -3 \\ 2 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -3 \\ 0 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 6\vec{i} - 8\vec{j} + 4\vec{k}$

b. Given vectors

$\vec{a} = 2\vec{i} + 4\vec{k} = (2, 0, 4)$ $\vec{b} = 3\vec{j} - 2\vec{k} = (0, 3 -2)$

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 4 \\ 0 & 3 & -2 \end{vmatrix} = -12\vec{i} + 4\vec{j} + 6\vec{k}$

c. Given,

$\vec{a} = 2\vec{i} + 3\vec{j} + \vec{k}$ $\vec{b} = -\vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ -1 & -2 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = 11\vec{i} - 7\vec{j} - \vec{k}$

2. If $\vec{a} = 3\vec{i} + 4\vec{j} - 5\vec{k}$ and $\vec{b} = 7\vec{i} - 3\vec{j} + 6\vec{k}$, calculate $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$ and $|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|$

Solution:

Given $\vec{a} = 3\vec{i} + 4\vec{j} - 5\vec{k}$ $\vec{b} = 7\vec{i} - 3\vec{j} + 6\vec{k}$

$\vec{a}+\vec{b} = (3\vec{i} + 4\vec{j} - 5\vec{k}) + (7\vec{i} - 3\vec{j} + 6\vec{k}) = 10\vec{i} + \vec{j} + \vec{k}$

$\vec{a} - \vec{b} = (3i + 4j - 5k) - (7\vec{i} - 3\vec{j} + 6\vec{k}) = -4\vec{i} + 7\vec{j} - 11\vec{k}$

$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 10 & 1 & 1 \\ -4 & 7 & -11 \end{vmatrix} = -18\vec{i} + 106\vec{j} + 74\vec{k}$

$|(\vec{a}+\vec{b}) \times (\vec{a} - \vec{b})| = \sqrt{(-18)^2 + (106)^2 + (74)+} = \sqrt{17036}$

3. If $\vec{a} = \vec{i} + \vec{j} + \vec{k}, \vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}$, find the value of $|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|$.

Solution:

Here, $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ $\vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}\vec{i}$

$(\vec{a}+\vec{b}) = (1+2)\vec{i} + (1+3)\vec{j} + (1+1)\vec{k} = 3\vec{i} + 4\vec{j} + 2\vec{k}$

$(\vec{a}-\vec{b}) = (1-2)\vec{i} + (1-3)\vec{j} + (1-1)\vec{k} = -\vec{i} - 2\vec{j} + 0\vec{k}$

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 2 \\ -1 & -2 & 0 \end{vmatrix} = (0+4)\vec{i} - \vec{j}(0+2) + \vec{k}(-6+4) = 4\vec{i} - 2\vec{j} - 2\vec{k}$$

$|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})| = \sqrt{4^2+(-2)^2+(-2)^2} = \sqrt{24} = 2\sqrt{6}$

4. Find the vector and the unit vector orthogonal to each of the following pair of vectors.
   a. $\vec{a} = 4\vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{b} = 5\vec{i} + \vec{j} - 4\vec{k}$
   b. $\vec{a} = 6\vec{i} + 3\vec{j} - 5\vec{k}$ and $\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$

Solution:

a. Given vectors $\vec{a} = 4\vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{b} = 5\vec{i} + \vec{j} - 4\vec{k}$

The vector orthogonal to each of given vectors is given by $\vec{a}\times\vec{b}$

$$\text{Now, } \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & 3 \\ 5 & 1 & -4 \end{vmatrix} = (8-3)\ \vec{i} - (-16-15)\vec{j} + (4+10)\vec{k} = 5\vec{i} + 31\vec{j} + 14\vec{k}$$

$|\vec{a}\times\vec{b}| = \sqrt{5^2+31^2+14^2} = \sqrt{1182}$

Unit vector is given as $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \dfrac{5\vec{i} + 31\vec{j} + 14\vec{k}}{\sqrt{1182}}$

b. Here,

$\vec{a} = (6, 3, -5)$ and $\vec{b} = (1, -4, 2)$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 3 & -5 \\ 1 & -4 & 2 \end{vmatrix}$$

$(6-20)\ \vec{i} - (12+5)\vec{j} + (-24-4)\vec{k} = -14\vec{i} - 17\vec{j} - 27\vec{k}$

$|\vec{a}\times\vec{b}| = \sqrt{(-14)^2+(-17)^2+(-27)^2} = \sqrt{1214}$

$\therefore$ Unit vector is $\dfrac{-14\vec{i} - 17\vec{j} - 27\vec{k}}{\sqrt{1214}}$

5. a. If θ be the angle between the vectors $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{b} = 4\vec{i} - 7\vec{k}$, find the value of $\sin\theta$.
   b. Find the sine of the angle between the vectors $\vec{a} = 3\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{b} = 2\vec{i} - 2\vec{j} + 9\vec{k}$
   c. Find the sine of the angle between the vectors $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 4)$.

Solution:

a. Given, $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$

$\vec{b} = 4\vec{i} - 7\vec{k}$

$a = |\vec{a}| = \sqrt{4 + 9 + 25} = \sqrt{38}$ $b = |\vec{b}| = \sqrt{4^2 + (-7)^2} = \sqrt{65}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 5 \\ 4 & 0 & -7 \end{vmatrix} = 21\vec{i} + 34\vec{j} + 12\vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{21^2 + 34^2 + 12^2} = \sqrt{1741}$

We know that $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|\ |\vec{b}|} = \dfrac{\sqrt{1741}}{\sqrt{38} . \sqrt{65}} = \sqrt{\dfrac{1741}{2470}}$

b. Given, $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 9)$

$a = |\vec{a}| = \sqrt{9 + 1 + 4} = \sqrt{14}$

$b = |\vec{b}| = \sqrt{4 + 4 + 81} = \sqrt{89}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 9 \end{vmatrix} = 13\vec{i} - 23\vec{j} - 8\vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{13^2 + (-23)^2 + (-8)^2} = \sqrt{762}$

$\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \dfrac{\sqrt{762}}{\sqrt{14 \times 89}} = \sqrt{\dfrac{762}{14 \times 89}} = \sqrt{\dfrac{381}{623}}$

c. Given rectors are $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 4)$

$a = |\vec{a}| = \sqrt{9 + 1 + 4} = \sqrt{14}$

$b = |\vec{b}| = \sqrt{4 + 4 + 16} = \sqrt{24}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} = 8\vec{i} - 8\vec{j} - \vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{64 + 64 + 64} = \sqrt{192}$

$\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{ab} = \sqrt{\dfrac{192}{14 \times 24}} = \dfrac{2}{\sqrt{7}}$

6. If the position vectors of the three points P, Q and R are respectively $\vec{i} + \vec{j} + 2\vec{k}$, $2\vec{i} + 3\vec{j} + \vec{k}$ and $3\vec{i} - \vec{j} + 4\vec{k}$, find a vector orthogonal to the plane PQR.

Solution:

Let O be the origin

Given $\vec{OP} = \vec{i} + \vec{j} + 2\vec{k}$

$\vec{OQ} = 2\vec{i} + 3\vec{j} + \vec{k}$

$\vec{OP} = 3\vec{i} - \vec{j} + 4\vec{k}$

$\vec{PQ} = \vec{OQ} - \vec{OQ} = (2, 3, 1) - (1, 1, 2) = (1, 2, -1) = \vec{i} + 2\vec{j} - \vec{k}$

$\vec{QR} = OR - \vec{OQ} = (3, -1, 4) - (2, 3, 1) = (1, -4, 3) = \vec{i} - 4\vec{j} + 3\vec{k}$

$$\vec{PQ} \times \vec{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 1 & -4 & 3 \end{vmatrix} = 2\vec{i} - 4\vec{j} - 6\vec{k}$$

Hence, $2\vec{i} - 4\vec{j} - 6\vec{k}$ is a vector perpendicular to both $\vec{PQ}$ and $\vec{QR}$ and hence perpendicular to the plane PQR.

7. Find the area of the triangle determined by the following pairs of vectors.

a. $\vec{a} = 3\vec{i} + \vec{j} + \vec{k}, \vec{b} = \vec{i} - 2\vec{j} - \vec{k}$. b. $\vec{a} = 3\vec{i} + 4\vec{j}$ and $\vec{b} = -5\vec{i} + 7\vec{j}$

Solution:

a. $\vec{a} = 3\vec{i} + \vec{j} + \vec{k}$ $\vec{b} = \vec{i} - 2\vec{j} - \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \vec{i} + 4\vec{j} - 7\vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{1 + 16 + 49} = \sqrt{66}$

$\therefore$ Area of triangle determined by $\vec{a}$ and $\vec{b}$ is given by $\frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{66}$ sq. units

b. Given vectors $\vec{a} = (3, 4, 0)$ and $\vec{b} = (-5, 7, 0)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 0 \\ -5 & 7 & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 41\vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{0^2 + 0^2 + 41^2} = 41$

Area of triangle determined by the vectors $\vec{a}$ and $\vec{b}$ is given by

$\frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2} \times 41 = 20\frac{1}{2}$ sq. unit

8. Prove that the area of the triangle whose vertices have $3\vec{i} - \vec{j} + 2\vec{k}$, $\vec{i} - \vec{j} - 3\vec{k}$ and $4\vec{i} - 3\vec{j} + 2\vec{k}$ as position vectors is $\frac{1}{2}\sqrt{141}$ square units.

Solution:

Let 0 be the origin. Let A, B and C be vertices of triangle

Then $\vec{OA} = 3\vec{i} - \vec{j} + 2\vec{k}$

$\vec{OB} = \vec{i} - \vec{j} - 3\vec{k}$

$\vec{OC} = 4\vec{i} - 3\vec{j} + 2\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = -2\vec{i} + 0\vec{j} - 5\vec{k}$

$\vec{BC} = \vec{OC} - \vec{OB} = 3\vec{i} - 2\vec{j} + 5\vec{k}$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & -5 \\ 3 & -2 & 5 \end{vmatrix} = -10\vec{i} - 5\vec{j} + 4\vec{k}$$

$|\vec{AB} \times \vec{BC}| = \sqrt{100 + 25 + 16} = \sqrt{141}$

$\therefore$ Area of triangle ABC $= \frac{1}{2}|\vec{AB} \times \vec{BC}| = \frac{1}{2}\sqrt{141}$ sq. units

9. Find the area of the parallelnram whose two adjacent sides are determined by the following pairs of vectors.

a. $\vec{a} = 7\vec{i} + 8\vec{j} - \vec{k}$ and $\vec{b} = 10\vec{i} - 11\vec{j} + 12\vec{k}$ b.

$\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} - 2\vec{j} + 4\vec{k}$

c. $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{b} = 3\vec{i} + 2\vec{j} + 2\vec{k}$

Solution:

a. Given, $\vec{a} = 7\vec{i} + 8\vec{j} - \vec{k}$ $\quad\quad \vec{b} = 10\vec{i} - 11\vec{j} + 12\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 8 & -1 \\ 10 & -11 & 12 \end{vmatrix} = (96 - 11)\vec{i} - (84 + 10)\vec{j} + (-77-80)\vec{k} = 85\vec{i} - 94\vec{j} - 157\vec{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{85^2 + 94^2 + 157^2} = \sqrt{40710}$$

$\therefore$ Area of parallelnram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\sqrt{40710}$ sq. units.

b. $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ $\quad\quad \vec{b} = \vec{i} - 2\vec{j} + 4\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & -2 & 4 \end{vmatrix} = (8+6)\vec{i} - (4 - 3)\vec{j} + (-2 - 2)\vec{k} = 14\vec{i} - \vec{j} - 4\vec{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{14^2 + 1^2 + 4^2} = \sqrt{196 + 1 + 16} = \sqrt{213}$$

$\therefore$ Area of parallelnram = $|\vec{a} \times \vec{b}| = \sqrt{213}$ sq units.

c. Given, $\vec{a} = (1, -2, 3)$ and $\vec{b} = (3, 2, 2)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ 3 & 2 & 2 \end{vmatrix} = (-4 -6)\vec{i} - (2 - 9)\vec{j} + (2 + 6)\vec{k}\vec{k} = -10\vec{i} + 7\vec{j} + 8\vec{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{100 + 49 + 64} = \sqrt{213}$$

$\therefore$ Area of parallelnram = $|\vec{a} \times \vec{b}| = \sqrt{213}$ sq units

10. Find the area of the parallelnram whose diagonals represent the vectors $\vec{i} + \vec{j} - \vec{k}$ and $\vec{i} - \vec{j} + \vec{k}$.

Solution:

Let $\vec{d}_1 = \vec{i} + \vec{j} - \vec{k}$ and $\vec{d}_2 = \vec{i} - \vec{j} + \vec{k}$ be two diagonals of a parallelnram.

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 0\vec{i} - 2\vec{j} - 2\vec{k}$$

$$|\vec{d}_1 \times \vec{d}_2| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

Area of parallelnram whose diagonals $\vec{d}_1$ and $\vec{d}_2$ is given by

$\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$ sq. units = $\frac{1}{2} . 2\sqrt{2}$ sq. units = $\sqrt{2}$ sq. units

11. a. If $|\vec{a}| = 15$, $|\vec{b}| = 4$ and $|\vec{a}\times\vec{b}| = 36$, find the value of $\vec{a}.\vec{b}$.

b. Given $|\vec{a}| = 9$, $|\vec{b}| = 5$ and $\vec{a}.\vec{b} = 36$, find $|\vec{a}\times\vec{b}|$.

c. Given any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, prove that
$\vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b}) = 0.$

d. If $\vec{a}+\vec{b}+\vec{c} = 0$, prove that $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$.

Solution:

a. Given $|\vec{a}| = 15$, $|\vec{b}| = 4$ and $|\vec{a}\times\vec{b}| = 36$

If θ be the angle between two vectors $\vec{a}$ and $\vec{b}$ then

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{36}{15\times4} = \frac{9}{15} = \frac{3}{5}$$

$$\therefore \quad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$$

Also, we know that $\cos\theta = \dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}||\vec{b}|}$

$$\therefore \quad \vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = 15\times4\times\frac{4}{5} = 48$$

$$\therefore \quad \vec{a}.\vec{b} = 48$$

b. Given, $|\vec{a}| = 9$, $|\vec{b}| = 5$ and $\vec{a}.\vec{b} = 36$

If θ be the angle between $\vec{a}$ and $\vec{b}$

$$\text{Then, } \cos\theta = \frac{\vec{a}.\vec{b}}{ab} = \frac{36}{9\times5} = \frac{4}{5}$$

$$\therefore \quad \sin\theta = \sqrt{1\ \cos^2\theta} = \frac{3}{5}$$

$$\text{Also, } |\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta = 9\times5\times\frac{3}{5} = 27$$

c. $\text{LHS} = \vec{a}\times(\vec{b}+\vec{c}) + \vec{b}\times(\vec{c}+\vec{a}) + \vec{c}\times(\vec{a}+\vec{b})$

$= \vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a} + \vec{c}\times\vec{b}$

$= \vec{a}\times\vec{b} - \vec{c}\times\vec{a} + \vec{b}\times\vec{c} - \vec{a}\times\vec{b} + \vec{c}\times\vec{a} - \vec{b}\times\vec{c} = 0$ RHS

d. Suppose

$\vec{a}+\vec{b}+\vec{c} = 0$

$\vec{a}+\vec{b} = -\vec{c}$ ... ... (i)

Taking cross product with $\vec{a}$ on both sides

$\vec{a}\times(\vec{a}+\vec{b}) = \vec{a}\times\vec{c})$

or, $\vec{a}\times\vec{a} + \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$

or, $0 + \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$

or, $\vec{a}\times\vec{b} = \vec{c}\times\vec{a}$ ... ... (ii)

Again, taking cross product with $\vec{b}$ on equation (i) both sides

$\vec{b}\times(\vec{a}+\vec{b}) = \vec{b}\times(-\vec{c})$

$\vec{b}\times\vec{a} + \vec{b}\times\vec{b} = -\vec{b}\times\vec{c}$

or, $-\vec{a}\times\vec{b} = 0 = -\vec{b}\times\vec{c}$

$\therefore \quad \vec{a}\times\vec{b} = \vec{b}\times\vec{c}$ ... ... (iii)

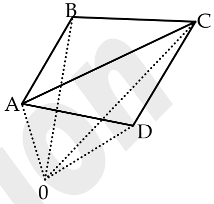
Combining (ii) and (iii) we get,

$\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ Proved.

12. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are the position vectors of the vertices of a quadrilateral ABCD, prove that the vector area of the quadrilateral ABCD is
$\frac{1}{2}[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{a}]$.

Solution:

Let 0 be the origin suppose A, B, C, D are vertices of a quadrilateral ABCD.



Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$ and $\vec{OD} = \vec{d}$

Now, $\vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a}$

$\vec{BC} = \vec{OC} - \vec{OB} = \vec{c} - \vec{b}$

$\vec{AC} = \vec{OC} - \vec{OA} = \vec{c} - \vec{a}$

$\vec{CD} = \vec{OD} - \vec{OC} = \vec{d} - \vec{c}$

Vector area of ΔABC

$= \frac{1}{2}\vec{AB} \times \vec{BC} = \frac{1}{2}(\vec{b} - \vec{a}) \times (\vec{c} - \vec{b}) = \frac{1}{2}[\vec{b}\times\vec{c} - \vec{b}\times\vec{b} - \vec{a}\times\vec{c} + \vec{a}\times\vec{b}]$

$= \frac{1}{2}[\vec{b}\times\vec{c} - 0 + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}] = \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}]$

Again, vector area of ΔACD

$= \frac{1}{2}(\vec{AC}\times\vec{CD}) = \frac{1}{2}[(\vec{c} - \vec{a}) \times (\vec{d} - \vec{c})]$

$= \frac{1}{2}[\vec{c}\times\vec{d} - \vec{c}\times\vec{c} - \vec{a}\times\vec{d} + \vec{a}\times\vec{c}] = \frac{1}{2}[\vec{c}\times\vec{d} + \vec{d}\times\vec{a} + \vec{a}\times\vec{c}]$

$\therefore$ Vector of quadrilateral ABCD = vector area of ΔABC + vector area of ΔACD

$= \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{c}+\vec{d} + \vec{d}\times\vec{a} + \vec{a}\times\vec{b}\times\vec{c}]$

$= \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{c}+\vec{d} + \vec{d}\times\vec{a} - \vec{c}\times\vec{a}] = \frac{1}{2}[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{d} + \vec{d}\times\vec{a}]$

13. If $\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - 3\vec{k}$ and $\vec{c} = \vec{i} - 2\vec{j} + 2\vec{k}$, then find

a. $(\vec{a} \times \vec{b}) \times \vec{c}$

b. $\vec{a} \times (\vec{b} \times \vec{c})$ and hence show that $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Solution:

Given,

$\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$ $\quad$ $\vec{b} = 2\vec{i} + \vec{j} - 3\vec{k}$ $\quad$ $\vec{c} = \vec{i} - 2\vec{j} + 2\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \vec{i} + 13\vec{j} + 5\vec{k}$$

$$(\vec{a}\times\vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 13 & 5 \\ 1 & -2 & 2 \end{vmatrix} = 36\vec{i} + 3\vec{j} - 15\vec{k} \;\ldots\ldots\ldots\; \text{(i)}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 1 & -2 & 2 \end{vmatrix} = -4\vec{i} - 7\vec{j} - 5\vec{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ -4 & -7 & -5 \end{vmatrix} = 19\vec{i} + 7\vec{j} - 25\vec{k} \text{ ... ... (ii)}$$

From (i) and (ii), Hence $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

14. If $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{c} = \vec{i} - \vec{j}$, then find $\vec{b}$ such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a} \cdot \vec{b} = 3$

Solution:

Given, $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ $\qquad \vec{c} = \vec{i} - \vec{j}$

Let $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$\vec{a}.\vec{b} = (\vec{i} + \vec{j} + \vec{k}) . (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$

$3 = b_1 + b_2 + b_3$ ... ... (i)

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \end{vmatrix} = (b_3 - b_2)\vec{i} + (b_1 - b_3)\vec{j} + (b_2 - b_1)\vec{k}$$

or, $\vec{c} = (b_3 - b_2)\vec{i} + (b_1 - b_3)\vec{j} + (b_2 - b_1)\vec{k}$

$\vec{i} - \vec{j} = (b_3 - b_2)\vec{i} + (b_1 - b_3)\vec{j} + (b_2 - b_1)\vec{k}$

Equating corresponding vectors

$b_3 - b_2 = 1$, $b_1 - b_3 = -1$ and $b_2 - b_1 = 0$

i.e. $b_2 - b_1 = 0$

$\therefore$ $b_1 = b_2$ ... ... (ii)

$b_3 = 1 + b_2$ ... ... (iii)

$b_3 = 1 + b_1$ ... ... (iv)

or, $b_1 + b_2 + b_3 = 3$

or, $b_1 + b_1 + 1 + b_1 = 3$

or, $3b_1 = 2$

$\therefore$ $b_1 = \frac{2}{3}$

$\therefore$ from (ii) $b_2 = \frac{2}{3}$

from (iv) $b_3 = 1 + \frac{2}{3} = \frac{5}{3}$

$\therefore$ $\vec{b} = (b_1, b_2, b_3) = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{5}{3}\vec{k} = \frac{1}{3}(2\vec{i} + 2\vec{j} + 5\vec{k})$

■

CHAPTER 13

CORRELATION AND REGRESSION

EXERCISE 13.1

1. Find the correlation coefficient between the two variables under the following conditions.

a. Cov (X, Y) = -16.5, Var(X) = 2.89 and Var(Y) = 100

b. N = 13, $\Sigma X = 117$, $\Sigma X^2 = 1.313$, $\Sigma Y = 260$, $\Sigma Y^2 = 6580$, $\Sigma XY = 2827$

c. N = 15, $\sigma_x = 3.2$, $\sigma_y = 3.4$ and $\Sigma(X - \bar{X})(Y - \bar{Y}) = 122$.

Solution:

a. Here, cov (x, y) = –16.5

var (x) = 2.89

var (y) = 100

Coefficient off correlation

$$r = \frac{\text{cov}(x, y)}{\sqrt{\text{var}(x)} \cdot \sqrt{\text{var}(y)}}$$

$$= \frac{-16.5}{\sqrt{2.89} \cdot \sqrt{100}}$$

$$= \frac{-16.5}{1.7 \times 10}$$

$$= \frac{-16.5}{17}$$

$$= -0.97$$

b. Here,

Given, N = 13 $\quad \Sigma x = 117$

$\Sigma x^3 = 1{,}313 \quad \Sigma y = 260$

$\Sigma y^2 = 6580 \quad \Sigma xy = 2827$

Coefficient of correlation

$$r = \frac{n\Sigma xy - \Sigma x. \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2} \cdot \sqrt{n\Sigma y - (\Sigma y)^2}}$$

$$= \frac{3 \times 2827 - 117 \times 260}{\sqrt{13 \times 1313} - \sqrt{13 \times 6580} - ....}$$

$$= \frac{36751 - 30420}{\sqrt{3380} - \sqrt{17940}}$$

$$= \frac{6331}{58.13 \times 133.94} = 0.81$$

c. Here, n = 15

$\sigma_x = 3.2 \qquad \sigma_y = 3.y$

$\Sigma(x - \bar{x})(y - \bar{y}) = 122$

$$\text{Coefficient of correlation (r)} = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y} = \frac{122}{15 \times 3.2 \times 3.y} = \frac{122}{163.2} = 0.75$$

2. a. Karl Pearson's coefficient of correlation between two variables X and Y is 0.28, their covariance is 0.76. If the variance of X is 9, find the standard deviation of Y series.

b. Given the following; correlation coefficient between X and Y = 0.85, Cov (X, Y) = 6.5, Var (X) = 6.1. Find the standard deviation of Y-series.

Solution:

a. Karl Pearson's coefficient of correlation between 'x' and 'y' (r) = 0.28

Cor (x, y) = 0.76, Var (x) = 9, σ_y = ?

Now, We have,

$$\text{Coefficient of correlation (r)} = \frac{\text{cov}(x, y)}{\sqrt{\text{var}(x)} \cdot \sqrt{\text{var}(y)}}$$

$$\text{or, } 0.28 = \frac{0.76}{\sqrt{9} \cdot \sqrt{\text{var}(y)}}$$

$$\text{or, } 0.36 = \frac{1}{3.\sqrt{\text{var}(y)}}$$

or, $1.105 \cdot \sqrt{\text{var}(y)} = 1$

or, $\sqrt{\text{var (y)}} = 0.904 \therefore \sigma_y = 0.904$

b. Correlation coefficient between x and y (r) = 0.85

Cov (x,y) = 6.5

Var (x) = 6.1

Standard derivation of y (σ_y) = ?

Now, we have,

Coefficient of correlation (r) = $\dfrac{\text{cov}(x,y)}{\sqrt{\text{var}(x)} \cdot \sqrt{\text{var}(y)}}$

or, $0.85 = \dfrac{6.5}{\sqrt{6.1} \cdot \sqrt{\text{var}(y)}}$

or, $0.13076 \times 2.4698 \cdot \sqrt{\text{var}(y)} = 1$

or, $\sqrt{\text{var}(y)} = \dfrac{1}{0.32295}$

or, $\sqrt{\text{var}(y)} = 3.096$

or, $\sigma_y = 3.096$

Hence, the required σ_y is 3.096

3. a. Calculate Karl Pearson's correlation coefficient between the two variables from the data given below.

Marks in Maths	48	35	17	23	47
Marks in Biolny	45	20	40	25	45

b. Calculate the coefficient of correlation from the following data of price and demand.

Data of price (Rs)	14	16	19	22	24	30
Demand (Rs)	24	22	20	24	23	28

c. Find the covariance and correlation coefficient between x and y for the following observations of

(x, y): (20, 7), (10, 15), (20, 12), (10, 16), (17, 17), (12, 10), (15, 11), (16, 8)

Solution:

a. Here,

Maths(x)	Biolny (y)	$x = (x - \bar{x})$	$y = (y - \bar{y})$	xy	x^2	y^2
48	45	14	10	140	196	100
35	20	1	–15	–15	1	225
17	40	–17	5	–85	289	25
23	25	–11	–10	110	121	100
47	45	13	10	130	169	100
$\Sigma x = 170$	$\Sigma y = 175$			$\Sigma xy = 280$	$\Sigma x^2 = 776$	$\Sigma y^2 = 550$

$\bar{x} = \dfrac{\Sigma x}{n} = \dfrac{170}{5} = 34,$ $\quad \bar{y} = \dfrac{\Sigma y}{N} = \dfrac{175}{5} = 35$

Karl Pearson's coefficient of correlation (r) = $\dfrac{\Sigma xy}{\sqrt{\Sigma x^2} \cdot \sqrt{\Sigma y^2}} = \dfrac{280}{\sqrt{776} \cdot \sqrt{550}}$

$= \dfrac{280}{\sqrt{776} \cdot \sqrt{550}} = \dfrac{280}{27.85 \times 23.45}$

$= \dfrac{280}{653.0325} = 0.42$

b. Calculation of correlation coefficient

Data of price (Rs.) (x)	Demand (Rs.) (y)	u = x – 19	v = v_1 – 20	u^2	v^2	uv
14	24	–5	–4	25	16	20
16	22	–3	–2	9	4	6
19	20	0	0	0	0	0
22	24	3	4	9	16	12
24	23	5	3	25	9	15
30	28	11	8	121	64	88
		11	9	189	109	141

∴ Required correlation coefficient is given by

$$r = \frac{n\Sigma uv - \Sigma u\Sigma v}{\sqrt{n\Sigma u^2 - (\Sigma u)^2}\sqrt{n\Sigma v^2 - (\Sigma v)^2}} = \frac{6 \times 141 - 11 \times 9}{\sqrt{6 \times 189 - (11)^2}\sqrt{6 \times 109 - (9)^2}}$$

$$= \frac{846 - 99}{\sqrt{1013}\sqrt{573}} = \frac{747}{31.83 \times 23.94} = \frac{747}{762.01} = 0.98$$

c. Calculation of co-varience and correlation coefficient

X	Y	$x-\bar{x}$	$(x-\bar{x})^2$	$y-\bar{y}$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
20	7	5	25	–5	25	–25
10	15	–5	25	3	9	–15
20	12	5	25	0	0	0
10	16	–5	25	4	16	–20
17	17	2	4	5	25	10
12	10	–3	9	–2	4	6
15	11	0	0	–1	1	0
16	8	1	1	–4	16	–4
Σx = 120	**Σy = 96**		**114**		**94**	**$\Sigma(x-\bar{x})(y-\bar{y})$=–48**

Here, $\bar{x} = \frac{\Sigma x}{n}$ and $\bar{y} = \frac{\Sigma y}{n}$

$\Rightarrow$ $\bar{x} = \frac{120}{8}$ and $\bar{y} = \frac{96}{8}$

∴ $\bar{x} = 15$ and $\bar{y} = 12$

∴ Co–varience of (x, y) = $\frac{\Sigma(x-\bar{x})(y-\bar{y})}{n} = \frac{-48}{8} = -6$

And, the correlation coefficient is

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2}\sqrt{(y-\bar{y})^2}} = \frac{-48}{\sqrt{114}\sqrt{94}} = \frac{-48}{10.67 \times 9.70} = \frac{-48}{103.50} = -0.463$$

4. a. From the following table calculate Karl Pearson's correlation coefficient between the two variables;

X	6	2	10	4	8
Y	9	11	?	8	7

Arithemetic means of X and Y series are 6 and 8 respectively.

b. From the following table calculate the missing data of X – series and correlation coefficient by Karl Pearson's method.

X	10	12	20	?	16	14
Y	9	12	15	18	14	16

The arithemetic means of X is 13.

Solution:

a. Here,

X	Y	$x = (X-\bar{X})$	$y = (Y-\bar{Y})$	xy	x^2	y^2
6	9	0	1	0	0	1
2	11	–4	3	–12	16	9
10	5	4	–3	–12	16	9
4	8	–2	0	0	4	0
8	7	2	–1	–2	4	1
$\Sigma x = 30$	$\Sigma y = 40$	$\Sigma x = 0$	$\Sigma y = 0$	$\Sigma xy = 26$	$\Sigma x^2 = 40$	$\Sigma y^2 = 20$

Hence, $\bar{y} = \dfrac{\Sigma y}{n}$, $\bar{x} = 6$

or, $8 = \dfrac{35 + a}{5}$

or, $40 - 35 \pm a$

$\therefore$ $a = 5$

Coefficient of correlation (r) $= \dfrac{\Sigma xy}{\sqrt{\Sigma x^2}\,.\,\sqrt{\Sigma y^2}} = \dfrac{-26}{\sqrt{40}\,.\,\sqrt{20}} = \dfrac{-26}{\sqrt{800}} = \dfrac{-26}{28.28} = -0.92$

b.

X	Y	$x=(X-\bar{X})$	$y=(Y-\bar{Y})$	x^2	y^2	xy
10	9	–3	–5	9	25	15
12	12	–1	–2	1	4	2
20	15	7	1	49	1	7
$x_1(6)$	18	–7	4	49	16	–28
16	14	3	0	9	0	0
14	16	1	2	1	4	2
$\Sigma x = 72 + x$	**$\Sigma y = 84$**	**0**	**0**	**118**	**50**	**–2**

It is given that,

$\bar{x} = 13$ and $\bar{y} = \dfrac{\Sigma y}{n}$

$\Rightarrow \dfrac{\Sigma x}{n} = 13$ and $\bar{y} = \dfrac{84}{6}$

$\Rightarrow \dfrac{72 + x_1}{6} = 13$ and $\bar{y} = 14$

$\therefore$ $x_1 = 6$

$\therefore$ The correlation coefficient is given by

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2}\,\sqrt{\Sigma y^2}} = \frac{-2}{\sqrt{118}\,\sqrt{50}} = \frac{-2}{10.86 \times 7.07} = \frac{-2}{76.78} = -0.026$$

5. Calculate the correlation coefficient for the following series of age of husband (X) and age of wife (Y).

X	41	44	45	48	40	42	44
Y	22	24	25	27	21	22	23

Solution:

X	Y	x^2	y^2	xy
41	22	1681	484	902
44	24	1936	576	1051

45	25	2025	625	1125
48	27	2304	729	1296
40	21	1600	441	840
42	22	1764	484	924
44	23	1936	529	1012
$\Sigma x=304$	$\Sigma y=164$	$\Sigma x^2 = 13246$	$\Sigma y^2=3868$	$\Sigma xy=7155$

No. of items (n) = 7

Coefficient of collection (r) $= \dfrac{n\Sigma xy - \Sigma x.\ \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\ .\ \sqrt{n\Sigma y^2 - (\Sigma y)^2}}$

$$= \frac{7\times 7155 - 304\times 164}{\sqrt{7\times 13246 - (304)^2}.\ \sqrt{7\times 3868.(-164)^2}}$$

$$= \frac{50085 - 49856}{\sqrt{306}\sqrt{180}} = \frac{229}{17.49 \times 13.41} = 0.976$$

6. The following data gives the marks obtained by 10 students in mathematics and English.

Student	1	2	3	4	5	6	7	8	9	10
Marks in math	45	70	65	30	90	40	50	75	85	60
Marks in Eng	35	90	70	40	95	40	60	80	80	50

Find the coefficient of correlation and interprent it.

Solution:

Math (x)	Eng(y)	xy	x^2	y^2
45	35	1575	2025	1225
70	90	6300	4900	8100
65	70	4550	4225	4900
30	40	1200	900	1600
90	95	8550	8100	9025
40	40	1600	1600	1600
50	60	3000	2500	3600
75	80	6000	5625	6400
85	80	6800	7225	6400
60	50	3000	3600	2500
$\Sigma x=610$	$\Sigma y=640$	$\Sigma xy=42575$	$\Sigma x^2=40700$	$\Sigma y^2=45350$

Coefficient of correlation (r) $= \dfrac{n\Sigma xy - \Sigma x.\ \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\ .\ \sqrt{n\Sigma y^2 - (\Sigma y)^2}}$

$$= \frac{10\times 42575 - 610\times 640}{\sqrt{10\times 40700 - (610)^2}\ .\ \sqrt{10\times 45360 - ...}}$$

$$= \frac{425750 - 390400}{\sqrt{34900}\ .\ \sqrt{43900}} = \frac{35350}{186.81\times 209.52}$$

$$= \frac{35350}{39140.4312} = 0.9033$$

7. A person while calculating the correlation coefficient between the variables X and Y obtained the following n = 30, $\Sigma X = 120$ $\Sigma X^2 = 600$, $\Sigma Y = 90$, $\Sigma Y^2 = 250$, $\Sigma XY = 356$. It was, however later discovered at the time of checking that it had copied down two pairs of observations as, (8, 10) and (12, 7) while correct values were (8, 12) and (10, 8) obtain the correct value of correlation coefficient between them.

Solution:

Here, Corrected $\Sigma x = 120 - 8 - 12 + 8 + 10 = 118$

Corrected $\Sigma y = 90 - 10 - 7 + 12 + 8 = 93$

Corrected $\Sigma x^2 = 600 - 8^2 - 12^2 + 8^2 + 10^2 = 556$

Corrected $\Sigma y^2 = 250 - 10^2 - 7^2 + 12^2 + 8^2 = 309$

Corrected $\Sigma xy = 356 - 8\times10 - 12\times7 + 8\times12 + 10\times8 = 368$

Now, Corrected value of f = 8

$$r_e = \frac{n\Sigma xy - \Sigma x.\ \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\ .\ \sqrt{n\Sigma y^2 - (\Sigma y^2)}} = \frac{30\times368 - 118\times93}{\sqrt{30\times556 - (118)^2}\sqrt{30\times309 - (93)^2}}$$

$$= \frac{11040 - 10974}{\sqrt{16680 - 13924}.\ \sqrt{9270 - 8649}} = \frac{66}{52.49 \times 24.91} = \frac{66}{1307.5259} = 0.05$$

8. Calculate Spearman's rank correlation coefficient between advertisement cost (*X*) and sales (*Y*) from the following data:

X:	39	65	62	90	82	75	25	98	36	78
Y:	47	53	58	86	62	68	60	91	51	84

Solution:

Here,

x	Rank (R_x)	y	Rank (R_y)	$d = R_x - R_y$	d^2
39	8	47	10	–2	4
65	6	53	8	–2	4
62	7	58	7	0	0
90	2	86	2	0	0
82	3	62	5	–2	4
75	5	68	4	1	1
25	10	60	6	4	16
98	1	91	1	0	0
36	9	51	9	0	0
78	4	84	3	1	1
					$\Sigma d^2 = 30$

$$\text{Rank } (\rho) = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)} = 1 - \frac{6 \times 30}{10 \times 99} = 1 - \frac{150}{990} = 0.8485$$

9. Two bank officers examined eleven loan applications and ranked them. Compute the rank Correlation coefficient.

Loan applicants	A	B	C	D	E	F	G	H	I	J	K
Officer I	1	7	4	2	3	6	5	9	10	8	11
Officer II	1	6	5	2	3	4	7	11	8	10	9

Solution:

Here,

Officer (x)	Officer (y)	xy	x^2	y^2
1	1	1	1	1
7	6	42	49	36
4	5	20	16	25
2	2	4	4	4
3	3	9	9	9
6	4	24	36	16
5	7	35	25	49

9	11	99	81	121
10	8	80	100	64
8	10	80	64	100
11	9	99	121	81
$\Sigma x = 66$	**$\Sigma y = 66$**	**$\Sigma xy = 493$**	**$\Sigma x^2 = 506$**	**$\Sigma y^2 = 506$**

Correlation coefficient (r) = $\dfrac{n\Sigma xy - \Sigma x.\ \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\ .\ \sqrt{n\Sigma y^2 - (\Sigma y^2)}}$

$$= \frac{11\times493 - 66\times66}{\sqrt{11\times506 - (66)^2}\ \sqrt{11\times506 - (66)^2}} = \frac{5423 - 4356}{\sqrt{1210}\ .\ \sqrt{1210}}$$

$$= \frac{1067}{1210} = 0.882$$

10. Seven methods of imparting business education were ranked by MBA students of two campuses as follows.

Methods of teaching	I	II	III	IV	V	VI	VII
Rank by student of A	2	1	3	5	4	6	7
Rank by student of B	1	3	2	4	7	5	6

Compute Spearman's rank correlation

Solution:

Calculation for spearman's rank correlation

Teaching method	**Rank of st A (R_A)**	**Rank of std. B (R_B)**	**$d = R_A - R_B$**	**d^2**
I	2	1	1	1
II	1	3	–2	4
III	3	2	1	1
IV	5	4	1	1
V	4	7	–3	9
VI	6	5	1	1
VII	7	6	1	1
				$\Sigma d^2 = 18$

Rank (ρ) = $1 - \dfrac{6\ \Sigma d^2}{n(n^2 - 1)} = 1 - \dfrac{6 \times 18}{7 \times 48} = 1 - \dfrac{108}{336} = \dfrac{228}{336} = 0.68$

11. From the following data of the martks obtained by 8 students in accountancy and statistics paper, calculate Spearman's rank correlation:

Marks in Accoutancy (X)	25	68	45	50	80	74	50	68
Marks in Statistics (Y)	36	40	57	40	72	75	60	40

Solution:

Calculation of correlation coefficient

X	Y	R_1	R_2	$d = R_1 - R_2$	d^2
25	36	8	8	0	0
68	40	3.5	6	–2.5	6.25
45	57	7	4	3	9
50	40	5.5	6	–0.5	0.25
80	72	1	2	–1	1
74	75	2	1	1	1
50	60	5.5	3	2.5	6.25
68	40	3.5	6	–2.5	6.25
					$\Sigma d^2 = 30$

Here , n = 8, $m_1 = 2$, $m_2 = 2$, $m_3 = 3$ R = ?

$$\text{Now, } R = 1 - \frac{6\left\{\sum d^2 + \frac{m_1(m_1^2 - 1)}{12} + \frac{m_2(m_2^2 - 1)}{12} + \frac{m_3(m_3^2 - 1)}{12}\right\}}{N(N^2 - 1)}$$

$$= 1 - \frac{6\left\{30 + \frac{2(4 - 1)}{12} + \frac{2(4 - 1)}{12} + \frac{3(9 - 1)}{12}\right\}}{8(64 - 1)}$$

$$= 1 - \frac{6(30 + 0.5 + 0.5 + 2)}{8 \times 63} = \frac{504 - 198}{504} = \frac{306}{504} = 0.60$$

EXERCISE 13.2

1. From the following results, find the regression coefficients.
   a. $\sigma_x = 20$, $\sigma_y = 15$, r = 0.48 b. $\sigma_x = 8$, $\sigma_y = 10$, r = -0.6

Solution:

a. Here, $\sigma_x = 20$, $\sigma_y = 15$, r = 0.48

The regression coefficients are $b_{yx} = r\frac{\sigma_y}{\sigma_x} = 0.48 \times \frac{15}{20} = 0.36$

and $b_{xy} = r\frac{\sigma_x}{\sigma_y} = 0.48 \times \frac{20}{15} = 0.64$

b. $\sigma_x = 8$, $\sigma_y = 10$, r = –0.6

The regression coefficients are

$b_{yx} = r\frac{\sigma_y}{\sigma_x} = -0.6 \times \frac{10}{8} = -0.75$ and $b_{xy} = r\frac{\sigma_x}{\sigma_y} = -0.6 \times \frac{8}{10} = -0.48$

2. Find the correlation coefficient between the two various under the following condition (if possible)
   a. $b_{xy} = 1.8$ and $b_{yx} = 0.35$ b. $b_{yx} = 0.84$ and $b_{xy} = 1.15$
   c. The two regression coefficients are 1.36 and 0.8.

Solution:

a. Here, $b_{yx} = 0.35$, $b_{xy} = 1.8$

Now, correlation coefficient (r) = $\sqrt{b_{yx} . b_{xy}} = \sqrt{0.35 \times 1.8} = \sqrt{0.63} = 0.7937$

b. We have, $\Sigma x = 60$, $\Sigma y = 40$, $\Sigma xy = 1150$

$\Sigma x^2 = 4160$, $\Sigma y^2 = 1720$, N = 10

Here, b_{yx}

$$= \frac{N\Sigma xy - \Sigma x\Sigma y}{N\Sigma x^2 - (\Sigma x)^2} = \frac{10 \times 1150 - 60 \times 40}{10 \times 4160 - (60)^2} = \frac{11500 - 2400}{4160 - 3600} = \frac{9100}{38000} = 0.2394$$

$$b_{xy} = \frac{N\Sigma xy - \Sigma x\Sigma y}{N\Sigma y^2 - (\Sigma y)^2} = \frac{10 \times 1150 - 60 \times 40}{10 \times 1720 - (40)^2} = \frac{9100}{15600} = 0.5833$$

And, $\bar{x} = \frac{\Sigma x}{N} = \frac{60}{10} = 6$, $\bar{y} = \frac{\Sigma y}{N} = \frac{40}{10} = 4$

Now, Regression equation of y on x is,

$y - \bar{y} = b_{yx}(x - \bar{x})$

$\Rightarrow$ y – 4 = 0.2394 (x – 6)

$\Rightarrow$ y – 4 = 0.2394x – 1.4364

$\Rightarrow$ y = 2.5636 + 0.2394x

Regression equation of x on y is,

$x - \overline{x} = b_{xy}(y - \overline{y})$

$\Rightarrow$ $x - 6 = 0.5833 (Y - 4)$

$\Rightarrow$ $x - 6 = 0.5833y - 2.3332$

$\Rightarrow$ $x = 3.6668 + 0.5833y$

And, the correlation coefficient is

$r = \sqrt{b_{yx} \times b_{xy}} = \sqrt{0.2394 \times 0.5833} = \sqrt{0.1396} = 0.3736$

c. We have,

$b_{yx} = 2.002$, $b_{xy} = -0.461$, $\overline{x} = 87.2$,

$\overline{y} = 127.2$

$\therefore$ Correlation coefficient is, $r = \sqrt{-2.002 \times -0.461} = -0.9606$

Since $b_{yx} < 0$, $b_{xy} < 0$ and $r > 0$

3. a. On the basis of the given information find the regression coefficient of X on Y

$\sum XY = 750$ $\sum X^2 = 2085$ $\sum Y^2 = 285$

$\sum X = 135$ $\sum Y = 45$ $N = 9$

b. Find the regression equations from the following data.

$\Sigma X = 60$, $\Sigma Y = 40$, $\Sigma XY = 1150$,

$\Sigma X^2 = 4160$, $\Sigma Y^2 = 1720$, $N = 10$

Also find the correlation coefficient between X and Y.

c. Regression coefficient of Y on X and X on Y are given as –2.002 and –0.461. Find the value of correlation coefficient between X and Y. If mean of X and Y are 87.2 and 127.2, estimate X when Y = 133.

d. Find the most likely price in Kathmandu corresponding to the price of Rs. 70 at Birgunj from the following data:

Average price in Birgunj	Rs. 65
Average price in Kathmandu	Rs. 67
Standard deviation of Birgunj price	2.5
Standard deviation of Kathmandu price	3.5
Correlation co-efficient between the price in two cities	+ 0.08

Solution:

a. Here, $\sum xy = 750$, $\sum x^2 = 2085$, $\sum y^2 = 285$, $\sum x = 135$, $\sum y = 45$, $N = 9$

Now, The regression coefficient of x only is

$$b_{xy} = \frac{n\sum xy - \sum x.\sum y}{n\sum y^2 - (\sum y)^2} = \frac{9 \times 750 - 135 \times 45}{9 \times 285 - (45)^2} = \frac{6750 - 6075}{2565 - 2025} = \frac{675}{540} = 1.25$$

b. Here, $\sum x = 60$, $\sum y = 40$, $\sum xy = 1150$, $\sum x^2 = 4160$, $\sum y^2 = 1720$, $n = 10$

Now, the point through which the regression lines intersect to each other is

$$(\overline{x}, \overline{y}) = \left(\frac{\sum x}{n}, \frac{\sum y}{n}\right) = \left(\frac{60}{10}, \frac{40}{10}\right) = (6, 4)$$

Since the equation of regression line of y on x is

$y - \overline{y} = b_{yx}(x - \overline{x})$ ... ... (i) and x on y is

$x - \overline{x} = b_{xy}(y - \overline{y})$ ... ... (ii) where

$$b_{yx} = \frac{n\sum xy - \sum x . \sum y}{n\sum x^2 - (\sum x^2)} = \frac{10 \times 1150 - 60 \times 40}{10 \times 4160 - 60^2} = \frac{11500 - 2400}{41600 - 3600} = \frac{9100}{38000} = 0.239$$

$$\text{and } b_{xy} = \frac{n\sum xy - \sum x. \sum y}{n\sum y^2 - (\sum y)^2} = \frac{10 \times 1150 - 60\times40}{10\times1720 - 40^2} = \frac{9100}{15600} = 0.583$$

Hence, from (i) and (ii), the required equations are

$y - 4 = 0.239\ (x - 6)$

or, $y = 2.566 + 0.239x$

and $x - 6 = 0.583\ (y - 4)$

or, $x = 3.668 + 0.583y$

Also, correlation coefficient (r) = $\sqrt{b_{yx}\ .\ b_{xy}} = \sqrt{0.239 \times 0.583} = \sqrt{0.139337} = 0.373$

c. Here, $b_{yx} = -2.002$ and $b_{xy} = -0.461$

Now, $r = \sqrt{b_{yx}\ .\ b_{xy}} = \sqrt{-2.002 \times -0.461} = \sqrt{0.922922} = -0.9607$

Also, $\bar{x} = 87.2$, $\bar{y} = 127.2$, $y = 133$, $x = ?$

Using, $x - \bar{x} = b_{xy}\ (y - \bar{y})$,

or, $x - 87.2 = -0.461\ (y - 127.2)$

or, $x - 87.2 = -461\ (133 - 127.2)$

or, $x - 87.2 = -2.6738$

or, $x = 84.5262$

d. Let average price in Birgunj $(\bar{x})$ = Rs. 65

Average price in Kathmandu $(\bar{y})$ = Rs. 67

$\sigma_x = 2.5$, $\sigma_y = 3.5$, $r = 0.08$

Now, $b_{yx} = r\ .\ \dfrac{\sigma_y}{\sigma_x} = 0.08 \times \dfrac{3.5}{2.5} = 0.112$

The equation of regression line of y on x is

$y - \bar{y} = b_{yx}\ (x - \bar{x})$

or, $y - 67 = 0.112\ (x - 65)$

or, $y - 67 = 0.112 \times -7.28$

or, $y = 59.72 + 0.112x$

If x = Rs. 70, then $y = 59.72 + 0.112 \times 70$ = Rs. 67.56

4. From the following pair of regression equations, find the regression coefficients, correlation coefficients and the means of x and y series.

a. $3x + 2y - 26 = 0;\ 6x + y - 31 = 0$ b. $3x + 4y = 65,\ 3x + y = 32$

Solution:

a. Here, the regression equations ae $3x + 2y - 26 = 0;\ 6x + y - 31 = 0$

or, $2y = -3x + 26$ and $6x = -y + 31$

or, $y = \dfrac{-3}{2}x + 13,\ x = -\dfrac{1}{6}y + \dfrac{31}{6}$

Implies the regression coefficient as $b_{yx} = -\dfrac{3}{2}$ and $b_{xy} = -\dfrac{1}{6}$

Now, correlation coefficient (r) = $\sqrt{b_{yx}\ .\ b_{xy}} = \sqrt{\dfrac{-3}{2} \times -\dfrac{1}{6}} = -0.5$

After solving the given equations, we get the intersection point (x, y) = (4, 7)

i.e, means of x = $\bar{x} = 4$ and means of y = $\bar{y} = 7$

b. We have, the given two regression equations are

$3x + 4y = 65$ and $3x + y = 31$

Since $(\bar{x}, \bar{y})$ lies on the given regression lines.

$3\bar{x} + 4\bar{y} = 65$ ... ... (i)

$3\bar{x} + \bar{y} = 32$ ... ... (ii)

Subtracting (ii) from (i), we get

$3\bar{y} = 33$

$\bar{y} = 11$

from (i), $3\bar{x} + 4\times 11 = 65 \Rightarrow 3\bar{x} = 65 - 44 = 21$

$\bar{x} = 7$

$\bar{y} = 11$

For the regression coefficients, we have

$3x + 4y = 65$

$\Rightarrow\ 4y = 65 - 3x$

$y = \frac{65}{4} - \frac{3}{4}x$ which is in the form of $y = a + bx$; where

$b = -\frac{3}{4}\ \therefore\ b = b_{yx} = -\frac{3}{4}$

Again, we have,

$3x + y = 32$

$\Rightarrow 3x = 32 - y$

$x = \frac{32}{3} - \frac{1}{3}y$ which is in the form of $x = a + by$; where $b = -\frac{1}{3}$

$b = b_{xy} = -\frac{1}{3}$

Correlation coefficient $(r) = \sqrt{b_{xy} + b_{xy}} = \pm\sqrt{-\frac{3}{4} \times -\frac{1}{3}}$

$r = -\frac{1}{2}$

5. **a.** The regression coefficient of x on y and y on x are 1.5 and 0.65 respectively. If the arithmetic's mean $\bar{x}$ and $\bar{y}$ are 36 and 52 respectively, find the two regression equations. Also find the value of y when x = 60.

Solution,

Here, $b_{xy} = 1.5$, $b_{yx} = 0.65$

$\bar{x} = 36$ and $\bar{y} = 52$

Regression equation of y on x is

$y - \bar{y} = b_{yx}(x - \bar{x})$

$\Rightarrow\ y - 52 = 0.65(x - 36)$

$\Rightarrow\ y = 0.65x - 23.4 + 52$

$y = 0.65x + 28.6$ ... ... (i)

And, the regression equation of x on y is

$x - \bar{x} = b_{xy}(y - \bar{y})$

$x - 36 = 1.5(y - 52)$

$x = 1.5y - 78 + 36$

$x = 1.5y - 42$ ... ... (ii)

When x = 60 then from (i),

$y = 0.65 \times 60 + 28.6 = 67.6$

b. Given the following, $\bar{x} = 20$, $\bar{y} = 120$ $(C.V.)_x = 25$, $(C.V.)_y = 28.83$, r = 0.8. Find x when y = 150.

Solution:

b. Given, $\bar{x} = 20$, $\bar{y} = 120$, $cov_x = 25$, $cov_y = 28.83$, $r = 0.8$, x = ? if y = 150

$cov_x = 25$, then $25 = \dfrac{\sigma_x}{\overline{x}} \times 100$

or, $\dfrac{25 \times 20}{100} = \sigma_x$

or, $\sigma_x = 5$

$Cov_y = 28.83 \Rightarrow 28.83 = \dfrac{\sigma_y}{\overline{y}} \times 100$

or, $\dfrac{28.83 \times 120}{100} = \sigma_y$

or, $\sigma_y = 34.596$

$\therefore \quad b_{xy} = r\dfrac{\sigma_x}{\sigma_y} = 0.8 \times \dfrac{5}{34.596} = 0.1156$

Now, the equation of regression line of x on y is

$x - \overline{x} = b_{xy}(y - \overline{y})$

or, $x - 20 = 0.1156(y - 120)$

or, $x = 0.1156y - 13.872 + 20$

$\therefore \quad x = 0.1156y + 6.128$

when $y = 150$, $x = 0.1156 \times 150 + 6.128 = 23.46$

6. From the following results, obtain the two regression equations and estimate the yield, when the rainfall is 29 cms and the rainfall, when the yield is 600 kg:

	Yield in kg.(Y)	**Rainfall in cms.(X)**
Mean	508.4	26.7
Standard deviation	36.8	4.6

Coefficient of correlation between yield and rainfall is +0.52.

Solution:

Here, average of rainfall $(\overline{x}) = 26.7$cm

Average of yield $(\overline{y}) = 508.4$kg

$\sigma_x = 4.6$, $\sigma_y = 36.8$, $r = 0.52$

For x on y:

$b_{xy} = r\dfrac{\sigma_x}{\sigma_y} = 0.52 \times \dfrac{4.6}{36.8} = 0.065$

So, the equation is $x - \overline{x} = b_{xy}(y - \overline{y})$

or, $x - 26.7 = 0.065(y - 508.4)$

or, $x - 26.7 = 0.065y - 33.046$

or, $x = 0.065y - 6.346$

When yield (y) = 600kg, $x = 0.065 \times 600 - 6.346 = 32.654$cm

For y on x

$b_{yx} = r\dfrac{\sigma_y}{\sigma_x} = 0.52 \times \dfrac{36.8}{4.6} = 4.16$

So, the equation is $y - \overline{y} = b_{yx}(x - \overline{x})$

or, $y - 5084 = 4.16(x - 26.7)$

or, $y - 5084 = 4.16x - 111.072$

or, $y = 4.16x + 397.328$

When rainfall (x) = 29cm

$y = 4.16 \times 29 + 397.328 = 517.968$kg

7. The advertisement expenses and sales of a new product are recorded as below:

Adv. exp (Rs. '000)	1	5	6	8	10
Sales (Rs. '000)	50	60	80	100	110

Estimate the sales when advertising expenses is Rs. 15000.

Solution:

x	y	xy	x^2
1 5 6 8 10	50 60 80 100 110	50 300 480 800 1100	1 25 36 64 100
$\sum x = 30$	$\sum y = 400$	$\sum xy = 2730$	$\sum x = 226$

Here, $\bar{x} = \frac{\sum x}{n} = \frac{30}{5} = 6$ and $\bar{y} = \frac{\sum y}{n} = \frac{400}{5} = 80$

$$b_{yx} = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} = \frac{5 \times 273 - 30 \times 400}{5 \times 226 - (30)^2} = \frac{13650 - 12000}{1130 - 900} = \frac{1650}{230} = 7.174$$

The regression equation of y on x is

$y - \bar{y} = b_{yx}(x - \bar{x})$

or, $y - 80 = 7.174(x - 6)$

or, $y - 80 = 7.174x - 43.04$

or, $y = 7.174x + 36.96$

When advertising expenses is Rs. 15000 i.e. $x = 15$,

$y = 7.174 \times 15 + 36.96 = 144.57$thousands

$\therefore$ Estimated sales is Rs. 144570.

8. Find the regression of X on Y from the following data:

X	2	4	5	6	8	11
Y	18	12	10	8	7	5

Estimate the value of X when Y = 12.

Solution:

Calculation regression equation of x on y

x	y	x^2	y^2	xy
2 4 5 6 8 11	18 12 10 8 7 5	4 16 25 36 64 121	324 144 100 64 49 25	36 48 50 48 56 55
36	60	266	706	293

Here, $\bar{x} = \frac{\sum x}{n}$ and $\bar{y} = \frac{\sum y}{n}$

$\Rightarrow \bar{x} = \frac{36}{6}$ and $\bar{y} = \frac{60}{6}$

$\therefore \bar{x} = 6 \quad \therefore \bar{y} = 10$

Again, $b_{xy} = \frac{n\sum xy - \sum x \sum y}{n\sum y^2 - (\sum y)^2} = \frac{6 \times 293 - 36 \times 60}{6 \times 706 - (60)^2} = \frac{1758 - 2160}{4236 - 3600} = \frac{-402}{636} = -0.6320$

$\therefore$ Regression equation of x on y is,

$x - \bar{x} = b_{xy}(y - \bar{y})$

$\Rightarrow x - 6 = -0.6320(y - 10)$

$\Rightarrow x - 6 = -0.6320y + 6.32$

$\therefore \quad x = 12.32 - 0.6320y$

When y = 12 then $x = 12.32 - 0.6320 \times 12$

$\therefore \quad x = 4.736$

9. While calculating the coefficient of correlation between two varioushes x and y the following results were obtained.

The number of observations $n = 25$, $\Sigma x = 125$, $\Sigma y = 100$, $\Sigma x^2 = 650$, $\Sigma y^2 = 460$, $\Sigma xy = 508$. It was however, later discovered at the time of checking that two pairs of observations (x, y) were copied (6, 14) and (8, 6) while the correct values were (8, 12) and (6, 8) respectively. Determine the correct value of coefficient of correlation. However find the correct equation of the two lines of regression.

Solution:

Here, $n = 25$, $\sum x = 125$, $\sum y = 100$, $\sum x^2 = 650$, $\sum y^2 = 460$, $\sum xy = 508$

But (8, 12) and (6, 8) were copied wrong as (6, 14) and (8, 6) respectively.

So, correct values are

$n = 25, \sum x = 125 + 8 - 6 + 6 - 8 = 125$

$\sum y = 100 + 12 - 14 + 8 - 6 = 100$

$\sum x^2 = 650 + 8^2 - 6^2 + 8^2 = 650$

$\sum y^2 = 460 + 12^2 - 14^2 + 8^2 - 6^2 = 436$

$\sum xy = 508 + 8\times12 + 6\times8 - 6\times14 - 8\times6 = 520$

Now, $b_{xy} = \dfrac{n\sum xy - \sum x.\sum y}{n\sum y^2 - (\sum y)^2} = \dfrac{25 \times 520 - 125 \times 100}{25 \times 436 - (100)^2} = \dfrac{13000 - 12500}{10900 - 100000} = \dfrac{5}{9}$

and $b_{yx} = \dfrac{n\sum xy - \sum x.\sum y}{n\sum y^2 - (\sum y)^2} = \dfrac{25 \times 520 - 12500}{25 \times 650 - (125)^2} = \dfrac{500}{625} = \dfrac{4}{5}$

Now, coefficient of correlation (r) $= \sqrt{b_{yx} \times b_{xy}} = \sqrt{\dfrac{5}{9} \times \dfrac{4}{5}} = \dfrac{2}{3}$

Now, $\overline{x} = \dfrac{\sum x}{n} = \dfrac{125}{25} = 5$

$\overline{y} = \dfrac{\sum y}{n} = \dfrac{100}{25} = 4$

The equation of regression line of x on y is

$x - \overline{x} = b_{xy}(y - \overline{y})$

or, $x - 5 = \dfrac{5}{9}(y - 4)$

or, $9x - 45 = 5y - 20$

or, $9x - 5y = 25$

$\therefore \quad 5y - 9x + 25 = 0$

and the equation of regression line of y on x is

$y - \overline{y} = b_{yx}(x - \overline{x})$

or, $y - 4 = \dfrac{4}{5}(x - 5)$

or, $5y - 20 = 4x - 20$

$\therefore \quad 5y - 4x = 0$

■

CHAPTER 14

PROBABILITY

EXERCISE 14.1

1. A card is drawn at random from well shuffled deck of 52 cards, find the probability that
   a. the card is either a club or diamond
   b. the card is not a king
   c. the card is either a face card or a club card.

Solution:

n = Total no. of cards = 52

a. No. of club = 13
   No. of diamond = 13
   m = No. of possible cases = 13 + 13 = 26
   P(Either a club or diamond) = $\frac{m}{n} = \frac{26}{52} = \frac{1}{2}$

b. There are four kings
   ∴ No. of possible cases = 52 – 4 = 48
   ∴ P(Not of king) = $\frac{48}{52} = \frac{12}{13}$

c. There are 12 face cards and 13 club cards.
   ∴ m = no. of cases = 12 + 13 – 3 = 22
   ∴ P(Either a face or a club) = $\frac{m}{n} = \frac{22}{52} = \frac{11}{26}$

2. From 20 tickets marked from 1 to 20, one is drawn at random. Find the probability that
   a. It is an odd number
   b. A multiple of 4 or 5

Solution:

a. P(Odd number) = ?
   Among 20 tickets, there are 10 tickets marked with odd number.
   ∴ P(Odd number) = $\frac{m}{n} = \frac{10}{20} = \frac{1}{2}$

b. P(A multiple of 4 or 5) = ?
   There are 5 tickets marked with multiple of 4 and 4 tickets marked with multiple of 5.
   = P(Multiple of 4) + P(multiple of 5) – P(Multiple of 4) × P(Multiple of 5)
   $= \frac{5}{20} + \frac{4}{20} - \frac{5}{20} \times \frac{4}{20} = \frac{2}{5}$

3. A problem in mathematics is given to four students A, B,C, and D their chances of solving it are 1/2 , 1/3, 1/4 and 1/5 respectively. Find the probability that the problem will
   a. be solved
   b. not be solved

Solution:

Given that,

P(A) = Probability that A solves the problem = $\frac{1}{2}$

P(B) = Probability that B solves the problem = $\frac{1}{3}$

P(C) = Probability that C solves the problem = $\frac{1}{4}$

P(D) = Probability that D solves the problem = $\frac{1}{5}$

$P(\bar{A})$ = Probability that A not solve the problem = $1 - \frac{1}{2} = \frac{1}{2}$

$P(\bar{B})$ = Probability that B not solve the problem $1 - \frac{1}{3} = \frac{2}{3}$

$P(\bar{C})$ = Probability that C not solve the problem = $1 - \frac{1}{4} = \frac{3}{4}$

$P(\bar{D})$ = Probability that D not solve the problem = $1 - \frac{1}{5} = \frac{4}{5}$

a. Probability that A, B, C, D not solve the problem = $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5}$

b. Probability that A, B, C, D solve the problem = $1 - \frac{1}{5} = \frac{4}{5}$

4. Suppose 4 cards are drawn from a well-shuffled deck of 52 cards.
   a. What is the probability that all 4 are spade?
   b. What is the probability that all 4 are black?

Solution:

a. There are 13 spades

Now, n = Total no. of possible cases = No. of selection of 4 cards out of 52 = ${}^{52}C_4$

= No. of favourable cases = No. of selection of 4 spades out of 13 = ${}^{13}C_4$

P(4 are spades) = ?

Now, P(4 are spades) = $\frac{m}{n} = \frac{{}^{13}C_4}{{}^{52}C_4} = \frac{13!}{9!\,4!} \times \frac{48! \times 41}{52!} = \frac{11}{4165}$

b. There are 26 black. So, we have to choose 4 black among 26 blacks.

Now, n = Total no. of possible cases

= No. of selection of 4 cards out of 52 = ${}^{52}C_4$

m = No. of favourable cases = No. of selection of 4 black out of 26 black = ${}^{26}C_4$

P(4 are black) = ?

Now, P(4 are black) = $\frac{m}{n} = \frac{{}^{26}C_4}{{}^{52}C_4} = \frac{46}{833}$

5. Two cards drawn successively one after other from well shuffled pack of 52 cards. If the cards are not replaced, find the probability that (a) all of them are kings. (b) one king and other is queen.

Solution:

a. Two cards can be drawn from a pack of 52 playing cards in ${}^{52}C_2$ ways

i.e. $\frac{52 \times 51}{2}$ = 1326 ways

The event that two kings appear in a single draw can appear in ${}^{4}C_2$ ways

$\therefore$ The probability that the two cards drawn from a pack of 52 cards are kings

$= \frac{6}{1326} = \frac{1}{221}$

b. One king and one queen can be selected as $\frac{4}{52} \times \frac{4}{51}$ ways.

One queen and one king can be selected as = $\frac{4}{52} \times \frac{4}{51}$ ways

Total no. of ways = $\frac{4}{52} \times \frac{4}{51} + \frac{4}{52} \times \frac{4}{51} = \frac{8}{663}$

6. Out of 9 candidates, 6 men and 3 women apply for two vacancies of a manufacturing company what is the probability that one man and one woman are selected in that vacancies?

Solution:

Here, Total no. of candidates = 9 Total no. of men = 6

Total no. of women = 3 Total no. of vacancy = 2

∴ Out of 2, one man and one woman can be selected in the following ways.

∴ $m = bc_1 \times 9c_1$

∴ Total no. of vacancy can be chosen from total no. of candidates as $n = 9c_2$

∴ $P(\text{1 man and 1 woman}) = \dfrac{6C_1 \times 9C_1}{9C_2} = \dfrac{18}{36} = 0.5$

7. A bog contains 7 white and 9 black balls two balls are drawn in succession at random with replacement. What is the probability that one of them is white and other is black?

Solution:

Since the bag consists of 7 white and 9 black balls.

∴ Total balls = 7 + 9 = 16

Total number of possible cases means the number of selection of 2 balls out of 16.

Since, the selection of 1 white and 1 black. So, the number of favourable cases is the selection of balls with 1 white and 1 black

∴ m = No. of favourable cases

= No. of selection of 1 white out of 7 and 1 black out of 9 = ${}^7C_1 \times {}^9C_1$

n = Total no. of possible cases= No. of selection of 2 balls out of 16. = ${}^{16}C_2$

∴ $P(\text{1 white and 1 black}) = \dfrac{m}{n} = \dfrac{{}^7C_1 \times {}^9C_1}{{}^{16}C_2} = \dfrac{63}{120}$

8. A bag contains 8 red and 6 white balls. Two balls are drawn randomly from the bag one after other without replacement. Find the probability that (a) both balls are white (b) both balls are red (c) one is red and 1 white.

Solution:

There are 6 + 8 = 14 balls (Total)

a. P(both white) = ?

$P(\text{First white}) = \dfrac{6}{14}$ and $P(\text{second white}) = \dfrac{5}{13}$

$P(\text{Both white}) = \dfrac{6}{14} \times \dfrac{5}{13} = \dfrac{15}{91}$

b. P(Both red) = ?

$P(\text{First red}) = \dfrac{8}{14}$, $P(\text{Second red}) = \dfrac{7}{13}$

∴ $P(\text{Both red}) = \dfrac{8}{14} \times \dfrac{7}{13} = \dfrac{4}{13}$

c. Since balls are drawn one after another without replacement.

P(One red and one white) = ?

∴ $P(\text{First red}) = \dfrac{8}{14}$, $P(\text{Second white}) = \dfrac{6}{13}$

$P(\text{First white}) = \dfrac{6}{14}$ $P(\text{Second red}) = \dfrac{8}{13}$

∴ $P(\text{One red and one white}) = \dfrac{6}{14} \times \dfrac{8}{13} + \dfrac{8}{13} \times \dfrac{6}{13} = \dfrac{48}{91}$

9. If A and B are two events such that P(A) = 0.40, P(B) = 0.80 and P(B/A) = 0.60. Find P(A/B) and P(A∪B)

Solution:

Given, P(A) = 0.40, P(B) = 0.80, P(B/A) = 0.60, P(A/B) = ? P(A∪B) = ?

$\therefore \quad P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

$\Rightarrow \quad P(A \cap B) = P(A) \,.\, P(B/A) = 0.40 \times 0.60 = 0.24$

$\therefore \quad P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.24}{0.80} = 0.30$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.40 + 0.80 - 0.3 = 0.90$

10. A box contained 6 red and 4 black balls. Two balls are drawn one at a time without replacing the first ball. Compute the following probabilities. $P(B_2/B_1)$, $P(R_2/B_1)$ and $P(R_2/R_1)$

Solution:

$P(R_2/B_1)$ = Probability of getting a red ball given that the first ball is black.

First black ball

n = Total no. of possible cases

Total no. of balls = 6 + 4 = 10

m = No. of favourable cases = No. of black balls = 4

$P(B_1) = \dfrac{m}{n} = \dfrac{4}{10}$

Second Red Ball

One black ball which is drawn is not replaced.

n = Total no. of possible cases = No. of remaining balls = 6 + 3 = 9

m = No. of favourable cases = No. of red balls = 6

$P(R_2/B_1) \ \dfrac{m}{n} = \dfrac{6}{9} = \dfrac{2}{3}$

$P(R_1)$ = Probability of getting a red ball $= \dfrac{m}{n} = \dfrac{6}{10}$

$P(R_2/R_1)$ Probability that second ball is red when first also red $= \dfrac{m}{n} = \dfrac{5}{9}$

$\therefore \quad P(R_1 \cap R_2) = P(R_1).P(R_2/R_1) = \dfrac{6}{10} \times \dfrac{5}{9} = \dfrac{1}{3}$

11. A lot contains 12 items of which 5 are defective. If 5 items are chosen from the lot at random. One after another without replacement. Find the probability that all the five are defective.

Solution:

We have,

No. of total items = 12

No. of defective items = 5

$\therefore \quad$ Probability of getting first item defective, $P(A) = \dfrac{5}{12}$

Since, second item is drawn without replacement of first items.

So, probability of getting second item defective $P(B) = \dfrac{4}{11}$

Similarly,

Probability of getting 3rd item defective, $P(C) = \dfrac{3}{10}$

Probability of getting 4^{th} item defective, $P(D) = \frac{2}{9}$

Probability of getting 5^{th} item defective, $P(E) = \frac{1}{8}$

Probability of getting all items defective

$= P(A) \times P(B) \times P(C) \times P(D) \times P(E) = \frac{5}{12} \times \frac{4}{11} \times \frac{3}{11} \times \frac{1}{9} \times \frac{1}{8} = \frac{1}{792}$

12. A bag contains 3 white, 2 black and 4 red balls. Two balls are drawn, the first replaced before the second is drawn, what is the probability that
    a. They will be of same colour? b. They will be of different colour?

Solution:

We have,

No. of white balls = 3 No. of black balls = 2

No. of red balls = 4 Total no. of balls = 9

Let P(W) = Probability of getting a white ball $= \frac{3}{9} = \frac{1}{3}$

P(B) = Probability of getting black ball $= \frac{2}{9}$

P(R) = Probability of getting red ball $= \frac{4}{9}$

a. P(They will be of same colour) = P(WW or BB or RR)

$= P(WW) + P(BB) + P(RR)$

$= P(W) \times P(W) + P(B) \times P(B) + P(R) \times P(R)$

$= \frac{1}{3} \times \frac{1}{3} + \frac{2}{9} \times \frac{2}{9} + \frac{4}{9} \times \frac{4}{9} = \frac{29}{81}$

b. Probability of getting different colours, there should be either WB or BW or BR or RB or WR or RW

$\therefore$ P(That they are of different colour)

= P(WB or BW or BR or RB or WR or RW)

$= P(W) \times P(B) + P(B) \times P(W) + P(B) \times P(R) + P(W) \times P(R) + P(R) \times P(W)$

$= \frac{1}{3} \times \frac{2}{9} + \frac{2}{9} \times \frac{1}{3} + \frac{2}{9} \times \frac{4}{9} + \frac{4}{9} \times \frac{2}{9} + \frac{1}{3} \times \frac{4}{9} + \frac{4}{9} \times \frac{1}{3} = \frac{52}{81}$

EXERCISE 14.2

1. For a binomial distribution if mean = 25 and variance = 5, find the value of p and q.

Solution:

We have, mean = np = 25 ... ... (i)

Variance = npq = 5 ... ... (ii)

from (i) and (ii)

25q = 5

or, $q = \frac{5}{25} = \frac{1}{5}$

$\therefore$ $p = 1 - q = 1 - \frac{1}{5} = \frac{4}{5}$

Hence, $p = \frac{4}{5}$, $q = \frac{1}{5}$

2. In a binomial distribution, find the mean and the standard deviation, if $p = \frac{3}{5}$ and n = 50.

Solution:

We have, $p = \frac{3}{5}$, $n = 50$

$\therefore \quad q = 1 - p = 1 - \frac{3}{5} = \frac{2}{5}$

$\therefore \quad \text{Mean} = np = 50 \times \frac{3}{5} = 30$

$\text{S.D} = \sqrt{npq} = \sqrt{50 \times \frac{3}{5} \times \frac{2}{5}} = 2\sqrt{3}$

3. Determine the binominal distribution for which the mean is 4 and standard deviation is $\sqrt{3}$.

Solution:

We have, mean = np = 4 ... ... (i)

$\text{S.D.} = \sqrt{npq} = \sqrt{3}$

or, npq = 3 ... ... (ii)

$\therefore$ from (i) and (ii)

$4q = 3$

$\Rightarrow q = \frac{3}{4}$

$\therefore \quad p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$

$\therefore$ from (i), $n \times \frac{1}{4} = 4 \Rightarrow n = 16$

$\therefore$ Binomial distribution $= (q + p)^n = \left(\frac{3}{4} + \frac{1}{4}\right)^{16}$

4. It is found that mean and variance of a binomial distribution are 7 and 11 respectively. Comment on the result.

Solution:

We have,

Mean = np = 7 ... ... (i)

Variance = npq = 11 ... ... (ii)

$\therefore$ from (i) and (ii)

$7 \times q = 11$

or, $q = \frac{11}{7} = 1.57 > 1$

Since, q is probability of failture, which cannot be greater than 1. So, the given statement is not correct.

5. Four coins are tossed simultaneously, what is the probability of getting
   a. 2 heads and 2 tails
   b. at least two heads
   c. at least one head

Solution:

Here, p = probability of getting ahead $= \frac{1}{2}$

q = probability of getting a tail $= \frac{1}{2}$

n = no. of trials = 4

p(r) = probability of r success in n trials $= n_{c_r} p^r q^{n-r}$

a. p(2) = probability of 2 heads in 4 trials = $^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} = \frac{4 \times 3}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{8}$

b. p(at least two heads) = p(2) + p(3) + p(4) = $4_{c2}\, p^2q^2 + 4_{c3}p^3q + 4_{c4}\, p^4$

$$= \frac{4 \times 3}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 \frac{1}{2} + 1\left(\frac{1}{2}\right)^4 = \frac{11}{16}$$

c. P(lat least one head) = p(1) + p(2) + p(3) + p(4)
$= {}^4C_1\, p1q^3 + {}^4C_2\, p^2q^2 + {}^4C_3\, p^3q + {}^4C_4\, p^4$
$= {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 + {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 + {}^4C_3 \left(\frac{1}{2}\right)^3 \frac{1}{2} + 1.\left(\frac{1}{2}\right)^4 = \frac{1}{4} + \frac{11}{16} = \frac{15}{16}$

6. A die is thrown 4 times. Getting 3 or 6 is considered to be a success. Find the probability of getting
   a. at least one success
   b. exactly two success.

Solution:

Let p be the probability of getting 3 or 6.

$\therefore\ p = \frac{2}{6} = \frac{1}{3}$, $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

n = no. of trials = 4

Now, probability of r success out of n trials is given by

$p(r) = n_{cr}\, p^rq^{n-r} = 4_{cr} \left(\frac{1}{3}\right)^r . \left(\frac{1}{3}\right)^{4-r} = r\, \frac{1}{81}\, 4_{cr}$

a. Probability of getting at least one success = p(≥ 1) = p(1) + p(2) + p(3) + p(4)
$= {}^4C_1\, p^1q^{4-1} + {}^4C_2\, p^2q^{4-2} + {}^4C_3\, p^3p^{4-3} + {}^4C_4\, p^4q^{4-4}$
$= \frac{4!}{3!\ 1!} \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^3 + \frac{4!}{2!2!} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 + \frac{4!}{3!\ 1!} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^1 + \frac{4!}{0!\ 4!} \left(\frac{1}{3}\right)^4$
$= \frac{4\times8}{81} + \frac{6\times4}{81} + \frac{4\times2}{81} + \frac{1}{81} = \frac{65}{81}$

b. Probability of exactly two success = p(2)
$= 4_{c2}\, p^2q^2 = \frac{4!}{2!\ 2!} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = \frac{4 \times 3}{2} \times \frac{4}{81} = \frac{8}{27}$

7. From a pack of 52 cards five cards are drawn successively with replacement. Find the probability that
   a. all cards are diamond
   b. only three are diamond
   c. none is diamond

Solution:

Let X represents the number of diamond cards among the five cards drawn. Since the drawing off cards is with replacement, the trials are Bornouli trial.
In a well-shuffled deck of 52 cards, there are 13 diamond cards.

$\Rightarrow\ p = \frac{13}{52} = \frac{1}{4}$, $q = 1 - \frac{1}{4} = \frac{3}{4}$

X has a Binomial distribution with n = 5 and $p = \frac{1}{4}$

$\therefore\ p(x = x) = {}^nC_x\, p^xq^{n-x}$, where x = 0, 1, 2, ... $n = {}^5C_x \left(\frac{3}{4}\right)^{5-x} \left(\frac{1}{4}\right)^x$

a. p(all 5 cards ae diamond) = p(x = 5) = ${}^5C_5 \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^5 = \frac{1}{1024}$

b. p(only 3 cards ae diamond) = p(x = 3) = $^5C_3 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^3 = \frac{45}{512}$

c. p(none is spade) = p(x = 0) = $^5C_0 \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^0 = \frac{243}{1024}$

8. Ten coins are tossed simultaneously. Find the probability of getting
   a. exactly six heads
   b. at least seven heads
   c. not more than 3 heads.

Solution:

Let p = the event of getting a head 10 coins being tossed simultaneously is the same as one coin being tossed 10 times.

$p(x = r) = {}^{10}C_r\, p^r.q^{n-r} = {}^{10}C_r \left(\frac{1}{2}\right)^{10}$

a. p(exactly 6 heads) = $^{10}C_6 \left(\frac{1}{2}\right)^{10} = \frac{10!}{6!\,4!} \times \frac{1}{1024} = \frac{105}{512}$

b. p(at least 7 heads) = p(7 heads or 8 heads or 9 heads or 10 heads)

$= 1 - [p(x = 0) + p(x = 1) + p(x = 2) + p(x = 3) + p(x = 4) + p(x = 5) + p(x = 6)] \left(\frac{1}{2}\right)^{10}$

$= 1 - (1 + 10 + 45 + 120 + 210 + 252 + 210) \frac{1}{1024} = \frac{176}{1024}$

c. p(not more than 3 heads) = $p(x \le 3)$

$= p(x = 0) + p(x = 1) + p(x = 2) + p(x = 3)$

$= ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3).\left(\frac{1}{2}\right)^{10} = (1 + 10 + 45 + 120)\,.\,\frac{1}{1024} = \frac{11}{64}$

9. If 4 dice are thrown, what is the probability of getting
   a. no sixes
   b. exactly 1 six
   c. exactly two sixes.

Solution:

Given, Probability of getting a six in one throw (p) = $\frac{1}{6}$

$\therefore\ q = 1 - p = \frac{5}{6}$

No. of trials (n) = 4

Now, probability of r success in 4 trials is given by

$p(r) = n_{c_r}\, p^r.q^{n-r} = {}^4C_r \left(\frac{1}{6}\right)^r . \left(\frac{5}{6}\right)^{4-r}$ ... ... (i)

a. p(no six) = p(0) = $^4C_0 \left(\frac{1}{6}\right)^0 \left(\frac{1}{6}\right)^{4-0} = \frac{625}{1296}$

b. p(exactly 1 six) = p(1) = $^4C_x \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^{4-1} = \frac{125}{324}$

c. p(exactly two sixes) = p(2) = $^4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{4-2} = \frac{25}{216}$

10. The overall percentage of failures in a certain exam is 40. What is the probability that out of a group of 6 candidates at least 4 passed the examination?

Solution:

Probability of fail = $\frac{40}{100} = \frac{2}{5} = q$ Probability of pass = $1 - \frac{2}{5} = \frac{3}{5} = p$

$n = 6,\ q = \dfrac{2}{5}$

$x \to$ R.V.

We have Binomial condition,

$P(x = r) = {}^{n}C_r\ p^r.q^{n-r}$

$p(x \geq 4) = ?$

$$\therefore\ p(x \geq 4) = p(x = 4) + p(x = 5) + p(x = 6)$$

$$= {}^{6}C_4 \left(\frac{3}{4}\right)^4 \left(\frac{2}{5}\right)^4 + {}^{6}C_5 \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right) + {}^{6}C_6 \left(\frac{3}{5}\right)^6 \left(\frac{2}{5}\right)^0$$

$$= \frac{6!}{4!\ 2!} \times \frac{3^4 \times 2^2}{5^6} + \frac{6!}{5!1!} \times \frac{3^5 \times 2}{5^6} + \frac{6!}{6!} \times \frac{3^6}{5^6} = \frac{1701}{3125}$$

11. Suppose that in a city 60% of all recorded births are male. If we select ten births from the population, what will be the probability that
    a. none of them is male
    b. exactly three of them are male
    c. more than four are male.

Solution:

Given, $p = 60\% = \dfrac{60}{100} = \dfrac{3}{5}$

$\therefore\ q = -1 - P = 1 - \dfrac{3}{5} = \dfrac{2}{5}$

n = number of trials = 10

Now, probability of r successes in 10 trials is given by

$$p(r) = {}^{10}C_r\ p^r.q^{10-r} = {}^{10}C_r \left(\frac{3}{5}\right)^r \left(\frac{2}{5}\right)^{10-r}$$

a. $P(\text{None of them male}) = p(0) = {}^{10}C_0 \left(\frac{3}{5}\right)^0 \left(\frac{2}{5}\right)^{10-r} = \dfrac{10!}{0!\ 10!} \times 1 \times \dfrac{2^{10}}{5^{10}} = 0.0001049$

b. $P(\text{Exactly three male}) = p(3) = {}^{10}C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^{10-3} = \dfrac{10!}{7!\ 3!} \times \dfrac{3^3 \times 2^7}{5^3 \times 5^7} = 0.04246$

c. $P(\text{More than 4 are male}) = P(r > 4) = p(5) + p(6) + p(7) + p(8) + p(9) + p(10)$

$$= {}^{10}C_5 \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right)^{10-5} + {}^{10}C_6 \left(\frac{3}{5}\right)^6 \left(\frac{2}{5}\right)^{10-6} + {}^{10}C_7 \left(\frac{3}{5}\right)^7 \left(\frac{2}{5}\right)^{10-7} + {}^{10}C_8 \left(\frac{3}{5}\right)^8$$
$$\left(\frac{2}{5}\right)^{10-8} + {}^{10}C_9 \left(\frac{3}{5}\right)^9 \left(\frac{2}{5}\right)^{10-9} + {}^{10}C_{10} \left(\frac{3}{5}\right)^{10} \left(\frac{2}{5}\right)^0$$

$$= \frac{10!}{5!\ 5!} \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right)^5 + \frac{10!}{6!\ 4!} \left(\frac{3}{5}\right)^9 + \frac{10!}{3!\ 7!} \left(\frac{3}{5}\right)^7 \left(\frac{2}{5}\right)^3 + \frac{10!}{8!\ 2!} \left(\frac{3}{5}\right)^8 \left(\frac{2}{5}\right)^2$$
$$+ \frac{10!}{9!\ 1!} \left(\frac{3}{5}\right)^9 \left(\frac{2}{5}\right) + \frac{10!}{10!} \left(\frac{3}{5}\right)^{10} = 0.9447$$

12. The probability of a man's hitting a target is $\dfrac{1}{5}$. If he fires 6 times, what is the probability of his hitting the target
    a. exactly once
    b. exactly twice.

Solution:

Here, p = Probability off hitting a target $= \dfrac{1}{5}$

$q = 1 - p = 1 - \dfrac{1}{5} = \dfrac{4}{5}$ n = No. of hitting = 6

p(r) = Probability of r successful hitting $= {}^{n}C_r\ p^r q^{n-r}$

a. p(Exactly once) = p(1) = ?

$$= {}^6C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{6-1} = \frac{6!}{5!\ 1!} \times \frac{1}{5} \times \left(\frac{4}{5}\right)^5 = 0.3932$$

b. p(Exactly twice) = p(2) = ${}^6C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{6-2} = \frac{6!}{4!\ 2!} \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^4 = 0.24576$

13. Assume that the probability a bomb dropped from an aeroplane will strike a target is $\frac{1}{4}$. If 5 bombs are dropped, find the probability that

a. none will strike the target
b. exactly three will strike the target
c. at least three will strike the target

Solution:

Given, p = Probability that a bomb dropped = $\frac{1}{4}$

$q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

n = no. of dropped = 5

a. P(None will strike target) = p(0) = $n_{c_r}\ p^r q^{n-r}$

$$= {}^5C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{5-3} = \frac{5!}{2!\ 3!} \times \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2 = \frac{243}{1024}$$

b. $P(3) = {}^5C_3\ p^3 q^{5-3}$

$$= {}^5C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^2$$

= 0.0879

c. p(At least three will strike target) = $p(x \le 3)$

$$= p(3) + p(4) + p(5) = {}^5C_3 \left(\frac{1}{4}\right)^3 . \left(\frac{3}{4}\right)^{5-3} + {}^5C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{5-4} \times {}^5C_5 \left(\frac{1}{4}\right)^5$$

$$= {}^5C_3 \left(\frac{1}{4}\right)^3 . \left(\frac{3}{4}\right)^2 + \frac{5!}{4!\ 1!} \times \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^1 + \frac{5!}{5!} \left(\frac{1}{4}\right)^5 = 0.1035$$

14. A company produces electronic chips by a process that normally average 20% defective product. A sample of four chips is selected at random and the parts are tested for certain characteristics, what is the probability that

a. no chip is defective
b. one chip is defective
c. more than one chip are defective

Solution:

Given, p = detective products = 20% = $\frac{20}{100} = \frac{!}{5}$

$q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$, n = 4

Now, the probability of r defective in 4 trials is given by

$p(r) = {}^4C_r\ p^r.q^{4-r} = {}^4C_r \left(\frac{1}{5}\right)^r \left(\frac{4}{5}\right)^{4-r}$

a. p(No chip is defective) = p(0)

$$= {}^4C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{4-0} = \frac{4!}{4!\ 0!} \times 1 \times \left(\frac{4}{5}\right)^4 = 0.4096$$

b. p(One chip is defective) = p(1)

$$= {}^4C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{4-1} = \frac{4!}{1!\ 3!} \times \frac{1}{5} \times \left(\frac{4}{5}\right)^3 = \frac{4 \times 4 \times 4 \times 4}{5 \times 5 \times 5 \times 5} = 0.4096$$

c. p(more than one chip care defective)

$$= p(2) + p(3) + p(4) = {}^4C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{4-2} + {}^4C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^{4-3} + {}^4C_4 \left(\frac{1}{5}\right)^4$$

$$= \frac{4!}{2!\ 2!} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^2 + \frac{4!}{3!\ 1!} \times \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right) + \frac{4!}{4!} \left(\frac{1}{5}\right)^4 = 0.1808$$

CHAPTER 15

DERIVATIVES

EXERCISE 15.1

1. Find the limit of the following function at given points.

a. $f(x) = \dfrac{\ln(1+x)}{x}$ at $x = 0$ b. $f(x) = \left(\dfrac{1-x}{1+x}\right)^{1/x}$ at $x = 0$

c. $f(x) = \dfrac{(1+x)^n - 1}{x}$ at $x = 0$

Solution:

a. Since, we have,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Now, $$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{1}{x}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right]$$

$$= \lim_{x\to 0}\left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right] = \left[1 - \frac{0}{2} + \frac{0^2}{3} - \frac{0^3}{4} + \ldots\right] = 1$$

$\therefore \lim\limits_{x\to 0} f(x) = 1$

b. Here, $y = \left(\dfrac{1-x}{1+x}\right)^{1/x}$

Taking ln on both sides,

$$\ln y = \frac{1}{x}\ln\left(\frac{1-x}{1+x}\right) \qquad [\because \ln m^n = n\ln m]$$

or, $$\ln y = \frac{1}{x}\ln\left[1 - \frac{2x}{1+x}\right] = \frac{1}{x}\,\frac{\ln\left[1 + \left(\frac{-2x}{1+x}\right)\right]}{\left(\frac{-2x}{1+x}\right)} \times \frac{-2x}{1+x} = \frac{\ln\left[1 + \left(\frac{-2x}{1+x}\right)\right]}{\frac{-2x}{1+x}} \times \frac{-2}{1+x}$$

Taking $\lim\limits_{x\to 0}$ on both sides, we have,

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln\left[1 + \left(\frac{-2x}{1+x}\right)\right]}{\frac{-2x}{1+x}} \times \frac{-2}{1+x} = 1 \times \lim_{x\to 0} -\frac{2}{1+x}\left[\because \lim_{x\to 0}\frac{\ln(1+x)}{x} = 1\right]$$

$$= \frac{-2}{1+0}$$

$\therefore \lim\limits_{x\to 0} \ln y = -2$

i.e, $\lim\limits_{x\to 0} y = e^{-2}$ $\qquad [\because \ln_x x = y \Leftrightarrow x = e^y]$

$$\therefore \lim_{x\to 0}\left(\frac{1-x}{1+x}\right)^{1/x} = e^{-2}$$

c. $$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{(1+x)^n - 1}{x}$$

$$= \lim_{x\to 0}\frac{(1+x-1)\,[(1+x)^{n-1} + (1+x)^{n-2} + (1+x)^{n-3} + \ldots + (1+x) + 1]}{x}$$

$$= \lim_{x\to 0} (1 + x)^{n-1} + (1+x)^{x-2} + (1+x)^{n-3} + \dots (1+x) + 1$$
$$= 1^{n-1} + 1^{n-2} + 1^{n-3} + \dots 1 + 1 = n-1 + 1 = n$$

2. a. A function f(x) is defined as follows $f(x) = \begin{cases} -x-3 \text{ for } x \le -2 \\ \frac{2}{3}x + \frac{1}{3} \text{ for } -2 < x < 1 \\ x^2 \text{ for } x \ge 1 \end{cases}$

Test the continuity of f(x) at x = -2 and x = 1.

b. Show that the following function is continuous at

x = 4. $f(x) = \begin{cases} \frac{x^2 - 16}{x - 4} & \text{for } x \neq 4 \\ 8 & \text{for } x = 4 \end{cases}$

c. A function f(x) is defined as follows

$$f(x) = \begin{cases} 3 + 2x & \text{for } -\frac{3}{2} \le x < 0 \\ 3 - 2x & \text{for } 0 \le x < \frac{3}{2} \\ -3 - 2x & \text{for } x \ge \frac{3}{2} \end{cases}$$

Test the continuity of f(x) at x = 0 and $x = \frac{3}{2}$

Solution:

a. Note: A function f(x) is said to be continuous at a point x = a if and only if,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

Here, To test the continuity of f(x) at x = –2, we proceed as follows:

Here, f(x) at x = –2 is

$f(-2) = -x - 3$ $\quad [\because f(x) = -x - 3 \text{ for } x \le -2]$

$= -2 - 3 = -5$ (a finite value)

Now, left hand limit of f(x) at x = –2 is,

$$\lim_{x\to -2^-} f(x) = \lim_{x\to 2^-} (-x-3) \quad [\because f(x) = -x - 3 \text{ for } x \le -2]$$
$$= -2 - 3 = -5$$

Finally,

Right hand limit of f(1) at x = –2 is,

$$\lim_{x\to -2^+} f(x) = \lim_{x\to 2^+} \left(\frac{2}{3}x + \frac{1}{3}\right) \quad \left[\because f(x) = \frac{2}{3}x + \frac{1}{3} \text{ for } -2 < x < 1\right]$$
$$= \frac{2}{3} \times (-2) + \frac{1}{3} = -\frac{4}{3} + \frac{1}{3} = \frac{-4 + 1}{3} = -\frac{3}{3} = -1$$

$$\therefore \lim_{x\to -2^-} f(x) = f(-2) \neq \lim_{x\to -2^+} f(x)$$

So, f(x) is discontinuous at a point x = –2.

2nd Part;

Again testing the continuity of f(x) at x = 1

For the functional value, $f(1) = x^2$ $[\because f(x) = x^2 \text{ for } x \ge 1]$

$= 1^2 = 1$

LHL at x = 1 is,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(\frac{2}{3}x + \frac{1}{3}\right) \quad \left[\because f(x) = \frac{2}{3}x + \frac{1}{3} \text{ for } -2 < x < 1\right]$$

$= \frac{2}{3} \times 1 + \frac{1}{3} = \frac{2}{3} + \frac{1}{3} = \frac{3}{3} = 1$

RHL at x = 1 is,

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (x^2)$ [$\because$ $f(x) = x^2$ for $x \geq 1$]

$= 1^2 = 1$

Here, we have,

$\lim_{x\to1^-} f(x) = f(1) = \lim_{x\to1^+} f(x)$

Thus, the given function is continuous at x = 1

b. Testing functional value at x = 4;

f(4) = 8 [$\because$ f(x) = 8 when x = 4]

Again, testing the limiting value at x = 4, we have,

$\lim_{x\to4} f(x) = \lim_{x\to4} \left(\frac{x^2 - 16}{x-4}\right)$ $\left[\because f(x) = \frac{x^2-16}{x-4} \text{ for } x \neq 4\right]$

$= \lim_{x\to4} \left(\frac{x^2-4^2}{x-4}\right)$ $\left[\because \text{at } x = 4, \frac{4^3-16}{4-4} = \frac{0}{0} \text{ form}\right]$

$= \lim_{x\to4} \frac{(x + 4)(x - 4)}{(x - 4)} = \lim_{x\to4} (x + 4)$ [$\because$ $x \neq 4$]

$= 4 + 4 = 8$

Here, we see, $\lim_{x\to4} f(x) = f(4) = 8$

Therefore, f(x) is continuous at x = 4

c. Testing the continuity of f(x) at x = 0,

For the functional value, f(0) = 3–2x

or, $f(0) = 3-2\times0$

$= 3$ (a finite value)

Again, LHL of f(x) at x = 0,

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} (3 + 2x)$ $\left[\because f(x) = 3+2x \text{ for } -\frac{3}{2} \leq x < 0\right]$

$= 3 + 2\times0 = 3$

Finally, RHL off f(x) at x = 0

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} (3 - 2x)$ [$\because$ $f(x) = 3 - 2x$ for $0 \leq x < 3/2$]

$= 3 - 2\times0 = 3$

Here, we see, $\lim_{x\to0^-} f(x) = f(0) = \lim_{x\to0^+} f(x)$

$\therefore$ The given function is continuous at point x = 0.

2nd part:

Testing the continuity of f(x) at x = 3/2

Functional value, $f\left(\frac{3}{2}\right) = -3-2x$ [$\because$ $f(x) = -3-2x$ for $x \geq 3/2$]

$= -3 \times -2 \times \frac{3}{2} = -6$

LHL at $x = \frac{3}{2}$, $\lim_{x\to3/2^-} f(x) = \lim_{x\to3/2^-} (3 - 2x)$ [$\because$ $f(x) = 3-2x$ for $0 \leq x < 3/2$]

$= 3 - 2\times \frac{3}{2}$

Here, we see,

$$\lim_{x\to 3/2^-} f(x) \neq x\left(\frac{3}{2}\right)$$

Therefore, the given function is discontinuous at $x = \frac{3}{2}$

3. a. Show that the function f(x) defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases} \text{ is continuous at } x = 0.$$

b. Examine for continuity at x = 0 for the function f(x) defined by

$$f(x) = \begin{cases} \frac{1 - \cos x}{x^2} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Solution:

a. Show that the function f(x) defined by,

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} \text{ for } r \neq 0 \\ 0 \text{ for } x \to 0 \end{cases} \text{ is continuous at } x = 0$$

Proof: Functional values, f(x) = 0 [$\because$ f(x) = 0 for x = 0]

Limiting value, $\lim_{x\to 0} f(x)$

$$= \lim_{x\to 0} x^2 \sin\frac{1}{x} \qquad \left[\because f(x) = x^2 \sin\frac{1}{x} \text{ for } x \neq 0\right]$$

= 0 × a finite value = 0

$\therefore$ $\lim_{x\to 0} f(x) = f(0)$. Thus the function is continuous at x = 0

Hence proved.

b. Functional values at x = 0;

f(0) = 1 (finite values) [$\because$ f(x) = 1 for x = 0]

Limiting values at x = 0,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1 - \cos x}{x^2} \qquad \left[f(x) = \frac{1-\cos x}{x^2} \text{ for } x \neq 0\right]$$

$$= \lim_{x\to 0} \frac{(1 - \cos x)}{x^2} \times \frac{1 + \cos x}{(1 + \cos x)}$$

$$= \lim_{x\to 0} \frac{1 - \cos^2 x}{x^2 (1 + \cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x^2} \times \frac{1}{1 + \cos x}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \times \lim_{x\to 0} \frac{1}{1 + \cos x}$$

$$= (1)^2 \times \frac{1}{2} = \frac{1}{2} \qquad \left[\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right]$$

Here, $\lim_{x\to 0} f(x) \neq f(0)$

$\therefore$ The given function is discontinuous at x = 0.

EXERCISE 15.2

1. Find from first principles the derivatives of (1 - 5).

a. i. $e^{\sin x}$ ii. $e^{\tan x}$ iii. e^{x^2}

b. i. $a \sin x/a$ ii. $\sin x^2$ iii. $\sqrt{\tan x}$

c. i. $\ln(\tan x)$ ii. $\ln(\sec x^2)$ iii. $\ln(\text{cosec } x)$

d. i. $\cot^{-1}x$ ii. $\ln \tan^{-1} x$ iii. $e^{\tan^{-1}x}$

e. i. 3^{x^2} ii. x^x iii. a^{2x}

Solution:

a. (i) $e^{\sin x}$.

$f(x) = e^{\sin x}$

We know by the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0}\frac{e^{\sin(x+h)}-e^{\sin x}}{h} \text{ ... ... (i)}$$

Put, $y = \sin x \Rightarrow y+k = \sin(x+h)$

$\Rightarrow k = \sin(x+h) - y$

where $k \to 0$ as $h \to 0$

$\Rightarrow \sin(x+h) = y + k$

Now, from (i)

$$\frac{d}{dx}(f(x)) = \lim_{k\to 0}\frac{e^{y+k}-e^y}{h} = \lim_{k\to 0}\frac{e^y(e^k-1)}{k}\times\frac{k}{h} = e^y . \lim_{k\to 0}\frac{e^k-1}{k}\times\frac{k}{h}$$

$$= e^y . \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \quad \left[\because \lim_{k\to 0}\frac{e^k-1}{k} = 1\right]$$

$$= e^y . \lim_{h\to 0}\frac{2\cos\frac{x+h+x}{2}\times\sin\frac{x+h-x}{2}}{h} = e^y . \lim_{h\to 0} 2\cos\left(x+\frac{h}{2}\right) . \frac{\sin\frac{h}{2}}{\left(\frac{h}{2}\right)\times 2}$$

$$= e^y\times\frac{1}{2}\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\left(\frac{h}{2}\right)}\times 2\cos\left(x+\frac{h}{2}\right) = \frac{1}{2}e^y\times\lim_{h\to 0} 2\cos\left(x+\frac{h}{2}\right)$$

$$= e^{\sin x}\cos\left(x+\frac{O}{2}\right) = e^{\sin x} . \cos x$$

ii. Let, $f(x) = e^{\tan x}$

By the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{\tan(x+h)}-e^{\tan x}}{h} \text{ ... ... (i)}$$

Put, $y = \tan x \Rightarrow y+k = \tan(x+h)$, where $k \to 0$ when $h \to 0$

$\Rightarrow k = \tan(x+h) - \tan x$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{k\to 0}\frac{e^{y+k}-e^y}{h}$$

$$= \lim_{k\to 0}\frac{e^y(e^k-1)}{h} = e^y . \lim_{k\to 0}\frac{e^k-1}{k}\times\frac{k}{h} = e^y . \frac{\tan(x+h)-\tan x}{h}$$

$$= e^y . \lim_{h\to 0}\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h} = \lim_{h\to 0}\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{h\cos x\cos(x+h)}$$

$$= \lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos x . \cos(x+h)} \quad [\because \sin A\cos B - \sin B\cos A = \sin(A-B)]$$

$$= e^y . \lim_{h\to 0} \frac{\sinh}{h} . \lim_{h\to 0} \frac{1}{\cos x . \cos(x+h)} = e^{\tan x} .1\times\frac{1}{\cos x \cos x} = e^{\tan x} . \sec^2 x$$

iii. Let, $f(x) = e^{x^2}$

Since, by the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{(x+h)^2} - e^{x^2}}{h} \quad \ldots\ldots\ldots (i)$$

Put, $y = x^2 \Rightarrow y+k = (x+h)^2$ where $k\to 0$ when $h\to 0$

$\Rightarrow \ k = (x+h)^2 - y = (x+h)^2 - x^2$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{k\to 0}\frac{e^{y+k}-e^y}{h} = \lim_{k\to 0}\frac{e^y(e^k-1)}{k}\times\frac{k}{h} = e^y \lim_{k\to 0}\frac{e^k-1}{k}\times\lim_{h\to 0}\frac{k}{h}$$

$$= e^y\times 1\times\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = e^y\times\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= e^y\times\lim_{h\to 0}\frac{2xh}{h} + \lim_{h\to 0}\frac{h^2}{h} = e^{x^2}\times 2x + 0 = 2x.e^{x^2}$$

b. i.Let, $f(x) = a\sin\frac{x}{a}$

Since by the definition of derivative,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a\sin\left(\frac{x+h}{a}\right) - a\sin\frac{x}{a}}{h} \quad \ldots\ldots\ldots (i)$$

Put, $y = \frac{x}{a} \Rightarrow y+k = \frac{x+h}{a}$, where $k\to 0$ when $h\to 0$

$\Rightarrow \ k = \frac{x+h}{a} - y \Rightarrow k = \frac{x+h}{a} - \frac{x}{a}$

from (i) we have,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{a\sin(y+k) - a\sin y}{h} = a\lim_{h\to 0}\frac{\left[2\cos\frac{y+k+y}{2} . \sin\frac{y+k-y}{2}\right]}{h}$$

$$= a\lim_{k\to 0}\left[\frac{2\cos(y+k/2) . \sin\frac{k}{2}}{h}\right] = 2a\left[\lim_{k\to 0}\cos\left(y+\frac{k}{2}\right) . \lim_{k\to 0}\frac{\sin\frac{k}{2}}{\frac{k}{2}}\times\frac{\frac{k}{2}}{h}\right]$$

$$= 2a\left[\cos y . \lim_{h\to 0}\frac{k}{2h}\right] \qquad [\because k\to 0 \text{ when } h\to 0]$$

$$= 2a\cos y . \lim_{h\to 0}\frac{\left(\frac{x+h}{a} - \frac{x}{a}\right)}{2h} = a\cos y\times\lim_{h\to 0}\frac{\left(\frac{x+h-x}{a}\right)}{h}$$

$$= a\cos y\times\frac{1}{a}\lim_{h\to 0}\frac{h}{h} = \cos y = \cos\frac{x}{a}$$

ii. Let, $f(x) = \sin x^2$

Since by the definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)^2 - \sin x^2}{h} \quad \ldots\ldots\ldots (i)$$

Put, $y = x^2 \Rightarrow y+k = (x+h)^2$ where $k\to 0$ as $h\to 0$

$\Rightarrow k = (x+h)^2 - x^2$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{k\to 0} \frac{\sin(y+k) - \sin y}{h} = \frac{\lim\limits_{k\to 0} 2\cos\frac{y+k+y}{2} . \sin\frac{y+k-y}{2}}{h}$$

$$= \lim_{k\to 0} \frac{2\cos\left(y+\frac{k}{2}\right) . \sin\frac{k}{2}}{h} = 2\lim_{k\to 0} \cos\left(y+\frac{k}{2}\right) . \frac{\sin\frac{k}{2}}{\frac{k}{2}} \times \frac{\frac{k}{2}}{h}$$

$$= 2\cos y \lim_{h\to 0} \frac{k}{2h} = \cos y \times \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \cos y \lim_{h\to 0} \frac{x^2 + h^2 + 2xh - x^2}{h}$$

$$= \cos y \lim_{h\to 0} \frac{h(h+2x)}{h} = \cos y \lim_{h\to 0} (h+2x) = 2x \cos y = 2x \cos x^2$$

iii. Let, $f(x) = \sqrt{\tan x}$

Since by the definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h} \times \frac{\sqrt{\tan(x+h)} + \sqrt{\tan x}}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}$$

$$= \lim_{h\to 0} (\tan(x+h) - \tan x) \times \frac{1}{h\left[\sqrt{\tan(x+h)} + \sqrt{\tan x}\right]}$$

$$= \lim_{h\to 0} \left(\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right) \times \frac{1}{h\left[\sqrt{\tan(x+h)} + \sqrt{\tan x}\right]}$$

$$= \lim_{h\to 0} \frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos x(\cos(x+h))} \times \frac{1}{h\left[\sqrt{\tan(x+h)} + \sqrt{\tan x}\right]}$$

$$= \lim_{h\to 0} \frac{\sin(x+h-x)}{\cos x \cos(x+h)} \times \frac{1}{h\left[\sqrt{\tan(x+h)} + \sqrt{\tan x}\right]}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \times \frac{1}{\cos(x+h)\cos x} \times \lim_{h\to 0} \frac{1}{\sqrt{\tan(x+h)} + \sqrt{\tan}}$$

$$= 1 \times \frac{1}{\cos^2 x} \times \frac{1}{\sqrt{\tan x} + \sqrt{\tan x}} = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

c. Let, $f(x) = \ln(\tan x)$

Since by the definition of derivative,

$$\frac{d(f(x))}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\ln\tan(x+h) - \ln(\tan x)}{h} \quad \text{... .... (i)}$$

Put, $y = \tan x \Rightarrow y+k = \tan(x+h)$ where $k \to 0$ as $h \to 0$

$k = \tan(x+n) - \tan x$

from (i), $\lim\limits_{k\to 0} \dfrac{\log(y+k) - \log y}{h}$

$$= \lim_{k\to 0} \frac{\ln\left(\frac{y+k}{y}\right)}{h} = \lim_{k\to 0} \frac{\ln\left(1+\frac{k}{y}\right)}{\frac{k}{y}} \times \frac{k/y}{h} = \frac{1}{y} \times \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h} = \frac{1}{y} \lim_{h\to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos x . \cos(x+h)}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\sin(x+h-x)}{h\cos x . \cos(x+h)} = \frac{1}{y} \lim_{h\to 0} \frac{\sin h}{h\cos x . \cos(x+h)} = \frac{1}{y} \times 1 \times \frac{1}{\cos^2 x}$$

$$= \frac{1}{\tan x} \times \frac{1}{\cos^2 x} = \frac{1}{\tan x} \cdot \sec^2 x$$

ii. $f(x) = f(x) = \ln \sec x^2$

Since by the definition of derivatives,

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\ln \sec(x+h)^2 - \ln \sec x^2}{h} \quad \dots\dots\dots (i)$$

Put, $y = \sec x^2 \Rightarrow y+k = \sec(x+h)^2$ where $k\to 0$, as $h\to 0$

$\Rightarrow k = \sec(x+h)^2 - \sec x^2$

from (i)

$$= \lim_{k\to 0} \frac{\ln(y+k) - \ln y}{h} = \lim_{k\to 0} \frac{\ln\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{k\to 0} \frac{\ln\left(\frac{y+k}{y}\right)}{h} = \lim_{k\to 0} \frac{\ln(1+k/y)}{k/y} \times \frac{k/y}{h} = 1 \times \lim_{h\to 0} \frac{\sec(x+h)^2 - \sec x^2}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\frac{1}{\cos(x+n)^2} - \frac{1}{\cos x^2}}{h} = \frac{1}{y} \lim_{h\to 0} \frac{\frac{1}{\cos(x+h)^2} - \frac{1}{\cos x^2}}{h}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{\cos x^2 - \cos(x+h)^2}{h \cdot \cos(x+h)^2 \cdot \cos x^2} = \frac{1}{y} \lim_{h\to 0} \frac{-2\sin \frac{x^2 + (x+h)^2}{2} \sin \frac{x^2 - (x+1)^2}{2}}{h \cdot \cos(x+h)^2 \cdot \cos x^2}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{-2\sin \frac{x^2 + (x+h)^2}{2} \cdot \sin \frac{x^2 - (x+h)^2}{2}}{h. \cos(x+h)^2 \cdot \cos x^2}$$

$$= \frac{1}{y} \lim_{h\to 0} \frac{-2\sin\left(\frac{x^2 + 2xh + h^2}{2}\right) \sin\left(\frac{-2xh - h^2}{2}\right)}{h. \cos(x+h)^2 \cdot \cos x^2}$$

$$= \frac{1}{y} x(-2) \frac{\sin x^2}{\cos x^2 \cos x^2} \times \lim_{h\to 0} (-1) \times \frac{\sin\left(\frac{2xh + h^2}{2}\right)}{h}$$

$$= \frac{2}{y} \tan x^2 \cdot \sec x^2 \times \lim_{h\to 0} \frac{\sin(2x+h)/2}{(2x+h)/2} \times \frac{(2x+h)}{2}$$

$$= \frac{2}{\sec x^2} \tan x^2 \cdot \sec x^2 \times 1 \times \frac{2x}{2} = 2x \tan x^2$$

iii. Let, $f(x) = \ln(\operatorname{cosec} x)$

Since by definition of derivatives,

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\ln \operatorname{cosec}(x+h) - \ln \operatorname{cosec} x}{h} \quad \dots\dots\dots (i)$$

Put $y = \operatorname{cosec} x \Rightarrow y+k = \operatorname{cosec}(x+h)$ where $k\to 0$ as $h\to 0$

$\Rightarrow k = \operatorname{cosec} x\,(x+h) - \operatorname{cosec} x$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{k\to 0} \frac{\ln(y+k) - \ln y}{h}$$

$$= \lim_{k\to 0} \frac{\ln\left(\frac{y+k}{y}\right)}{h} = \lim_{k\to 0} \frac{\ln(1+k/y)}{k/y} \times \frac{k/y}{k} = \lim_{h\to 0} \frac{k}{yh}$$

$$= \frac{1}{y}\lim_{h\to 0}\frac{\frac{1}{\sin(x+h)}-\frac{1}{\sin x}}{h} = \frac{1}{y}\lim_{h\to 0}\frac{\sin x-\sin(x+h)}{\sin x\sin(x+h)\times h}$$

$$= \frac{1}{y}\lim_{h\to 0}\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{h\times\sin x\sin(x+h)}$$

$$= \frac{1}{y}\lim_{h\to 0}\frac{2\cos(x+h/2)\sin(-h/2)}{h\times\sin x\sin(x+h)} = \frac{2}{y}\frac{\cos x}{\sin x\,.\,\sin x}\times\lim_{h\to 0}(-1)\frac{\sin h/2}{h/2\times 2}$$

$$= \frac{2}{\operatorname{cosec}x}\,.\,\cot x\,.\,\operatorname{cosec}x\times\left(\frac{-1}{x}\right) = \frac{-\operatorname{cosec}x\,.\,\operatorname{col}x}{\operatorname{cosec}x} = -\cot x$$

d. i. Let, $f(x) = \cot^{-1}x$

$\therefore\quad f(x+h) = \cot^{-1}(x+h)$

We know, $f'(x) = \lim_{h\to 0}\dfrac{\cot^{-1}(x+h)-\cot^{-1}x}{h}$

Let $\cot^{-1}x = y$ and $\cot^{-1}(x+h) = y+k$

$\therefore\quad y+k-y = k = \cot^{-1}(x+h)-\cot^{-1}x$

When $h\to 0$ then $k\to 0$

Also, $x+h = \cot(y+k)$ and $x = \cot y$

$x+h-x = h = \cot(y+k)-\cot y$

Now, $f'(x) = \lim_{k\to 0}\dfrac{y+k-y}{h} = \lim_{k\to 0}\dfrac{k}{\cot(y+k)-\cot y} = \lim_{k\to 0}\dfrac{k}{\frac{\cot(y+k)}{\sin(y+k)}-\frac{\cos y}{\sin y}}$

$$= \lim_{k\to 0}\frac{k}{\frac{\sin y(\cos y+k)-\cos y\sin(y+k)}{\sin(y+k)\,.\,\sin y}}$$

$$= \lim_{k\to 0}\frac{k\sin(y+k)\,.\,\sin y}{\sin(y+k+y)} = \lim_{k\to 0}\frac{-k}{\sin k}\times\sin(y+k)\times\sin y$$

$$= -1\times\sin(y+0)\,.\,\sin y = -\sin^2 y = \frac{-1}{\operatorname{cosec}^2 y} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$$

ii. Let, $f(x) = \ln\tan^{-1}x$

Since by definition of derivatives,

$\dfrac{d}{dx}(f(x)) = \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\dfrac{\log\tan^{-1}(r+h)-\log\tan^{-1}x}{h}$ ... ... (i)

Put, $y = \tan^{-1}x \Rightarrow y+k = \tan^{-1}(x+h)$ where $k\to 0$ as $h\to 0$

or, $\tan y = x$ or, $\tan(y+k)-\tan y = h$

Now, from (i)

$$\frac{d}{dx}(f(r) = \lim_{k\to 0}\frac{\ln(y+k)-\log y}{h} = \lim_{k\to 0}\frac{\ln\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{k\to 0}\frac{\ln(1+k/y)}{k/y}\times\frac{k}{yh} = \frac{1}{y}\times\lim_{k\to 0}\frac{k}{\tan(y+k)-\tan y}$$

$$= \frac{1}{y}\lim_{k\to 0}\frac{k\times\cos(y+k)\,.\,\cos y}{\sin(y+k)\,.\,\cos y-\sin y\,.\,\cos(y+k)} = \frac{1}{y}\lim_{k\to 0}\frac{k\cos(y+k)\cos y}{\sin(y+k-y)}$$

$$= \frac{1}{y}\lim_{k\to 0}\frac{k\cos y\,.\,\cos(y+k)}{\sin k} = \frac{1}{y}\left(\lim_{k\to 0}\frac{k}{\frac{\sin k}{k}\times k}\right)\lim_{k\to 0}\cos y\,.\,\cos(y+k)$$

$$= \frac{1}{\tan^{-1}x}\times 1\times\cos^2 y = \frac{\cos^2 y}{\tan^{-1}x} = \frac{1}{\tan^{-1}x\sec^2 y} = \frac{1}{\tan^{-1}x(1+\tan^2 y)} = \frac{1}{\tan^{-1}x(1+x^2)}$$

iii. Let, $f(x) = e^{\tan^{-1}x}$

Since by definition of derivatives

$$\frac{d}{dx} f(x) = \lim_{k\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{k\to 0} \frac{e^{\tan^{-1}(x+h)} - e^{\tan^{-1}x}}{h}$$

Put, $y = \tan^{-1} x \Rightarrow y+k = \tan^{-1}(x+h)$ where $k \to 0$ as $h \to 0$

or, $\tan y = x \Rightarrow \tan(y+k) = x+h$

$$\frac{d}{dx}(f(x)) = \lim_{k\to 0} \frac{e^{y+k} - e^{y}}{h} = \left(\lim_{K\to 0} \frac{e^{y}(e^{k}-1)}{k}\right) \times \lim_{k\to 0} \frac{k}{h}$$

$$= e^{y} . \lim_{k\to 0} \frac{k}{\tan(y+k) - \tan y}$$

$$= e^{y} . \lim_{k\to 0} \frac{ky \cos(y+k) \cos y}{\sin(y+k)\cos y - \sin y . \cos(y+k)}$$

$$= e^{y} . \lim_{k\to 0} \frac{k \cos y \cos(y+k)}{\sin(y+k-y)} = e^{y} . \lim_{k\to 0} \frac{ky \cos y \cos(y+k)}{\frac{\sin k}{k} \times k}$$

$$= e^{y} . \lim_{k\to 0} \frac{\cos y \cos(y+k)}{\frac{\sin k}{k}} = e^{y} . \cos^{2} y = e^{\tan^{-1}}x . \frac{1}{\sec^{2} y}$$

$$= e^{\tan^{-1}x} . \frac{1}{1+\tan^{2} y} = e^{\tan^{-1}x} . \frac{1}{1+x^{2}}$$

e.i. Let, $f(x) = 3^{x^2} = e^{\ln 3\, x^2} = e^{x^2 \ln 3}$.

Since by definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} e^{(x+h)^2 \ln 3} - e^{x^2 \ln 3} \quad \ldots\ldots\ldots \text{(i)}$$

Put, $y = x^2 \ln 3 \Rightarrow y + k = (x+h)^2 \ln 3$ where $k \to 0$ as $h \to 0$

$\Rightarrow k = (x+h)^2 \ln 3 - x^2 \ln 3$

from (i)

$$\frac{d}{dx}(f(x)) = \lim_{k\to 0} \frac{e^{y+k} - e^{y}}{h} = \lim_{k\to 0} \frac{e^{y}(e^{k}-1)}{k} \times \frac{k}{h}$$

$$= e^{y} \lim_{k\to 0} \left(\frac{e^{k}-1}{k}\right) \times \lim_{h\to 0} \frac{(x+h)^2 \ln 3 - x^2 \ln 3}{h}$$

$$= e^{y} . \lim_{h\to 0} \frac{x^2 \ln 3 + h^2 \ln 3 + 2xh \ln 3 - x^2 \ln^3}{h}$$

$$= e^{y} . \lim_{h\to 0} \frac{h(h\ln 3 + 2x\ln 3)}{h} = e^{x^2 \ln 3} \times 2x\ln 3$$

$$= e^{x^2 \ln 3}\, 2x\ln 3 - e^{\ln 3 x^2}\, 2\times \ln 3 = 3^{x^2}\, 2x\ln 3$$

ii. Let, $f(x) = x^x = e^{\ln x^x} = e^{x \ln x}$

Since by definition of derivatives

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{(x+h)\log(x+h)} - e^{x\log x}}{h} \quad \ldots\ldots\ldots \text{(i)}$$

Put, $y = x\ln x \Rightarrow y+k = (x+h)\ln(x+h)$ where $x\to 0$ cos $h\to 0$

From (i)

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{e^{y+k} - e^{y}}{h} = \lim_{h\to 0} \frac{e^{y}(e^{k}-1)}{k} \times \frac{k}{h}$$

$$= e^{y}\left(\lim_{k\to 0} \frac{e^{k}-1}{k}\right) \times \lim_{h\to 0} \frac{(x+h)\ln(x+h) - x\log x}{h}$$

$$= e^{y} . \lim_{h\to 0} \frac{x\ln(x+h) + h\ln(x+h) - x\ln x}{h}$$

$$= e^y \cdot \lim_{h\to 0}\left[\frac{x[\log(x+h)-\log x]}{h} + \frac{h\log(x+h)}{h}\right]$$

$$= e^y\left[\lim_{h\to 0}\left\{\frac{x\log(1+h/x)}{\frac{h}{x}x}\right\} + \lim_{h\to 0}\frac{h\log(x+h)}{h}\right]$$

$$= e^y[1 + \ln(x+0)] = e^{x\ln x}[1+\ln x] = x^x[1+\ln x]$$

iii. Let $f(x) = a^{2x} = e^{\ln a^{2x}} = e^{2x\ln a}$

Since by definition of derivatives,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{e^{2(x+h)\log a} - e^{2x\log a}}{h}$$

$$= \lim_{h\to 0}\frac{e^{2x\log a}\cdot e^{2h\log a} - e^{2x\log a}}{h} = \lim_{h\to 0} e^{2x\ln a}\frac{[e^{2h\log a}-1]}{h}$$

$$= e^{2x\ln a}\times\lim_{h\to 0}\frac{e^{h\log a}-1}{\frac{2h\log a}{2\log a}} = e^{\ln a^{2x}}\times 2\ln a\left[\lim_{h\to 0}\frac{e^{2h\log a}-1}{2h\log a}\right] = 2a^{2x}\ln a$$

2. Find the derivative of $y = \dfrac{x\sqrt{x^2+a^2}}{2} + \dfrac{a^2}{2}\ln(x + \sqrt{x^2+a^2})$

Solution:

Differentiate both sides w.r.to x we get,

$$\frac{dy}{dx} = \left[\frac{1}{2}\sqrt{x^2+a^2} + x\times\frac{1}{2\sqrt{x^2+a^2}}\times 2x\right]$$

$$+\left[\frac{a^2}{2}\frac{1}{x+\sqrt{x^2+a^2}}\left\{1 + \frac{1}{2\sqrt{x^2+a^2}}\times 2x\right\}\right]$$

$$= \frac{1}{2}\left[\frac{x^2+a^2+x^2}{\sqrt{x^2+a^2}}\right] + \frac{a^2}{2}\left[\frac{1}{x+\sqrt{x^2+a^2}}\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2x^2+a^2}{\sqrt{x^2+a^2}}\right] + \frac{a^2}{2}\times\frac{1}{\sqrt{x^2+a^2}}$$

$$= \frac{1}{2}\left[\frac{2x^2+a^2+a^2}{\sqrt{x^2+a^2}}\right]$$

$$= \frac{1}{2}\left[\frac{2x^2+2a^2}{\sqrt{x^2+a^2}}\right]$$

$$= \sqrt{x^2+a^2}$$

3. Find the derivative of following with respect to x.

a. $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

b. $\cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$

c. $\sin^{-1}(3x - 4x^3)$

d. $\sin^{-1}\sqrt{1-x^2}$

e. $\sin^{-1}(2x\sqrt{1-x^2})$

f. $\sec^{-1}(\tan x)$

g. $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$

h. $\sin^{-1}\dfrac{2x}{1+x^2} + \sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right)$ *[Hint: Put $x = \tan\theta$]*

Solution:

a. Let $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

Put $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

Now, $y = \tan^{-1}\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta$

$y = 2\tan^{-1}x$

Now, differentiating on both sides, we get

$\dfrac{dy}{dx} = 2\dfrac{d\tan^{-1}x}{dx} = 2 \times \dfrac{1}{1+x^2} \qquad \dfrac{dy}{dx} = \dfrac{2}{1+x^2}$

b. Let $y = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Differentiating on both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d\cos^{-1}\dfrac{2x}{1+x^2}}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-\left(\dfrac{2x}{1+x^2}\right)^2}} \times \left[\frac{(1+x^2)\times 2 - 2x \times 2x}{(1+x^2)^2}\right]$$

$$= -\frac{(1+x^2)}{\sqrt{(1+x^2)^2 - 4x^2}}\left[\frac{2+2x^2-4x^2}{(1+x^2)^2}\right] = -\frac{(2-2x^2)}{\sqrt{1+2x^2+x^4-4x^2}} \times \frac{1}{(1+x^2)}$$

$$= -\frac{(1-x^2)}{(1-x^2)} \times \frac{2}{(1+x^2)}$$

$$\frac{dy}{dx} = -\frac{2}{(1+x^2)}$$

c. Let $y = \sin^{-1}(3x - 4x^3)$

Put $x = \sin\theta$

$y = \sin^{-1}(3\sin\theta - 4\sin 3\theta)$

$y = \sin^{-1}(\sin 3\theta)$

$y = 3\theta = 3\sin^{-1}x \therefore \dfrac{dy}{dx} = \dfrac{3}{\sqrt{1-x^2}}$

d. Let $y = \sin^{-1}\sqrt{1-x^2}$

Put $x = \cos\theta$

$\therefore \quad y = \sin^{-1}\sqrt{1-\cos^2\theta} = \sin^{-1}\sin\theta = \theta$

$y = \cos^{-1}x$

Now, differentiating on both sides, we get

$\dfrac{dy}{dx} = \dfrac{d(\cos^{-1}x)}{dx} \quad \therefore \dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$

e. Let $y = \sin^{-1}2x\sqrt{1-x^2}$

Put $x = \cos\theta$ then

$y = \sin^{-1}2\cos\theta\sqrt{1-\cos^2\theta} = \sin^{-1}(2\cos\theta \times \sin\theta) = \sin^{-1}\sin 2\theta$

$y = 2\theta = 2\cos^{-1}x$

Now, differentiating on both sides, we get

$\dfrac{dy}{dx} = 2 \times -\dfrac{1}{\sqrt{1-x^2}}$

$\therefore \quad \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$

f. Let $y = \sec^{-1}(\tan x)$

Differentiating on both sides w.r.t. x, we get

$\frac{dy}{dx} = \frac{d\sec^{-1}(\tan x)}{dx} = \frac{1}{\tan x\sqrt{\tan^2 x - 1}} \cdot \sec^2 x \therefore \frac{dy}{dx} = \frac{\sec^2 x}{\tan x\sqrt{\tan^2 x - 1}}$

g. Put, $x = \tan\theta$

Now, $y = \tan^{-1}\frac{\sqrt{1+\tan^2\theta} - 1}{\tan\theta} = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$y = \tan^{-1}\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2} = \tan^{-1}\tan\theta/2 = \theta/2 = \frac{1}{2}\tan^{-1}x$

Differentiate both sides w.r.to x,

$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{(1+x^2)} = \frac{1}{2(1+x^2)}$

h. Put, $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

L.H.S. $\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} + \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right) = \sin^{-1}\sin 2\theta + \sec^{-1}\left(\frac{\sec^2\theta}{1-\frac{\sin^2\theta}{\cos^2\theta}}\right)$

$= 2\theta + \sec^{-1}\left(\frac{1}{\cos^2\theta - \sin^2\theta}\right) = 2\theta + \sec^{-1}\sec 2\theta = 4\theta = 4\tan^{-1}x$

Differentiate both sides by x, we get,

$\frac{dy}{dx} = \frac{4}{1+x^2}$ R.H.S. Proved.

4. a. If $y = x^y$, then prove that $\frac{dy}{dx} = \frac{y^2}{x(1 - y\ln x)}$

b. If $x^p \cdot y^q = (x+y)^{p+q}$ then prove that $\frac{dy}{dx} = \frac{y}{x}$

c. If $\sin y = x\cos(a+y)$ show that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\cos a}$

d. If $y = e^{x+e^{x+e^{x\cdots}}}$, show that $\frac{dy}{dx} = \frac{y}{1-y}$

Solution:

a. To prove $\frac{dy}{dx} = \frac{y^2}{x(1-y\ln x)}$

We have,

$y = x^y$

Taking ln on both sides, we get

$\ln y = y\ln x$

Now, differentiating on both sides we get

$\frac{1}{y}\frac{dy}{dx} = \ln x\frac{dy}{dx} + y.\frac{1}{x}$

$\Rightarrow \left(\frac{1}{y} - \ln x\right)\frac{dy}{dx} = \frac{y}{x}$

b. To prove $\frac{dy}{dx} = \frac{y}{x}$

We have, $x^p.y^q \ (x+y)^{P+q}$

Taking ln on both sides, we get

$\ln(x^p.y^q) = \ln(x+y)^{p+q}$

$\Rightarrow p\ln x + q\ln y = (p+q)\ln(x+y)$

Now, differentiating on both sides, we get

$p.\frac{1}{x} + q\frac{1}{y}\frac{dy}{dx} = (p+q)\cdot\frac{1}{(x+y)}\left(1+\frac{dy}{dx}\right)$

$\Rightarrow \frac{p}{x} + \frac{q}{y}\frac{dy}{dx} = \frac{p+q}{x+y} + \frac{p+q}{x+y}\frac{dy}{dx}$

$\Rightarrow (1 - y\ln x)\dfrac{dy}{dx} = \dfrac{y^2}{x}$

$\therefore \dfrac{dy}{dx} = \dfrac{y^2}{x(1 - y\ln x)}$

$\Rightarrow \left(\dfrac{q}{y} - \dfrac{p+q}{x+y}\right)\dfrac{dy}{dx} = \dfrac{p+q}{x+y} - \dfrac{p}{x}$

$\Rightarrow \dfrac{(qx + qy - py - qy)}{y(x+y)} \cdot \dfrac{dy}{dx} = \dfrac{px + qx - px - py}{x(x+y)}$

$\Rightarrow \dfrac{(qx - py)}{y(x+y)}\dfrac{dy}{dx} = \dfrac{(qx - py)}{x(x+y)}$

$\therefore \dfrac{dy}{dx} = \dfrac{y}{x}$

c. Differentiate both sides w.r. to x we ge,t

$\cos y\dfrac{dy}{dx} = -x\sin(a+y)\dfrac{dy}{dx} + 1.\cos(a+y)$

or, $(\cos y + x\sin(a+y)\dfrac{dy}{dx} = \cos(a+y)$

or, $\dfrac{dy}{dx} = \dfrac{\cos(a+y)}{\cos y + x\sin(a+y)}$

or, $\dfrac{dy}{dx} = \dfrac{\cos(a+y)}{\cos y + \dfrac{\sin(a+y)}{\cos(a+y)}} = \dfrac{\cos^2(a+y)}{\cos(a+y-y)} = \dfrac{\cos^2(a+y)}{\cos a}$ proved.

d. Let, $y = e^{x+e^{x+e^{x+\cdots}}}$. Then we have,

$y = e^{x+y}$

Taking ln on both sides we get,

$\ln y = (x+y)\ln e$

Differentiate both sides w.r. to x, we get

$\dfrac{1}{y}\dfrac{dy}{dx} = 1 + \dfrac{dy}{dx}$

or, $\left(\dfrac{1}{y} - 1\right)\dfrac{dy}{dx} = 1$

or, $\left(\dfrac{1-y}{y}\right)\dfrac{dy}{dx} = 1$

or, $\dfrac{dy}{dx} = \dfrac{y}{1-y}$

5. Find the derivative with respect to x of following

a. $x^{\sin x}$

b. $\sin x^{\cos x}$

c. $(\sin x)^{\cos x} + (\cos x)^{\sin x}$

d. $x^{\tan x} + (\tan x)^x$

Solution:

a. Let $y = x^{\sin x}$

$\ln y = \sin x \ln x$

Differentiating on both sides, we get

$\dfrac{1}{y}\cdot\dfrac{dy}{dx} = \ln x \cdot \cos x + \sin x\ \dfrac{1}{x}$

$\Rightarrow \dfrac{dy}{dX} = y\left[\ln x\cos x + \dfrac{\sin x}{x}\right]$

b. Let $y = (\sin x)^{\cos x}$

$\ln y = \cos x \cdot \ln(\sin x)$

$\dfrac{1}{y}\dfrac{dy}{dx} = -\ln(\sin x)\cdot\sin x + \cos x \times \cos x$

$\therefore \dfrac{dy}{dx} = (\sin x)^{\cos x}$ [cosx . cosx – sinx, Ln(sinx)]

$\therefore \dfrac{dy}{dx} = x^{\sin x}\left[\dfrac{\sin x}{x} + \cos x \,.\, \ln x\right]$

c. Let, $y = u + v$ where, $u = (\sin x)^{\cos x}$ and $v = (\cos x)^{\sin x}$

Now, $u = (\sin x)^{\cos x}$

Taking ln on both sides,

or, $\ln u = \cos x \ln \sin x$

Differentiate both sides w.r. to x, we get,

$\dfrac{1}{u}\dfrac{dy}{dx} = \dfrac{\cos x \,.\, \cos x}{\sin x} + \ln \sin x \,.\, (-\sin x) = \dfrac{\cos^2 x}{\sin x} - \sin x \ln \sin x$

or, $\dfrac{dy}{dx} = (\sin x)^{\cos x}[\cos x \,.\, \cot x - \sin x \,.\, \ln \sin x]$

Similarly, $\dfrac{1}{v}\dfrac{dv}{dx} = \sin x \,.\, \dfrac{-\sin x}{\cos x} + \cos x \ln(\cos x)$

$\dfrac{dv}{dx} = (\cos x)^{\sin x}[\cos x \ln \cos x - \sin x \,.\, \tan x]$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dx} + \dfrac{dv}{dx}$

$= (\sin x)^{\cos x}[\cos x.\cos x - \sin x.\ln \sin x] + (\cos x)^{\sin x}[\cos x.\ \ln\cos x - \sin x.\tan x]$

d. Let, $y = u + v$, where, $u = x^{\tan x}$ and $v = (\tan x)^x$.

if, $u = (x)^{\tan x}$

Taking ln on both sides,

$\ln u = \tan x \ln x$

Differentiate both sides w.r. to x, we get,

$\dfrac{1}{u}\dfrac{dy}{dx} = \tan x\,\dfrac{1}{x} + \ln x \,.\, \sec^2 x$

$\dfrac{dy}{dx} = (x)^{\tan x}\left[\dfrac{\tan x}{x} + \log x \,.\, \sec^2 x\right]$

Again similarly,

$\dfrac{1}{v}\dfrac{dv}{dx} = x.\dfrac{\sec^2 x}{\tan x} + \ln \tan x.1$

$\dfrac{dv}{dx} = (\tan x)^x\left[\ln \tan x + \dfrac{x\sec^2 x}{\tan x}\right]$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dx} + \dfrac{dv}{dx} = (x)^{\tan x}\left[\dfrac{\tan x}{x} + \ln x \sec^2 x\right] + (\tan x)^x\left(\ln \tan x + \dfrac{x\sec^2 x}{\tan x}\right)$

EXERCISE 15.3

Find the derivative with respect to x of the following.

1. $e^{\cosh x/a}$
2. $\ln \tan hx$
3. $\tanh(\text{arc } \sin x)$
4. $\text{sech}^{-1} x - \text{cosech}^{-1} x$
5. $x^{\cosh x}$
6. $x^{\sin h\, x^2/a}$
7. $x^{\cos h^{-1} x/a}$
8. $(\ln x)^{\sin hx}$
9. $(\sin hx)^{\cosh^{-1} x}$
10. $(\cos hx)^{\cos hx}$
11. $\left(\tan h\dfrac{x}{a}\right)^{\ln x}$
12. $(\sin h^{-1}x + \cosh^{-1}x)^x$
13. $\left(\sin h\dfrac{x}{a} + \cos h\dfrac{x}{a}\right)^{nx}$

Solution:

1. Let, $y = e^{\cosh x/a}$

Differentiate both sides w.r. to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\cosh x/a}\right) = e^{\cos 3hx/a} . \frac{1}{a}\sinh x/a\ \frac{d(x/a)}{dx}$$

$$= \frac{1}{a}\sinh\frac{x}{a}\ e^{\cosh x/a}$$

2. Let, y = ln tanhx

Differentiate both sides w.r.to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}(\ln \tanh x) = \frac{1}{\tanh x}\frac{d}{dx}(\tanh x)$$

$$= \frac{1}{\tanh x} . \operatorname{sech}^2 x = \frac{\cosh x}{\sinh x . \cosh^2 x} = \frac{1}{\sinh x \cosh x} = \frac{2}{\sinh 2x} = 2\operatorname{cosech} 2x$$

3. Let, y = tanh $(\sin^{-1}x)$

Differentiate both sides w.r.to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}(\tanh \sin^{-1}x) = \operatorname{sech}^2 \sin^{-1}x\ \frac{d}{dx}(\sin^{-1}x) = \operatorname{sech}^2\left(\sin^{-1}x\ \frac{x}{\sqrt{1-x^2}}\right)$$

4. Let, y = $\operatorname{sech}^{-1}x = -\operatorname{cosch}^{-1}x$

Differentiate both sides w.r. to x, we get,

$$\frac{dy}{dx} = \frac{d}{dx}[\operatorname{sech}^{-1}x - \operatorname{cosech}^{-1}x]$$

$$= \frac{-1}{x\sqrt{1-x^2}} + \frac{1}{x\sqrt{x^2+1}}$$

$$= \frac{1}{x}\left[\frac{1}{\sqrt{x^2+1}} - \frac{1}{\sqrt{1-x^2}}\right]$$

5. Let, y = $x^{\cosh x}$

Taking ln on both sides we have,

lny = coshx lnx

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y}\frac{dy}{dx} = \cosh x\ \frac{1}{x} + \ln x . \sinh x$$

$$\frac{dy}{dx} = x^{\cosh x}\left[\frac{\cosh x + x\sinh x \log x}{x}\right]$$

6. Let, y = $x^{\sinh x^2/a}$

Taking ln on both sides we get,

$$\ln y = \sinh\frac{x^2}{a}\ln x$$

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y}\frac{dy}{dx} = \sinh x^2/a\ \frac{1}{x} + \ln x.\cos x\ \frac{x^2}{a} = \frac{2x}{a}$$

$$\frac{dy}{dx} = x^{\sinh x^2/a}$$

$$\left[\frac{\sinh x^2/a}{x} + \frac{2x\ln x}{a}\cosh x^2/a\right]$$

7. Let, y = $x^{\cosh^{-1}x/a}$

Taking ln on both sides we get,

lny = $x^{\cosh^{-1}x/a}$ lnx

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y}\frac{dy}{dx} = \cosh^{-1}x/a\ \frac{1}{x} + \ln x\ \frac{1}{\sqrt{\frac{x^2}{a^2}-1}} \times \frac{1}{a}$$

$$\frac{dy}{dx} = x^{\cosh^{-1}x/a}\left[\frac{1}{x}\cosh^{-1}x/a + \frac{\log x}{\sqrt{x^2-a^2}}\right]$$

8. Let, y = $(\ln x)^{\sinh x}$

Taking ln on both sides we get,

lny = sinhx ln (lnx)

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y}\frac{dy}{dx} = \sinh x\ \frac{1}{\log x} . \frac{1}{x} + \ln(\ln x).\cosh x$$

or, $\frac{dy}{dx} = (\ln x)^{\sinh x}$

$$\left[\frac{\sinh x}{x\log x} + \cosh x \log(\log x)\right]$$

9. Let, y = $(\sinh x)^{\cosh^{-1}x}$

Taking ln on both sides w.r. to x, we get,

lny = $\cosh^{-1}x$ ln(sinhx)

Differentiate both sides w.r. to x, we get,

or, $\frac{1}{y}\frac{dy}{dx} = \cosh^{-1}x . \frac{1}{\sinh x} . \cosh x + \ln$ $(\sinh x)\frac{1}{\sqrt{x^2-1}}$

$$\text{or, } \frac{dy}{dx} = (\sinh x)^{\cosh^{-1}x} \left[\cosh^{-1}x \coth x + \frac{\log \sinh x}{\sqrt{x^2 - 1}}\right]$$

10. Let, $y = (\cosh x)^{\cosh x}$

Taking ln on both sides we get,

$\ln y = \cosh x \ln(\cosh x)$

Differentiate both sides w.r. to x, we get,

$$\text{or, } \frac{1}{y}\frac{dy}{dx} = \cosh x \frac{\sinh x}{\cosh x} + \ln(\cosh x) . \sinh x$$

$$\text{or, } \frac{dy}{dx} = (\cosh x)^{\cosh x} [\cosh x \tanh x + \sinh x \ln (\cosh x)]$$

$$\therefore \frac{dy}{dx} = (\cosh x)^{\cosh x} [\text{Sinh}x + \sinh x \ln (\cosh x)]$$

11. Let, $y = (\tanh \frac{x}{a})^{\ln x}$

Taking ln on both sides w.r. to x, we get,

$\ln y = \ln x \ln (\tanh\frac{x}{a})$

Differentiate both sides w.r. to x, we get,

$$\frac{1}{y}\frac{dy}{dx} = \ln x \frac{\text{sech}^2 x/a}{\tanh x/a} \times \frac{1}{a} + \ln \tanh \frac{x}{a} . \frac{1}{x}$$

$$\text{or, } \frac{dy}{dx} = (\tanh \frac{x}{a})^{\ln x} \left[\frac{\cos x . \cosh x/a}{\cosh^2 x/a. \sinh x/a}\frac{1}{a} + \ln \tanh \frac{\ }{a} . \frac{1}{x}\right]$$

$$\text{or, } \frac{dy}{dx} = (\tanh \frac{x}{a})^{\ln x} \left[\frac{2 \log x}{2 \sinh x/a \cosh x/a} \times \frac{1}{a} + \ln \tanh x/a . \frac{1}{x}\right]$$

$$\text{or, } \frac{dy}{dx} = (\tanh \frac{x}{a})^{\ln x} \left[\frac{2}{a} \text{cosech} 2\, x/a . \log x + \ln \tanh x/a . \frac{1}{x}\right]$$

$$\text{or, } \frac{dy}{dx} = (\tanh \frac{x}{a})^{\ln x} \left[\frac{2}{a} \ln x \text{ cosech } 2x/a + \frac{1}{x} \ln \tanh x/a\right]$$

12. Let, $y = (\sinh^{-1}x + \cosh^{-1}x)^x$

Taking ln on both sides w.r. to x, we get,

$\ln y = x \ln (\sinh^{-1}x + \cosh^{-1}x)$

Differentiate both sides w.r. to x, we get,

$$\text{or, } \frac{1}{y}\frac{dy}{dx} = \frac{x}{(\sinh^{-1}x + \cosh^{-1}x)} \left[\frac{d}{dx}(\sinh^{-1}x + \cosh^{-1}x)\right] + \ln (\sinh^{-1}x + \cosh^{-1}x)$$

$$\text{or, } \frac{1}{y}\frac{dy}{dx} = \frac{x}{(\sinh^{-1}x + \cosh^{-1}x)} \left(\frac{1}{\sqrt{1 + x^2}} + \frac{1}{\sqrt{x^2 - 1}}\right) + \ln (\sinh^{-1}x + \cosh^{-1}x)$$

$$\text{or, } \frac{dy}{dx} = (\sinh^{-1}x + \cosh^{-1}x)^x \left[\frac{x}{(\sinh^{-1}x + \cosh^{-1}x)} \left(\frac{1}{\sqrt{1 + x^2}} + \frac{1}{\sqrt{x^2 - 1}}\right)\right.$$
$$\left. + \ln (\sinh^{-1}x + \cosh^{-1}x)\right]$$

13. Let, $y = (\sinh \frac{x}{a} + \cosh \frac{x}{a})^{nx}$

Taking ln on both sides we get,

$\ln y = nx \ln (\sinh\frac{x}{a} + \cosh \frac{x}{a})$

Differentiate both sides w.r. to x, we get,

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} hx \times \frac{\left(\cosh\frac{x}{a} + \sinh\frac{x}{a}\right)}{\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)} \times \frac{1}{a} + n\ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{nx}{a} + n\ln\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right]$$

$$= \left(\sinh\frac{x}{a} + \cosh\left(\frac{x}{a}\right)\right)^{nx}\left[\frac{nx + na\ln(\sinh x\frac{x}{a} + \cosh\frac{x}{a})}{a}\right]$$

$$= n\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}\left[\frac{x}{a} + \ln(\sinh\frac{x}{a} + \cosh\frac{x}{a})\right]$$

EXERCISE 15.4

1. Find the slope and inclination with the x-axis of the tangent of following curves.
   a. $y = x^3 + 2x + 7$ at $x = 1$
   b. $x^2 - y^2 = 9$ at $(3, 0)$
   c. $y = -3x - x^4$ at $x = -1$

Solution:

a. Given, $y = x^3 + 2x + 7$

Differentiate both sides w.r. to x, we get,

$$\frac{dy}{dx} = 3x^2 + 2$$

Slope at x = 1 is, $\left.\frac{dy}{dx}\right|_{x = 1}$

$$\therefore \frac{dy}{dx} = 3\times1^2 + 2 = 5$$

Since slope m = tanθ, θ is angle from x–axis.

$\therefore$ Tanθ = 5

$\Rightarrow$ $\theta = \tan^{-1}5$

b. Given, $x^2 - y^2 = 9$

Differentiate both sides w.r. to x, we get,

$$2x - 2y\frac{dy}{dx} = 0$$

or, $\frac{dy}{dx} = \frac{x}{y}$

Slope at (3, 0) is, $\left.\frac{dy}{dx}\right|_{(3,0)} = \frac{3}{0} = \infty$

Again, $\tan\theta = \infty$

$$\theta = \tan^{-1}\infty = \tan^{-1}(1)\tan\frac{\pi}{2} = \frac{\pi}{2}$$

c. Given, $y = -3x - x^4$

Differentiate both sides w.r. to x,

$$\frac{dy}{dx} = -3 - 4x^3$$

Slope at x = –1 is, $\frac{dy}{dx}_{x=-1} = -3-4(-1)^3 = -3+4 = 1$

Again, $\tan\theta = 1$

$$\theta = \tan^{-1}1 = \tan{-1}\tan\frac{\pi}{4} = \frac{\pi}{4}$$

2. Obtain the equation to the tangent to the parabola $y^2 = 8x$ at (2, -4).

Solution:

Since we know the equation of tangent be the parabola $y^2 = 4ax$ at (x_1, y_1) is, $yy_1 = 2a(x + x_1)$

Now, $y^2 = 8x = 4 \times 2x$ and $(x_1, y_1) = (2, -4)$

$\therefore$ Required equation of tangent is

$y(-4) = 4(x + 2)$

or, $-4y = 4x + 8$

or, $-y = x + 2$

or, $x + y + 2 = 0$

Given, $y^2 = 8x$

$2y\frac{dy}{dx} = 8$

$\frac{dy}{dx} = \frac{4}{y}$

$\left.\frac{dy}{dx}\right|_{(2,\,-4)} = \frac{4}{-4} = -1$

Now, required equation of tangent,

$y - y_1 = m(x - x_1)$

or, $y + 4 = -1(x - 2)$

or, $y + 4 = -x + 2$

or, $x + y + 2 = 0$

3. Find the equation of tangent and normal to the curve

a. $y = 2x^2 - 3x - 1$ at $(1, -2)$

b. $y = x^3$ at $(2, 8)$

c. $x^2 - y^2 = 16$ at $(6, 3)$

d. $x^2 + 3xy + y^2 = 11$ at $(2, 1)$

e. $x^{2/3} + y^{2/3} = 2$ at $(1, 1)$

f. $y^2 = 8x$ at $(2, -4)$

Solution:

a. Given $y = 2x^2 - 3x - 1$

Differentiate both sides w.r. to x, we get

$\frac{dy}{dx} = 4x - 3$

Slope of tangent say (m_1) at $(1, -2)$ is $m_1 = \frac{dy}{dx} = 4 - 3 = 1$

Then slope of normal is say m_2 is given by

$m_1 \times m_2 = -1$

$m_2 = -\frac{1}{1} = -1$

Now, equation of tangent is,

$y - (-2) = 1(x - 1)$

$\Rightarrow y + 2 = x - 1$

$\Rightarrow x - y - 3 = 0$

Again equation of normal at $(1, -2)$ is,

$y - (-2) = -1(x - 1) \Rightarrow y + 2 = -x + 1$

$\Rightarrow x + y + 1 = 0$

b. Given, $y = x^3$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 3x^2$

Slope (m_1) $\left.\frac{dy}{dx}\right|_{(2,\,8)} = 3\times2^2 = 12$

Equation of tangent at $(2, 8)$ is

$y - 8 = 12(x - 2)$

$\Rightarrow y - 8 = 12x - 24$

$\Rightarrow 12x - y - 16 = 0$

c. Given, $x^2 - y^2 = 16$

Differentiate both sides w.r. to x, we get

$2x - 2y\frac{dy}{dx} = 0$

or, $2y\frac{dy}{dx} = 2x$

or, $\frac{dy}{dx} = \frac{x}{y}$

$\left.\frac{dy}{dx}\right|_{(6,\,3)} = \frac{6}{3} = 2$

Again, Slope of normal is $= -\frac{1}{12}$

Now, equation of normal is,

$y - 8 = -\frac{1}{12}(x - 2)$

$\Rightarrow$ $12y - 96 = -x + 2$

$\Rightarrow$ $x + 12y - 98 = 0$

d. Given, $x^2 + 3xy + y^2 = 11$

Differentiate both sides w.r. to x, we get,

$2x + 3\left[x\frac{dy}{dx} + y\right] + 2y\frac{dy}{dx} = 0$

$\Rightarrow$ $2x + 3x\frac{dy}{dx} + 3y + 2y\frac{dy}{dx} = 0$

$\Rightarrow$ $\frac{dy}{dx}(3x + 2y) = -(2x + 3y)$

$\therefore$ $\frac{dy}{dx} = \frac{-(2x + 3y)}{3x + 2y}$

At (2, 1), $\frac{dy}{dX} = \frac{-(4 + 3)}{6 + 2} = -\frac{7}{8}$

Equation of the tangent at (2, 1) is

$y - 1 = -\frac{7}{8}(x - 2)$

$\Rightarrow$ $8y - 8 = -7x + 14$

$\therefore$ $7x + 8y = 22$

And, the equation of the normal at (2, 1) is,

$y - 1 = \frac{8}{7}(x - 2)$

$\Rightarrow$ $8x - 16 = 7y - 7$

$\therefore$ $8x - 7y = 9$

f. $y^2 = 8x$ at (2, –4)

Solution:

We have, $y^2 = 8x$

$\Rightarrow$ $2y\frac{dy}{dx} = 8$

$\therefore$ $\frac{dy}{dx} = \frac{4}{y}$

At (2, –4), $\frac{dy}{dx} = 1$

Equation of the tangent at (2, –4)

$y + 4 = -1(x - 2)$

$\Rightarrow$ $y + 4 = -x + 2$

$\therefore$ $x + y + 2 = 0$

And, the equation of the normal at (2, –4) is

$y + 4 = 1(x - 2)$

Now, equation of tangent at (6, 3) is,

$y - 3 = 2(x - 6)$

or, $y - 3 = 2x - 12$

or, $2x - y - 9 = 0$

Again, equation of normal is,

$y - 3 = -\frac{1}{2}(x - 6)$

or, $2y - 6 = -x + 6$

or, $x + 2y - 12 = 0$

e. Differentiate w.r. to x, we get,

$\frac{2}{3}x^{(2/3-1)} + \frac{2}{3}y^{(2/3-1)}\frac{dy}{dx} = 0$

or, $\frac{2}{3}x^{-1/2} + \frac{2}{3}y^{-1/2}\frac{dy}{dx} = 0$

or, $y^{-1/2}\frac{dy}{dx} = x^{-1/2}$

or, $\frac{dy}{dx} = \frac{x^{1/2}}{y^{-1/2}} = \frac{y^{1/2}}{x^{1/2}} = \frac{\sqrt{y}}{\sqrt{x}}$

$\left.\frac{dy}{dx}\right|_{(1,\,1)}$ is 1

Now, equation of tangent at (1, 1) is,

$y - 1 = 1(x - 1)$

$\Rightarrow$ $y-1 = x-1 \Rightarrow x-y = 0$

Again, equation of normal is,

$y - 1 = -1(x - 1)$

$\Rightarrow$ $y - 1 = -x + 1$

$\Rightarrow$ $x + y - 2 = 0$

$\therefore$ x – y – 6 = 0

4. Find the points on the curve where the tangents are parallel to the x-axis.
   a. $y = 2x - x^2$ b. $y = 2x^2 - 6x + 9$ c. $x^2 + y^2 = 16$

Solution:

a. Differentiate both sides w.r. to x, we get, $\frac{dy}{dx} = 2 - 2x$

If the tangent are parallel to x–axis, then the slope must be zero

$\therefore \frac{dy}{dx} = 2 - 2x = 0$

$\Rightarrow$ 2 = 2x $\Rightarrow$ x = 1

If x = 1, then $y = 2 - 1^2 = 1$

$\therefore$ The required point is (1, 1)

b. Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 4x - 6]$

Now, $\frac{dy}{dx} = 0 \Rightarrow 4x = 6$

$\Rightarrow x = \frac{3}{2}$

If $x = \frac{3}{2}$, then $y = 2 \times \frac{9}{4} - 6 \times \frac{3}{2} + 9 = \frac{9}{2}$

$\therefore$ Required point is $\left(\frac{3}{2}, \frac{9}{2}\right)$

c. Differentiate both sides w.r. to x, we get,

$2x + 2y\frac{dy}{dx} = 0$

or, $2y\frac{dy}{dx} = -2x$

or, $\frac{dy}{dx} = -\frac{x}{y}$

Now, $\frac{dy}{dx} = 0 \Rightarrow -\frac{x}{y} = 0 \Rightarrow -x = 0 \Rightarrow x = 0$

If x = 0 then y = ±4

Therefore the required point (0, ±4)

5. Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangent are
   a. parallel to x-axis b. parallel to y-axis.

Solution:

Given, $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Differentiate both sides w.r. to x, we get,

$\frac{2x}{9} + \frac{2y}{16}\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{2x}{9} \times \frac{16}{2y} = \frac{-16x}{9y}$

a. For, parallel to x–axis

$\frac{dy}{dx} = 0$

$\Rightarrow \frac{-16x}{9y} = 0$

$\Rightarrow$ x = 0

If x = 0 then, $y^2 = 16$

y = ± 4

Therefore required points (0, ± 4)

b. For parallel to γ-axis,

$\frac{dy}{dx} = \infty$

$\Rightarrow \frac{-16x}{9y} = \infty$ which is possible only when y = 0

Then we have,

$\frac{x^2}{9} = 1$

$\therefore$ x = ±3

Therefore required points (± 3, 0)

6. Show that the tangents to the curve $y = 2x^3 - 3$ at the points where $x = 2$ and $x = -2$ are parallel.

Solution:

Given, $y = 2x^3 - 3$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 6x^2$

Slope at x = 2 i.e, $\frac{dy}{dx}_{x=2}$ is $\frac{dy}{dx} = 6\times2^2 = 24$.

Again slope at x = –2 is, $\frac{dy}{dx}_{x=-2} = 6\times(-2)^2 = 24$

Hence, if x = 2 and x = –2 slope are equal it means the tangent are parallel.

7. a. Find the equation of tangent line to the curves $y = x^2 - 2x + 7$ which is parallel to the line $2x - y + 9 = 0$
   b. Find the point on the curve $y^2 = 4x + 1$ at which the tangent is perpendiculars to the line $7x + 2y = 1$.

Solution:

a. Given, curve, $y = x^2 - 2x + 7$

Differentiate both sides w.r. to x, we get,

$\frac{dy}{dx} = 2x - 2$(slop of tangent)

Again slop of the line $2x - y + 9 = 0$, obtained by comparing to $y = mx + c$ i.e. $y = 2x + 9$. Therefore the slope of given line is 2.

If the required tangent is parallel to the given line then slope must be equal

$\therefore\ 2x - 2 = 0$

or, $x = 2$

Put, x = 2 in $y = x^2 - 2x + 7$ we get,

$y = 2^2 - 2\times2 + 7 = 7$

$\therefore$ Required point is, (2, 7)

Now, the equation of tangent is,

$y - 7 = 2(x - 2)$

or, $y - 7 = 2x - 4$

or, $2x - y + 3 = 0$

b. Given, curve, $y^2 = 4x + 1$,

$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

Again slope of $7x + 2y = 1$

or, $2y = -7x + 1$

$y = \frac{-7}{2}x + \frac{1}{2}$

$\therefore$ Slope of this box ix $\frac{-7}{2}$.

If the tangent to the curve $y^2 = 4x+1$ is perpendicular to the line $7x + 2y = 1$, product of slope must be –1.

$\therefore\ \frac{2}{y} \times -\frac{7}{2} = -1 \Rightarrow 14 = 2y \Rightarrow y = 7$

Putting y = 7 in $y^2 = 4x + 1$ we get,

$49 = 4x + 1$

$48 = 4x \Rightarrow x = 12$

$\therefore$ The required point is (12, 7)

8. Show that equation of tangent to the curve $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 = 2$ at (a, b) is $\frac{x}{a} + \frac{y}{b} = 2$

Solution:

Given, curve is, $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 = 2$

Differentiate both sides w.r. to x, we get,

$$\frac{3x^2}{a^3} + \frac{3y^2}{b^3}\frac{dy}{dx} = 0$$

or, $\frac{dy}{dx} = \frac{-x^2}{a^3} \times \frac{b^3}{y^2}$

Now, $\left.\frac{dy}{dx}\right|_{(a,\, b)} = -\frac{a^2b^3}{a^3b^2} = -\frac{b}{a}$

Now, equation of tangent at (a, b) is,

$$y - b = -\frac{b}{a}(x - a)$$

$\Rightarrow$ ay – ab = –bx + ab

$\Rightarrow$ ay + bx = 2ab. Dividing both sides by ab

We get, $\frac{x}{a} + \frac{y}{b} = 1$ proved.

9. Find the angle of intersection of the following curves.

a. $y = x^3$ and $6y = 7 - x^2$ at (1, 1)
b. $y = x^3$ and $y = 2x$
c. $y = 6 - x^2$ and $x^3 = 4y$ at (2, 4)
d. $x^2 + y^2 = 5$ and $y^2 = 4x$

Solution:

a. Solving we get,

$6x^3 + x^2 - 7 = 0$

or, $6x^3 - 6x^2 + 7x^2 - 7x + 7x - 7 = 0$

or, $6x^2(x - 1) + 7x(x - 1) + 7 = (x - 1) = 0$

or, $(x - 1)(6x^2 + 7x + 7) = 0$

$\therefore$ x = 1 as $6x^2 + 7x + 7 = 0$ does not have any real values.

If x = 1 then y = 1

Now, from $y = x^3$

$\Rightarrow \frac{dy}{dx} = 3x^2$

Again, from, $6y - 7 + x^2 = 0$

$$6\frac{dy}{dx} + 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{6} = -\frac{x}{3}$$

$\therefore 6\frac{dy}{dx} = -2x$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{3}$

$\therefore \frac{dy}{dx}$ at (1, 1) (say m_2) = $\frac{-1}{3}$

If θ be the angle between two curves, then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{3 + \frac{1}{3}}{1 + 3 \times \left(-\frac{1}{3}\right)}\right| = \left|\frac{\frac{10}{3}}{0}\right| = \infty$$

$\therefore \theta = \tan^{-1}\infty$

$\therefore \theta = \frac{\pi}{2}$

b. Solving we get,

$x^3 = 2x$ [$\because$ $x \neq 0$ otherwise it doesn't remains curves]

$\Rightarrow x^2 = 2 \quad \therefore x = \pm\sqrt{2}$

Now, if $x = \sqrt{2}$. Then $y = \pm 2\sqrt{2}$

From $y = x^3$

$\frac{dy}{dx} = 3x^2$

$\left.\frac{dy}{dx}\right|_{(\sqrt{2},\, 2\sqrt{2})}$ (m_1 say) $= 3(\sqrt{2})^2 = 6$

From, $y = 2x$

$\frac{dy}{dx} = 2 \qquad \therefore \left.\frac{dy}{dx}\right|_{(\sqrt{2},\, 2\sqrt{2})}$ (m_2 say) $= 2$

Now, by using the formula,

$\tan\theta \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{6-2}{1+6\times2}\right| = \left|\frac{4}{1+12}\right| = \left|\frac{4}{13}\right|$

$\therefore \quad \theta = \tan^{-1}\frac{4}{13}$

c. Here, $y = 6 - x^2$

$\frac{dy}{dx} = -2x$

Say $m_1 = \frac{dy}{dx}(2, 4) = -2\times2 = -4$

and, $x^3 = 4y$

$3x^2 = \frac{4dy}{dx} \qquad \therefore \quad \frac{dy}{dx} = \frac{3}{4}x^2$

say $m_2 = \frac{dy}{dx}(2, 4) = \frac{3}{4}\times 4 = 3$

If θ be the angle then,

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{-4-3}{1+(-4)(3)}\right| = \left|\frac{-7}{11}\right|$

$\therefore \quad \theta = \tan^{-1}\frac{7}{11}$

d. Solving we ge,t

$x^2 + 4x - 5 = 0$

or, $x^2 + 5x - x - 5 = 0$

or, $x(x + 5) - 1(x + 5) = 0$

$\therefore \quad x = 1, -5$

If $x = 1$ then $y^2 = 4 \Rightarrow y = 2$

If $x = -5$ then $y^2 = -20$ (does not give real values)

Now from $x^2 + y^2 = 5$

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

m_1 (say) $= \frac{dy}{dx}(1, 2) = -\frac{1}{2}$

Again, $y^2 = 4x$

$dy\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

or, m_2(say) $= \frac{dy}{dx}_{(1,2)} = \frac{2}{2} = 1$

$\therefore \quad \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\frac{-1}{2} - 1}{1 + \left(-1\frac{1}{2}\right)(1)}\right| = \left|\frac{\frac{-3}{2}}{\frac{1}{2}}\right| = |3|$

$\therefore \quad \theta = \tan^{-1}(3)$

EXERCISE 15.5

1. Verify the Rolle's theorem for each of the following functions.
   a. $f(x) = x^2 + 2$ in [-2, 2]
   b. $f(x) = x^3 - 4x$ in [0, 2]
   c. $f(x) = \sin 2x$ in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
   d. $f(x) = \cos 2x$, in $[-\pi, \pi]$
   e. $f(x) = \sqrt{25 - x^2}$ in [-5, 5]
   f. $f(x) = (x - 1)(x - 2)(x - 3)$ in [1, 3]
   g. $f(x) = \sin x + \cos x$ in $[0, 2\pi]$

Solution:

a. Here, $f(x) = x^2 + 2$

i. Since the polynomial function is continuous. Hence given function is continuous in [–2, 2].

ii. Again, $f(x) = x^2 + 2$
$f'(x) = 2x$ which gives real values for all values of x in (2, 2)
Hence f(x) is differentiable in (–2, 2)

iii. Since $f(a) = f(-2) = (-2)^2 + 2 = 6$ and $f(b) = f(2) = 2^2 + 2 = 6$

$\therefore$ f(a) = f(b).

Here f(x) satisfies all the condition of Rolle's theorem so these exists $C \in (a, b)$ such that $f'(c) = 0 \Rightarrow 2c = 0$

Now, $f(c) = 2 = 0$

$\Rightarrow c = 0 \in (-2, 2)$

b. Here $f(x) = x^3 - 4x$

Here, $f'(x) = 3x^2 - 4$. Which is defined for all values in (0, 2). Hence the given function is differentiable in (0, 2). Again since differentiable function is continuous. So the given function is continuous on [0, 2].

Now, $f(a) = f(0) = o3 - 4\times0 = 0$

and $f(b) = f(2) = 2^3 - 4\times2 = 8 - 8 = 0$

$\therefore$ f(a) = f(b)

So by Rolle's theorem there exits $c \in (a, b)$ s.t. $f'(c) = 0$

Now, $f'(x) = 3x^2 - 4$

$f'(c) = 3x^2 - 4$

$f'(c) = 0 \Rightarrow 3c^2 - 4 = 0 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm\sqrt{\frac{4}{3}}$

$\therefore \quad c = \sqrt{\frac{4}{3}} \in (0, 2).$

c. Since we know sine function is continuous in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and differentiable in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now, $f(a) = f\left(\frac{\pi}{2}\right) = \sin 2 \times \frac{\pi}{2} = \sin(-\pi) = -\sin\pi = 0$

and $f(b) = f\left(\frac{\pi}{2}\right) = \sin 2\frac{\pi}{2} = \sin\pi = 0$

Thus, by Rolle's theorem there exists $c \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in which $f'(c) = 0$

$\Rightarrow 2\cos 2c = 0$

$\Rightarrow \cos 2c = 0 = \cos\frac{\pi}{2}$

$\Rightarrow 2c = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

d. Since cosine function is continuous in $[-\pi,\pi]$ and is also differentiable $(-\pi,\pi)$
Now, $f(a) = f(-\pi) = \cos 2(-\pi) = \cos 2\pi = 1$
$f(b) = f(\pi) = \cos 2\pi = 1$
Hence by Rolle's Theorem $c \in (-\pi, \pi)$ such that $f'(c) = l$
$\Rightarrow$ $f'(c) = -\sin 2c = 0$
$\Rightarrow$ $\sin 2c = \sin 0$
$\Rightarrow$ $2c = 0$
$\Rightarrow$ $c = 0 \in (-\pi, \pi)$

e. Since given function is continuous in $[-5, 5]$ and also gives real values for all values in $(-5, 5)$ and hence differentiable.
Now, $f(-5) = \sqrt{25 - 25} = 0$, $f(b) = f(5) = \sqrt{25 - 25} = 0$
$\therefore$ $f(a) = f(b)$
Now, by Rolle's theorem, $\ni c \in (-5, 5)$ s.t. $f'(c) = 0$

$\Rightarrow$ $f'(c) = \dfrac{-2x}{2\sqrt{25 - x^2}} = 0$

$\Rightarrow$ $-2x = 0 \Rightarrow x = 0 \in (-5, 5)$

f. $f(x) = (x - 1)(x - 2)(x - 3)$
$= (x - 1)(x^2 - 5x + 6) = x^3 - x^2 - 5x^2 + 5x + 6x - 6 = x^3 - 6x^2 + 11x - 6$
Since polynomial function is continuous. So given function is continuous in [1, 3] and is also defined for all values in (1, 3) so is differentiable.
Now, $f(1) = f(3) = 0$
Now by Rolle's theorem,
$f'(c) = 3c^2 - 12c + 11 = 0$

$\Rightarrow$ $c = \dfrac{6 \pm \sqrt{3}}{3}$

$\therefore$ $c = \dfrac{6 + \sqrt{3}}{3} \in (1, 3)$

g. Since the sum of two continuous function is continuous. So given function is continuous in $[0, 2\pi]$.
The given function defined all values in $[0, 2\pi]$. Hence differentiable in $(0, 2\pi)$
Again, $f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$
and $f(2\pi) = \sin 2\pi + \cos 2\pi = 0+1 = 1$
Now by Rolle's theorem $\ni c \in (0, 2\pi)$ s.t. $f'(c) = 0$
$\Rightarrow$ $f'(c) = \cos c - \sin c = 0$
$\Rightarrow$ $\text{Cos}\, c = \sin c$

$\Rightarrow$ $\tan c = 1 = \tan\frac{\pi}{4}$

$\therefore$ $c = \frac{\pi}{4}$

Which is positive for $c = \frac{\pi}{4} \in (0, 2\pi)$

2. By using Rolle's theorem find a point on each of the curves given by the following where the tangent is parallel to x-axis.

   a. $f(x) = 6x - x^2$ in [0, 6]   b. $f(x) = 2x^2 - 4x$ in [0, 2]

Solution:

a. Being a polynomial function continuous in [0,6]. So it gives real values in (0, 6). Therefore the given function is differentiable in (0, 6)

$f(a) = 6\times0 - 0^2 = 0$

and $f(b) = 6\times6 - 6^2 = 0$

$\therefore$ $f(a) = f(b)$

So by Rolle's theorem, $\exists\ c \in (0, 6)$ s.t. $f'(c) = 6 - 2c$

$\Rightarrow$ $f'(c) = 6 - 2c = 0$

$\Rightarrow$ $2c = 6 \Rightarrow c = 3$

Thus the tngent to the given curve is parallel to x–axis at x = 3.

$\therefore$ If x = 3 . Then $y = 6\times3 - 3^2 = 18 - 9 = 9$

Therefore the required points is (3, 9)

b. Given $f(x) = 2x^2 - 4x$

Since the polynomial function is continuous in [0, 2]

Also the given function gives definite values for all values in (0, 2). Hence the function is also differentiable in (0, 2)

Now, $f(0) = 2\times0^2 - 4\times0 = 0$

$f(2) = 2\times2^2 - 4\times2 = 8 - 8 = 0$

$\therefore$ $f(0) = f(2)$

Here, all the conditioned of Rolle's theorem is satisfied os $\ni$ $c\in(0, 2)$ s.t. $f'(c) = 0$

$\Rightarrow$ $f'(c) = 4c - 4 = 0$

$\Rightarrow$ $4c = 4 \Rightarrow C = 1\in(0, 2)$

Thus, the tangent to the curve $2x^2 - 4x$ is parallel to x–axis at the point x=1.

$\therefore$ When x=1, $y = 2x^2 - 4x = 2\times1 - 4\times1 = -2$

So the required point is (1, –2)

3. Verify the mean value theorem for each of the following function in the given interval.

   a. $f(x) = 3x^2 - 2$ in [2, 3]   b. $f(x) = x^2$ in [1, 2]   c. $f(x) = x(x - 1)(x - 2)$ in $[0, \frac{1}{2}]$

   d. $f(x) = e^x$ in [0, 1]   e. $f(x) = \sqrt{x^2 - 4}$ in [2, 4]

Solution:

a. Since, being the polynomial function $f(x) = 3x^2 - 2$ is continuous in [2, 3]

Also $f(x) = 3x^2 - 2$ is defined for all values in (2, 3). Hence is differentiable in (2, 3).

Again, $f(2) = 3\times2^2 - 2 = 12 - 2 = 10$

and $f(3) = 3\times3^2 - 2 = 27 - 2 = 25$

$\therefore$ $f(2) \neq f(3)$

Hence all the condition of mean value theorem satisfied. So by the theorem $\exists$

$c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(2)}{3 - 2} = \frac{25 - 10}{1} = 15 \text{ ... ... } (*)$$

We have, $f(x) = 3x^2 - 2$ ... ... (**)

$f'(c) = 6c$

$\therefore$ from (*) and (**)

$$6c = 15 \Rightarrow c = \frac{15}{6} \in (2, 3)$$

b. Since quadratic function is continuous for all values of x so the given function is continuous in [1, 2]

$f(x) = x^2$ have a definite values in [1,2] so is differentiable in (1, 2)

Now, $f(1) = 1$ and $f(2) = 4$

$\therefore$ $f(1) \neq f(2)$

All condition of M.V.T satisfied so $\exists$ $c \in (a, b)$

Such that, $f'(c) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{4 - 1}{1} = 3$ ... ... (*)

Again, $f'(x) = 2x$ $\therefore$ $f'(c) = 2c$ ... ... (**)

from (*) and (**)

$$2c = 3 \Rightarrow c = \frac{3}{2} = 1.5 \in (1, 2)$$

c. $f(x) = x(x^2 - 3x + 2) = x^3 - 3x^2 + 2x$

Since being polynomial function is continuous so the given function is continuous in [0, ½]. All gives definite values in (0, ½). So is differentiable in (0, ½)

Now, $f(0) = 0$ and $f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 2\times\frac{1}{2} = \frac{1}{8} - \frac{3}{4} + 1 = \frac{1 - 6 + 8}{8} = \frac{3}{8}$

$\therefore$ $f(0) \neq f\left(\frac{1}{2}\right)$

Therefore M.V. theorem applicable, so $\exists$ $c \in (a, b)$ such that f'(c)

$$= \frac{f(b) - f(a)}{b - a} = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0} = \frac{\frac{3}{8} - 0}{\frac{1}{2}} = \frac{3}{8} \times 2 = \frac{3}{4}$$

Also, $f'(x) = 3x^2 - 6x + 2$

$f'(c) = 3c^2 - 6c + 2$ ... ... (**)

from (*) and (**) we have,

$$3c^2 - 6c + 2 = \frac{3}{4}$$

$\Rightarrow$ $12c^2 - 24c + 8 = 3$

$\Rightarrow$ $12c^2 - 24c + 5 = 0$

$$\Rightarrow c = \frac{24 \pm \sqrt{576 - 240}}{24} = \frac{24 \pm \sqrt{16 \times 21}}{24} = \frac{4(6 \pm \sqrt{21})}{24} = \frac{6 \pm 4.58}{6} \text{ (Appro.)}$$

Taking positive sing,

$$c = \frac{1.42}{6} = 0.23 \in (0, ½)$$

d. Since exponential function is continuous

$\therefore$ The given function is continuous in [0, 1]

Also differential in (0, 1)

Now, $f(0) = e^0 = 1$

$f(1) = e^1 = 2.718$ (Approx)

$\therefore$ f(0) $\neq$ f(1)

Now by M.V. theorem $\ni$ c$\in$(0,b) s.t.

$$f'(c) = \frac{f(b) - f(a)}{b -} = \frac{f(1) - f(0)}{1 - 0} = \frac{2.718 - 1}{1} = 1.718 \text{ ... ... } (*)$$

and $f'(x) = e^x$

$f'(c) = e^c$ ... ... (**)

$\therefore$ from (*) and (**)

$e^c = 1.718$

Taking ln on both sides,

c lne = ln (1.718)

$\therefore$ c = 0.236 $\in$ (0, 1)

e. Given function is continuous in [2, 4]

Also differentiable in (2, 4)

Now, f(2) = 0

And $f(4) = 2\sqrt{3}$

$\therefore$ f(2) $\neq$ f(4)

So by MVT. These exists c$\in$ [2, 4] such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \frac{f(4) - f(2)}{4 - 2}$$

$$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \frac{2\sqrt{3} - 0}{2} = \sqrt{3}$$

$\Rightarrow$ $c = \sqrt{c^2 - 4} \times \sqrt{3}$

$\Rightarrow$ $c^2 = (c^2 - 4)3$

$\Rightarrow$ $c^2 = 3c^2 - 12$

$\Rightarrow$ $2c^2 = 12$

$\Rightarrow$ $c^2 = 6$

$\Rightarrow$ $c = \sqrt{6}$

= 2.44 $\in$ [2, 4]

Hence, MVT is verified.

4. Show that the mean value theorem is not applicable to the function $f(x) = \dfrac{1}{x}$ in (–1, 1).

Solution:

Given function $f(x) = \dfrac{1}{x}$ in (–1, 1) since the given function is not defined at x = 0 $\in$ (–1, 1). Hence the function is not differentiable at x = 0$\in$(–1, 1). To satisfy the M.V.T, f(x) should be differentiable for all x$\in$(–1, 1)

Moreover the graph of $f(x) = \dfrac{1}{x}$ is,

Here, we cannot draw a tangent at x = 0. So, the function is not differentiable. Hence M.V. theorem for the underlying function in the defined interval is not applicable.

5. Find the points on the curve $f(x) = (x - 3)^2$ where the tangent is parallel to the chord joining the points (3, 1) and (4, 4).

Solution:

Let, the chord joining the ending points be, (a, f(a)) = (3, 0) and (b, f(b)) = (4, 1)

Since $f(x) = (x - 2)^2$ is continuous in [3, 4]

Also exist for all values in (3, 4) and hence differentiable.
Also, $f(a) \neq f(b)$
By M.V. theorem $\ni c \in (3, 4)$ s.l.

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Now the slope of the chord joining (3, 1), (4, 4) is, $\dfrac{f(b) - f(a)}{b - a} = \dfrac{1 - 0}{4 - 3} = 1$ ... ... (*)

Since, $f(x) = (x - 3)^2$

$f'(x) = \dfrac{dy}{dx} = 2(x - 3)$

$f'(c) = 2(c - 3)$ ... ... (**)

from (*) and (**)

$2c - 6 = 1 \Rightarrow 2c = 7 \Rightarrow c = \dfrac{7}{2} \in (3, 4)$

If $x = \dfrac{7}{2}$ then $y = \left(\dfrac{7}{2} - 3\right)^2 = \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

$\therefore$ The tangent at $\left(\dfrac{7}{2}, \dfrac{1}{4}\right)$ is parallel to be chord joining (3, 0) and (4, 1).

6. Find the point on the curve $f(x) = x^3 - x^2 + 2$ where the tangent is parallel to the line joining the points (1, 2) and (3, 20).

Solution:

Since the given function is continuous on [1, 3] being polynomial and $f'(x) = 3x^2 - 2x$ exist for all (1, 3) and $f(a) = 2$ and $f(b) = 20$

$\therefore$ $f(a) \neq f(b)$

So by M.V. theorem $\ni c \in (1, 3)$ s.t.

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Now the slope of chord joining (1, 2) and (3, 20)

i.e, $\dfrac{f(b) - f(a)}{b - a}$ is, $\dfrac{20 - 2}{3 - 1} = \dfrac{18}{2} = 9$ ... ... (*)

And, $f'(c) = 3c^2 - 2c$ ... ... (**)

From (*) and (**)

$3c^2 - 2c = 9 \Rightarrow 3c^2 - 2c - 9 = 0$

Solving $c = \dfrac{1 \pm 2\sqrt{7}}{3} = \dfrac{1 + 2 \times 2.64}{3}$ (Appro.)

$= \dfrac{1+5.29}{3}$ (Appro.) (Taking positive sign) $= \dfrac{6.29}{3} \in (1, 3)$

If $x = 2.1$ then, $y = (2.1)^3 - (2.1)^2 + 2 = 9.26 - 4.41 + 2 = 6.85$

$\therefore$ The required point is (2.1, 6.85)

EXERCISE 15.6

By using L Hospital's rule, evaluate:

1. a. $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x^2 - 9}$ b. $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a}$ c. $\lim\limits_{x \to 0} \dfrac{3x - \sin x}{x}$ d. $\lim\limits_{x \to \infty} \dfrac{5x^2 + 4x - 3}{2x^2 - 3x + 5}$

e. $\lim\limits_{x \to 0} \dfrac{e^x + e^{-x} - 2\cos x}{\sin^2 x}$ f. $\lim\limits_{x \to 0} \dfrac{x - \sin x}{\tan^3 x}$ g. $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x^3}$

h. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan 5x}{\tan x}$ i. $\lim\limits_{x \to 0} \dfrac{(e^x - 1)\tan x}{x^2}$ j. $\lim\limits_{x \to 0^+} \dfrac{\ln \tan x}{\ln x}$

Solution:

a. Here, $\lim\limits_{x\to 3}\dfrac{x^3-27}{x^2-9} = \lim\limits_{x\to 3}\dfrac{3x^2}{2x}$ (Differentiate w.r.to x) $= \dfrac{3\times 3^2}{2\times 3} = \dfrac{9}{2}$

b. $\lim\limits_{x\to a}\dfrac{x^2-a^n}{x-a} = \lim\limits_{x\to a}\dfrac{nx^{n-1}}{1}$ [Differentiate w.r.to x] $= na^{n-1}$

c. $\lim\limits_{x\to 0}\dfrac{3x-\sin x}{x} = \dfrac{3-\cos x}{1}$ [Differentiate w.r. to x] $= \dfrac{3-\cos 0}{1} = \dfrac{3-1}{1} = 2$

d. $\lim\limits_{x\to\infty}\dfrac{5x^2+4x-3}{2x^2-3x+5} = \left(\dfrac{\infty}{\infty}\text{ form}\right)$

$= \lim\limits_{x\to\infty}\dfrac{10x+4}{4x-3}$ (Differentiate w.r.to x] $\left(\dfrac{\infty}{\infty}\text{ form}\right) = \dfrac{10}{4} = \dfrac{5}{2}$

e. $\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}-2\cos x}{\sin^2 c} = \lim\limits_{x\to 0}\dfrac{e^x-e^{-x}+2\sin x}{2\sin x\cos x}$

$= \lim\limits_{x\to 0}\dfrac{e^x+e^{-x}+2\cos x}{2\cos 2x} = \dfrac{1+1+2}{2} = \dfrac{4}{2} = 2$

f. Since, $\lim\limits_{x\to 0}\dfrac{x-\sin x}{\tan^3 x}$

$= \lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3\times\left(\dfrac{\tan x}{x}\right)^3} = \lim\limits_{x\to 0}\left(\dfrac{x}{\tan x}\right)^3\times\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$

$= 1\times\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3} = \lim\limits_{x\to 0}\dfrac{1-\cos x}{3x^2}$ $(\dfrac{0}{0}$ form) [Differentiate w.r. to x]

$= \lim\limits_{x\to 0}\dfrac{0+\sin x}{6x}$ [Differentiate w.r.to x] $= \lim\limits_{x\to 0}\left[\dfrac{\sin x}{x}\right]\times\dfrac{1}{6} = 1\times\dfrac{1}{6} = \dfrac{1}{6}$

g. Here, $\lim\limits_{x\to 0}\dfrac{x-\tan x}{x^3} = (\dfrac{0}{0}$ form)

By L–Hospital rule, differentiate numerator and denominator w.r.to x, we get,

$\lim\limits_{x\to 0}\dfrac{1-\sec^2 x}{3x^2}$ $(\dfrac{0}{0}$ form) [Differentiate w.r. to x]

$= \lim\limits_{x\to 0}\dfrac{-\tan^2 x}{3x^2} = -\dfrac{1}{3}\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^2 = -\dfrac{1}{3}\times 1 = -\dfrac{1}{3}$

h. Here, $\lim\limits_{x\to\pi/2}\dfrac{\tan 5x}{\tan x}\left(\dfrac{\infty}{\infty}\text{ form}\right)$

Using L-Hospital rule,

$\lim\limits_{x\to\pi/2}\dfrac{5\sec^2 5x}{\sec^2 x}$ [Differentiate w.r. to x]

$= \lim\limits_{x\to\pi/2}\dfrac{5\cos^2 x}{\cos^2 5x} = \lim\limits_{x\to\pi/2}\dfrac{-5\times 2\cos x\sin x}{-5\,2\cos 5x\sin 5x}$ [Differentiate w.r. to x]

$= \lim\limits_{x\to\pi/2}\dfrac{\sin 2x}{\sin 10x} = \lim\limits_{x\to\pi/2}\dfrac{2\cos 2x}{10\cos 10x} = \dfrac{1}{5}\left(\dfrac{\cos 2\pi}{\cos 5\pi}\right) = \dfrac{1}{5}\left(\dfrac{-1}{-1}\right) = \dfrac{1}{5}$

i. Here, $\lim\limits_{x\to 0}\dfrac{(e^x-1)\tan x}{x^2} = \lim\limits_{x\to 0}\dfrac{(e^x-1)}{x}\times\lim\limits_{x\to 0}\dfrac{\tan x}{x}$

$= \lim\limits_{x\to 0} \dfrac{e^x - 1}{x} \times 1$ [% form] $= \dfrac{e^x}{1}$ [Differentiate w.r.t. to x] $= e^0 = 1$

j. Since, $\lim\limits_{x\to 0} \dfrac{\log \tan x}{\log x} \left(\dfrac{-\infty}{\infty} \text{ form}\right)$

By using L–Hospital rule

$$= \lim_{x\to 0} \frac{\frac{1}{\tan x} \times \sec^2 x}{(1/x)} = \lim_{x\to 0} \frac{\text{Sec}^2 x}{\tan x}$$

$$= \lim_{x\to 0} \frac{\sec^2 x + x \times 2\sec x.\sec x\ .\ \tan x}{\sec^2 x}$$

$$= \frac{1 + 0}{1} = \frac{1}{1}$$

$= 1$

2. a. $\lim\limits_{x\to\infty} \dfrac{x^4}{e^x}$

b. $\lim\limits_{x\to+\infty} \dfrac{\ln (x^2 + 1)}{\ln (x^3 + 1)}$

c. $\lim\limits_{x\to 0^+} x^x$

d. $\lim\limits_{x\to 0^+} \sin x \ln x^2$

e. $\lim\limits_{x\to 0} \left[\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right]$

Solution

a. Since, $\lim\limits_{x\to\infty} \dfrac{x^4}{e^\infty} \left(\dfrac{\infty}{\infty} \text{ form}\right)$

Using L–Hospital rule,

$= \lim\limits_{x\to\infty} \dfrac{4x^3}{e^x}$ [Differentiate num and dere. w.r.to x]

Again $\dfrac{\infty}{\infty}$ form, using L–Hospital rule,

$= \lim\limits_{x\to\infty} \dfrac{12x^2}{e^x}$ [Differentiate w.r. to x]

Again $\dfrac{\infty}{\infty}$ form, using L–Hospital rule,

$$= \lim_{x\to\infty} \frac{24x}{e^x}$$

Again $\dfrac{\infty}{\infty}$ form, using L–Hospital rule,

$$= \lim_{x\to\infty} \frac{24}{e^x} = \frac{24}{e^\infty} = \frac{24}{\infty} = 0$$

b. Since, $\lim\limits_{x\to\infty} \dfrac{\log (x^2 + 1)}{\log (x^3 + 1)} \left(\dfrac{\infty}{\infty} \text{ form}\right)$

Using L–Hospital rule,

$= \lim\limits_{x\to\infty} \dfrac{2x(x^3 + 1)}{3x^2(x^2 + 1)}$ [Differentiate w.r. to x] $= \dfrac{2}{3} \lim\limits_{x\to\infty} \dfrac{x^3 + 1}{x(x^2 + 1)}$

Again $\dfrac{\infty}{\infty}$ form, using L–Hospital rule,

$$= \frac{2}{3}\ \frac{3x^2}{3x^2+1} = \frac{2}{3} \lim_{x\to\infty} \left(\frac{3x^2+1}{3x^2+1} - \frac{1}{3x^2+1}\right)$$

$$= \frac{2}{3} \lim_{x\to\infty} \left(1 - \frac{1}{3x^2+1}\right) = \frac{2}{3}(1-0) = \frac{2}{3}$$

c. Since, $\lim\limits_{x\to 0} x^x$ (0^0 forms)

Using L–Hospital rule, for this let,

$y = x^x \Rightarrow \ln y = x\ln x$

Taking limit as x tends to 0

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} x \ln x$$

or, $\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{\log x}{\frac{1}{x}} = \lim\limits_{x\to 0} \dfrac{1/x}{-1/x^2}$ [Differentiate w.r. to x] $= \lim\limits_{x\to 0} -x$

$$\therefore \lim_{x\to 0} \ln y = 0$$

$$\therefore \lim_{x\to 0} y = e^0$$

$$\therefore \lim_{x\to 0} x^x = 1$$

d. Since, $\lim\limits_{x\to 0} \sin x \ln x^2$ ($0.\infty$ forms)

$= \lim\limits_{x\to 0} \dfrac{\log x^2}{\operatorname{cosec} x} = \lim\limits_{x\to 0} \dfrac{2x/x^2}{-\operatorname{cosec}x.\cot x}$ [Differentiate w.r. to x]

$$= \lim_{x\to 0} \frac{-2}{x \operatorname{cosec}x \,.\, \cot x}$$

$$= \lim_{x\to 0} \frac{-2}{x \times \frac{1}{\sin x} \times \frac{\cos x}{\sin x}}$$

$$= \lim_{x\to 0} \frac{-2\sin^2 x}{x\cos x} \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-4 \sin x.\cos x}{\cos x - x\sin x} = \frac{-4\times 0}{1} = \frac{0}{1} = 0$$

e. We have,

$$\lim_{x\to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right] \ (\infty - \infty \text{ forms})$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{2\sin x\cos x - 2x}{2x\sin^2 x + 2x^2 \sin 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\sin 2x - 2x}{2x\sin^2 x + 2x^2 \sin 2x} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{2\cos 2x - 2}{2\sin^2 x + 4x\sin x\cos x + 4x\sin 2x + 4x^2\cos 2x}$$

$$= \lim_{x\to 0} \frac{2\cos 2x - 2}{2\sin^2 x + 6x\sin 2x + 4x^2\cos 2x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-4\sin 2x}{2\sin 2x + 6\sin 2x + 12\cos 2x + 8x\cos 2x - 8x^2\sin 2x}$$

$$= \lim_{x\to 0} \frac{-4\sin 2x}{8\sin 2x + 12\cos 2x + 8x\cos 2x - 8x^2\sin 2x} \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-8\cos 2x}{16\cos 2x - 24\sin 2x + 8\cos 2x - 16x\sin 2x - 16x\sin 2x - 16x^2\sin 2x}$$

$$= -\frac{8}{24} = -\frac{1}{3}$$

EXERCISE 15.7

1. Find Δy and dy of the following
   a. $y = x^3 + 3$ for $x = 2$ and $\Delta x = 0.1$ b. $y = \sqrt{x}$, for $x = 4$ and $\Delta x = 0.41$
   c. $y = x^2 + 3x$ when $x = 2$ and $\Delta x = 0.2$

Solution:

a. Since we know, $\Delta y = f(x + \Delta x) - f(x)$ and $dy = f'(x)\,dx$.

$\therefore$ $dy = 3x^2\,dx = 3\times 2^2 \times 0.1 = 12\times 0.1 = 1.2$

Again, $\Delta y = f(x + \Delta x) - f(x) = f(2 + 0.1) - f(2) = f(2.1) - f(2) = (2.1)^3 + 3 - (2^3 + 3)$
$= 9.261 + 3 - 11 = 12.261 - 11 = 1.261$

b. Now, $dy = f'(x)\,dx$

$= \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2\sqrt{4}} \times 0.41 = \frac{1}{4} \times 0.41 = 0.1025$

and, $\Delta y = f(x + \Delta x) - f(x) = f(4+0.41) - f(4) = \sqrt{4.41} - \sqrt{4} = 2.1 - 2 = 0.1$

c. Since, $dy = f'(x)\,dx = (2x + 3)dx = (2\times 2+3) \times 0.2 = -7 \times 0.2 = 1.4$

and, $\Delta y = f(x + \Delta x) - f(x) = f(2 + 0.2) - f(2) = f(2.2) - f(2) = (2.2)^2 + 3\times 2.2 - (2^2 + 3\times 2)$
$= 4.84 + 6.6 - 10 = 11.44 - 10 = 1.44$

2. Find the approximate change in the volume of a cube of side xm caused by increasing the side by 2%.

Solution:

Here, side of cube = xm

$\therefore$ $dx = 2\%$ of $x = \frac{2}{100}x = 0.02x$

Now, the volume of cube having side x,
$v = x^3$
Now the change in volume,
$dv = 3x^2dx = 3x^2(0.02x) = 0.06x^3$

3. If $y = x^4 - 10$ and if x changes form 2 to 1.99. What is the exact and approximates change in y?

Solution:

Since, $x = 2$ and $x + \Delta x = 1.99 \Rightarrow \Delta x = 1.99 - 2 = -0.01$

Now, $\frac{dy}{dx} = 4x^3$

$\Rightarrow dy = 4x^3dx$

At, $x = 2$, $dy = 4\times(2)^3 \times (-0.01) = -0.32$

Again, if $x = 2$, then $y = x^4 - 10 = 2^4 - 10 = 6$

$y + \Delta y = (x + \Delta x)^4 - 10$

or, $\Delta y = (x + \Delta x)^4 - 10 - x^4 + 10 = (2 - 0.01)^4 - 2^4$ (For $x = 2$)
$= -0.317$

4. If the radius of a sphere changes from 3cm to 3.01cm. Find the approximate increase in its volume.

Solution:

Let, x = 3cm then r + Δr = 3.01

$\Rightarrow$ Δr = 3.01 – 3

$\Rightarrow$ Δr = 0.01

Since volume of sphere,

$r = \frac{4}{3}\pi r^3$

$\Rightarrow dv = \frac{4}{3}\pi \cdot 3r^2 dr$

$\Rightarrow dv = \frac{4}{3}\pi \times 3 \times 3^2 \times 0.01$

$= 0.36\pi$

5. Find the approximate increase in the surface area of a cube of the edge from 10 to 10.01. Calculate percent error in the surface area.

Solution:

Let, a = 10 then a + Δa = 10.01

Δa = 10.01 – 10 = 0.01

Since surface area of cube is

$A = 6a^2 = 12a\ da = 12 \times 10 \times 0.01 = 120 \times 0.01 = 1.2$

Again for percent error

Since we know that percentage error $= \frac{\text{Change}}{\text{orginal}} \times 100 = \frac{6(10.01-10)^2}{6\times 10^2} \times 100$

$= (0.01)^2 = 0.0001\%$

6. A circular cupper plate is heated so that its radious increases from 5cm to 5.06 cm. Find the approximate increase in area and also the actual increase in area.

Solution:

Let, r = 5. Then r + Δr = 5.06

Δr = 5.06 – 5 = 0.06

Now, $A = \pi r^2$

$dA = 2\pi r\ dr = 2\pi \times 5 \times 0.06 = 0.6\pi$

Again, actual increase in area,

$= \pi(5.06)^2 - \pi(5)^2 = \pi(25.603 - 25) = \pi \times 0.603 = 0.603\pi$

7. The radious of sphere is found by measurement to be 209cm with possible error of 0.02 of a centimeter. Find the consequent error in the surface.

Solution:

Here, r = 20cm and Δr = 0.02

Then, $A = 4\pi r^2 = 4 \times \frac{22}{7} \times (20)^2 = \frac{3500}{7} = 5028.58$

Now since, $\frac{\Delta A}{A} = 2\frac{\Delta r}{r}$

$\Delta A = 2 \times \frac{0.02}{20} \times 5028.58 = 10.05$ sqcm

■

CHAPTER 16

ANTI-DERIVATIVES

EXERCISE 16.1

1. Evaluate

a. $\int \frac{dx}{4x^2 + 9}$ b. $\int \frac{xdx}{x^4 + 3}$ c. $\int \frac{(2x + 3)}{4x^2 + 1} dx$ d. $\int \frac{x^2 \, dx}{x^6 - 9}$

e. $\int \frac{dx}{x^2 + 6x + 8}$ f. $\int \frac{\cos x \, dx}{\sin^2 x + 4 \sin x + 5}$ g. $\int \frac{xdx}{x^4 - x^2 - 1}$

h. $\int \frac{e^x \, dx}{e^{2x} + 2e^x + 5}$ i. $\int \frac{dx}{9x^2 + 12x + 13}$ j. $\int \frac{dx}{1 - 6x - 9x^2}$

k. $\int \frac{3x + 5}{x^2 + 4x + 20} dx$ l. $\int \frac{2x + 2}{3 + 2x - x^2} dx$ m. $\int \frac{6x + 2}{9x^2 + 6x + 26} dx$

Solution

a. $\int \frac{dx}{4x^2 + 9}$

$\int \frac{dx}{(2x)^2 + 3^2}$

Put y = 2x

$\frac{dy}{2} = dx$

Now, $= \frac{1}{2} \int \frac{dy}{y^2 + 3^2} = \frac{1}{2} \frac{1}{3} \tan^{-1} \frac{y}{3} + c = \frac{1}{6} \tan^{-1} \frac{2x}{3} + c$

b. $\int \frac{xdx}{x^4 + 3} = \int \frac{xdx}{(x^2)^2 \ (\sqrt{3})^2}$

Put $y = x^2$

$\frac{dy}{2} = xdx,$

Now, given integral reduces into

$= \frac{1}{2} \int \frac{dy}{y^2 + (\sqrt{3})^2} = \frac{1}{2} \cdot \frac{1}{\sqrt{3}} \tan^{-1} \frac{y}{\sqrt{3}} + c = \frac{1}{2\sqrt{3}} \tan^{-1} \frac{x^2}{\sqrt{3}} + c$

c. $\int \frac{2x + 3}{4x^2 + 1} dx$

$= \int \frac{2x}{4x^2 + 1} + \frac{3}{4x^2 + 1} dx$

$$= \frac{1}{4}\int \frac{8x}{4x^2+1}\,dx + 3\int \frac{1}{4(x^2+\frac{1}{4})}\,dx$$

$$= \frac{1}{4}\ln(4x^2+1) + \frac{3}{4} \times \frac{1}{\frac{1}{2}}\tan^{-1}\frac{x}{\frac{1}{2}} + C$$

$$= \frac{1}{4}\ln(4x^2+1) + \frac{3}{4}\tan^{-1}2x + C$$

d. $\int \frac{x^2dx}{x^6-9} = \int \frac{x^2dx}{(x^3)^2-3^2}$

Put $y = x^3$

$\frac{dy}{3} = x^2\,dx$

Now, $\frac{1}{3}\int \frac{dy}{y^2-3^2} = \frac{1}{3}\frac{1}{2.3}\ln\frac{y-3}{y+3} + c = \frac{1}{18}\ln\frac{x^3-3}{x^3+3} + c$

e. $\int \frac{dx}{x^2+6x+8} = \int \frac{dx}{x^2+2.3x+9-9+8} = \int \frac{dx}{(x+3)^2-1^2}$

Put $y = x + 3$

$dy = dx$

Now, $\int \frac{dy}{y^2-1^2} = \frac{1}{2.1}\ln\frac{y-1}{y+1} + c = \frac{1}{2}\ln\frac{x+3-1}{x+3+1} + c = \frac{1}{2}\ln\frac{x+2}{x+4} + c$

f. $\int \frac{\cos x dx}{\sin^2x+4\sin x+5}$

Put $y = \sin x \Rightarrow dy = \cos x.dx$

$$= \int \frac{dy}{y^2+4y+5} = \int \frac{dy}{y^2+2.2y+4-4+5} = \int \frac{dy}{(y+2)^2+1^2} = \tan^{-1}\frac{y+2}{1} + c$$

$$= \tan^{-1}(\sin x + 2) + c$$

g. $\int \frac{xdx}{x^4-x^2-1}$

$\int \frac{xdx}{(x^2)^2-x^2-1}$

Put $y = x^2$

$dy = 2x.dx$

$\frac{dy}{2} = xdx$

Now, $= \frac{1}{2}\int \frac{dy}{y^2-y-1}$

$$= \frac{1}{2}\int \frac{dy}{y^2-2.y.\frac{1}{2}+\frac{1}{4}-\frac{1}{4}-1} = \frac{1}{2}\int \frac{dy}{\left(y-\frac{1}{2}\right)^2-\frac{5}{4}}$$

$$= \frac{1}{2}\int \frac{dy}{\left(y - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2} = \frac{1}{2}\,\frac{1}{2.\frac{\sqrt{5}}{2}} \ln \frac{y - \frac{1}{2} - \frac{\sqrt{5}}{2}}{y - \frac{1}{2} + \frac{\sqrt{5}}{2}} + c = \frac{1}{2\sqrt{5}} \ln \frac{2x^2 - 1 - \sqrt{5}}{2x^2 - 1 + \sqrt{5}} + c$$

$$= \frac{1}{2\sqrt{5}} \ln \frac{2x^2 - 1 - \sqrt{5}}{2x^2 - 1 + \sqrt{5}} + c$$

h. $\int \frac{e^x\,dx}{e^{2x} + 2e^x + 5}$

Put $e^x = y$
$dy = e^x \,.\, dx$

Now, $= \int \frac{dy}{y^2 + 2y + 5} = \int \frac{dy}{y^2 + 2.y.1 + 1 - 1 + 5} = \int \frac{dy}{(y + 1)^2 + (2)^2}$

$$= \int \frac{dy}{(y + 1)^2 + (2)^2} = \frac{1}{2} \tan^{-1} \frac{y + 1}{2} + c = \frac{1}{2} \tan^{-1} \frac{e^x + 1}{2} + c$$

i. $\int \frac{dx}{9x^2 + 12x + 13} = \int \frac{dx}{(3x)^2 + 2.3x\,.\,2 + 4 - 4 + 13} = \int \frac{dx}{(3x + 2)^2 + 3^2}$

Put $y = 3x + 2 \Rightarrow \frac{dy}{3} = dx$

Now, $\int \frac{dx}{(3x + 2)^2 + 3^2}$

$$= \frac{1}{3}\int \frac{dy}{y^2 + 3^2} = \frac{1}{3}\,.\,\frac{1}{3} \tan^{-1} \frac{y}{3} + c = \frac{1}{9} \tan^{-1} \frac{3x + 2}{3} + c$$

j. $\int \frac{dx}{1 - 6x - 9x^2} = -\int \frac{dx}{(3x)^2 + 6x - 1}$

$$= -\int \frac{dx}{(3x)^2 + 2.3x\,.\,1 + 1 - 1 - 1} = -\int \frac{dx}{(3x + 1)^2 - (\sqrt{2})^2}$$

Put $y = 3x + 1$
$\frac{dy}{3} = dx$

Now, $-\frac{1}{3}\int \frac{dy}{y^2 - (\sqrt{2})^2} = -\frac{1}{3}\,.\,\frac{1}{2.\sqrt{2}} \ln \frac{y - \sqrt{2}}{y + \sqrt{2}} + c = -\frac{1}{6\sqrt{2}} \ln \frac{3x + 1 - \sqrt{2}}{3x + 1 + \sqrt{2}} + C$

k. $\int \frac{3x + 5}{x^2 + 4x + 20}\,dx$

$$= 3\int \frac{x + \frac{5}{3}}{x^2 + 4x + 20}\,dx$$

$$= \frac{3}{2}\int \frac{2x + \frac{10}{3}}{x^2 + 4x + 20}\,dx$$

$$= \frac{3}{2}\int \frac{(2x + 4) - \frac{2}{3}}{x^2 + 4x + 20} dx$$

$$= \frac{3}{2}\int \frac{2x + 4}{x^2 + 4x + 20} - \int \frac{dx}{x^2 + 4x + 20}$$

$$= \frac{3}{2}\int \frac{(2x + 4)}{x^2 + 4x + 20} - \int \frac{dx}{(x + 2)^2 + 4^2}$$

$$= \frac{3}{2}\ln(x^2 + 4x + 20) - \frac{1}{4}\tan^{-1}\left(\frac{x + 2}{4}\right) + C$$

l. $\displaystyle\int \frac{(2x + 2)}{(3 + 2x - x^2)} dx$

$$I = \int \frac{(2x + 2)}{(3 + 2x - x^2)} dx = -\int \frac{-2x + 2 - 4}{3 + 2x - x^2} dx$$

$$= -\int \frac{2 - 2x}{3 + 2x - x^2} dx + 4\int \frac{1}{3 + 2x - x^2} dx$$

$$= -\ln(3 + 2x - x^2) + 4\int \frac{1}{(2)^2 - (x - 1)} dx \backslash$$

$$= -\ln(3 + 2x - x^2) + \frac{4}{2.2}\ln\frac{2 + x - 1}{2 - x + 1} + c$$

$$= -\ln(3 + 2x - x^2) + \ln\frac{x + 1}{3 - x} + c = \ln\left(\frac{x + 1}{3 - x}\right) - \ln(3 + 2x - x^2) + c$$

m. $\displaystyle\int \frac{6x + 2}{9x^2 + 6x + 26} dx = \frac{1}{3}\int \frac{18x + 6}{9x^2 + 6x + 26} dx = \frac{1}{3}\ln(9x^2 + 6x + 26) + c$

2. a. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 4}}$ b. $\displaystyle\int \frac{dx}{\sqrt{x^2 + x - 2}}$ c. $\displaystyle\int \frac{dx}{\sqrt{2x^2 + 3x + 4}}$

d. $\displaystyle\int \frac{dx}{\sqrt{2ax + x^2}}$ e. $\displaystyle\int \frac{dx}{\sqrt{5 - x + x^2}}$ f. $\displaystyle\int \frac{dx}{\sqrt{6 + x - x^2}}$

g. $\displaystyle\int \frac{dx}{\sqrt{2 - 2x - x^2}}$ h. $\displaystyle\int \frac{dx}{\sqrt{9x^2 + 6x + 10}}$ i. $\displaystyle\int \frac{x\,dx}{\sqrt{a^4 + x^4}}$

j. $\displaystyle\int \frac{x\,dx}{\sqrt{x^4 + 2x^2 + 10}}$ k. $\displaystyle\int \frac{2x + 3}{\sqrt{x^2 + 4x + 20}} dx$ l. $\displaystyle\int \frac{x - 2}{\sqrt{2x^2 - 3x + 5}} dx$

m. $\displaystyle\int \frac{x}{\sqrt{7 + 6x - x^2}} dx$ n. $\displaystyle\int \frac{dx}{\sqrt{(x + a)(x + b)}}$ o. $\displaystyle\int \frac{dx}{(11 + x)\sqrt{2 + x}}$

p. $\displaystyle\int \frac{dx}{(4x + 3)\sqrt{x + 3}}$ q. $\displaystyle\int \sqrt{\frac{1 + x}{1 - x}} dx$

Solution:

a. $\displaystyle\int\frac{dx}{\sqrt{x^2-4}}=\int\frac{dx}{\sqrt{x^2-2^2}}=\ln\left(x+\sqrt{x^2-4}\right)+c$

b. $\displaystyle\int\frac{dx}{\sqrt{x^2+x-2}}=\int\frac{dx}{\sqrt{x^2+2x\,.\,\frac{1}{2}+\frac{1}{4}-\frac{1}{4}-2}}$

$\displaystyle=\int\frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}}=\ln\left[\left(\frac{2x+1}{2}\right)+\sqrt{x^2+x-2}\right]+c$

c. $\displaystyle\int\frac{dx}{\sqrt{2x^2+3x+4}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{x^2+\frac{3}{2}x+2}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{x^2+2\,.\,\frac{3}{4}\,.\,x+\frac{9}{16}-\frac{9}{16}+2}}$

$\displaystyle=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{\sqrt{23}}{4}\right)}}^{2}=\frac{1}{\sqrt{2}}\ln\left(x+\frac{3}{4}+\sqrt{x^2+\frac{3}{2}x+2}\right)+c$

d. $\displaystyle\int\frac{dx}{\sqrt{2ax+x^2}}=\int\frac{dx}{\sqrt{a^2+2ax+x^2-a^2}}=\int\frac{dx}{\sqrt{(a+x)^2-a^2}}=\ln\left(a+x+\sqrt{2ax+x^2}\right)+c$

e. $\displaystyle\int\frac{dx}{\sqrt{5-x+x^2}}=\int\frac{dx}{\sqrt{x^2-x+5}}=\int\frac{dx}{\sqrt{x^2-2.x\,.\,\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+5}}$

$\displaystyle=\int\frac{dx}{\sqrt{x^2-2.x\,.\,\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+5}}=\int\frac{dx}{\sqrt{x-\frac{1}{2}+\left(\frac{\sqrt{19}}{2}\right)^2}}$

$\displaystyle=\int\frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{19}}{2}\right)^2}}=\ln\left(\frac{2x-1}{2}+\sqrt{x^2-x+5}\right)+c$

f. $\displaystyle\int\frac{dx}{\sqrt{-(x^2-x-6)}}=\int\frac{dx}{\sqrt{-(x^2-2.x.\,\frac{1}{2}+\frac{1}{4}+\frac{1}{4}-6)}}$

$\displaystyle=\int\frac{dx}{\sqrt{-\left[x-\frac{1}{2}\right]^2-\frac{25}{4}}}$

$\displaystyle=\int\frac{dx}{\sqrt{(5/2)^2-(x-\frac{1}{2})^2}}$

$\displaystyle=\sin^{-1}\frac{\left(x-\frac{1}{2}\right)}{\left(\frac{5}{2}\right)}+c$

$$= \sin^{-1}\left(\frac{2x-1}{5}\right) + c$$

g. $\displaystyle\int \frac{dx}{\sqrt{2-2x-x^2}} = \int \frac{dx}{\sqrt{-(x^2+2x-2)}} = \int \frac{dx}{\sqrt{-(x^2+2x-2)}} = \int \frac{dx}{\sqrt{-(x^2+2x+1-1-2)}}$

$$= \int \frac{dx}{\sqrt{-\{(x+1)^2-(\sqrt{3})^2\}}} = \int \frac{dx}{\sqrt{(\sqrt{3})^2-(x+1)^2}} = \sin^{-1}\frac{x+1}{\sqrt{3}} + c$$

h. $\displaystyle\int \frac{dx}{\sqrt{9x^2+6x+10}} = \int \frac{dx}{\sqrt{(3x)^2+6x+10}} = \int \frac{dx}{\sqrt{(3x)^2+2.3x.1+1-1+10}}$

$$= \int \frac{dx}{\sqrt{(3x+1)^2+3^2}}$$

Put $y = 3x + 1$

$\frac{dy}{3} = dx$

Now, $\displaystyle\frac{1}{3}\int \frac{dx}{\sqrt{y^2+3^2}}$

$= \frac{1}{3}\ln[\, y + \sqrt{y^2+3^2}\,] + C$

$= \frac{1}{3}\ln\,[(3x+1) + \sqrt{(3x+1)^2+9}\,] + C$

$= \frac{1}{3}\ln\,[(3x+1) + \sqrt{9x^2+6x+10}\,] + C$

i. $\displaystyle\int \frac{xdx}{\sqrt{a^4+x^4}}$

$\displaystyle\int \frac{xdx}{\sqrt{(a^2)^2+(x^2)^2}}$

Put $x^2 = y$

$xdx = \frac{dy}{2}$

Now, $\displaystyle\frac{1}{2}\int \frac{dy}{\sqrt{a^4+y^2}} = \frac{1}{2}\ln\left(y+\sqrt{a^4+y^2}\right) + c = \frac{1}{2}\ln\left(x^2+\sqrt{a^4+x^4}\right) + c$

j. $\displaystyle\int \frac{xdx}{\sqrt{x^4+2x^2+10}} = \int \frac{dx}{\sqrt{(x^2)^2+2x^2+10}}$

Put $x^2 = y$

$x.dx = \frac{dy}{2}$

Now, $\displaystyle\frac{1}{2}\int \frac{dy}{\sqrt{y^2+2y+10}} = \frac{1}{2}\int \frac{dy}{\sqrt{y^2+2.y.1+1-1+10}} = \frac{1}{2}\int \frac{dy}{\sqrt{(y+1)^2+3^2}}$

$= \frac{1}{2}\ln\left(y+1+\sqrt{x^4+2x^2+10}\right) + c = \frac{1}{2}\ln\left(x^2+1+\sqrt{x^4+2x^2+10}\right) + c$

k. $\int \frac{2x+3}{\sqrt{x^2+4x+20}}\,dx = \int \frac{2x+3+1-1}{\sqrt{x^2+4x+20}}\,dx = \int \frac{2x+4}{\sqrt{x^2+4x+20}}\,dx - \int \frac{dx}{\sqrt{x^2+4x+20}}$

$= 2\sqrt{x^2+4x+20} - \int \frac{dx}{\sqrt{x^2+2.2x+4-4+20}} = 2\sqrt{x^2+4x+20} - \int \frac{dx}{\sqrt{(x+2)^2+4^2}}$

$= 2\sqrt{x^2+4x+20} - \ln\left(x+2+\sqrt{x^2+4x+20}\right) + c$

l. $\int \frac{x-2}{\sqrt{2x^2-8x+5}}\,dx = \frac{1}{4}\int \frac{4x-8}{\sqrt{2x^2-8x+5}}\,dx$

$= \frac{1}{4}\,2\sqrt{2x^2-8x+5} + c = \frac{1}{2}\sqrt{2x^2-8x+5} + c$

m. $\int \frac{x\,dx}{\sqrt{7+6x-x^2}}$

This equation can be written as

$p\int \frac{6-2x}{\sqrt{7+6x-x^2}}\,dx + q\int \frac{1}{\sqrt{7+6x-x^2}}\,dx$ ... . (i)

By comparing

$-2p = 1$ $\qquad$ $6p+q = 0$

$\therefore\ p = \frac{-1}{2}$ $\qquad$ $6 \times \left(\frac{-1}{2}\right) + q = 0$

$-3 + q = 0$

$\therefore\ q = 3$

Put the value of p and q in equation (i)

$= \frac{-1}{2}\int \frac{6-2x}{\sqrt{7+6x-x^2}}\,dx + 3\int \frac{1}{\sqrt{7+6x-x^2}}\,dx$

$= \frac{-1}{2}\int (6-2x)\,(7+6x-x^2)^{-1/2}\,dx + 3\int \frac{1}{\sqrt{-(x^2-6x-7)}}\,dx$

$= \frac{-1}{2} \times \frac{2}{1}(7+6x-x^2)^{1/2} + 3\int \frac{1}{\sqrt{\{x^2-2.3x+9-9-7}}\,dx$

$= -\sqrt{7+6x-x^2} + 3\int \frac{dx}{\sqrt{-\{(x-3)^2-(4)^2\}}}$

$= -\sqrt{7+6x-x^2} + 3 \times \sin^{-1}\left(\frac{x-3}{4}\right) + c$

n. $I = \int \frac{dx}{\sqrt{(x+a)\,(x+b)}} = \int \frac{1}{\sqrt{x^2+bx+ax+ab}}\,dx = \int \frac{1}{\sqrt{x^2+(a+b)\,.\,x+ab}}\,dx$

$= \int \frac{1}{\sqrt{(x)^2+2.x\left(\frac{a+b}{2}\right)+\left(\frac{a+b}{2}\right)^2-\left(\frac{a+b}{2}\right)^2+ab}}\,dx$

$$= \int \frac{1}{\sqrt{\left(x + \frac{a+b}{2}\right)^2 - \left(\frac{a^2 + 2ab + b^2 - 4ab}{4}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{a+b}{2}\right)^2 - \left(\frac{a^2 - 2ab + b^2}{4}\right)}} = \int \frac{1}{\sqrt{\left(x + \frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2}}\,dx$$

$$= \ln\left(x + \frac{a+b}{2} + \sqrt{(x+a)\,(x+b)}\right) + c$$

o. $\displaystyle\int \frac{dx}{(11+x)\sqrt{2+x}}$

Put $z^2 = 2 + x$

$2z \,.\, dz = dx$

Then $\displaystyle I = \int \frac{2z\,.\,dz}{(9+z^2)z} = 2\int \frac{dz}{z^2 + 3^2} = \frac{2}{3}\tan^{-1}\frac{z}{3} + c = \frac{2}{3}\tan^{-1}\frac{\sqrt{2+x}}{3} + c$

p. $\displaystyle J = \int \frac{dx}{(4x+3)\sqrt{x+3}}$

Put $x + 3 = y^2$

$\therefore\quad dx = 2y.dy \;\therefore\; x = y^2 - 3$

$$J = \int \frac{2ydy}{[4(y^2-3)+3]y} = 2\int \frac{dy}{4y^2 - 9} = 2\int \frac{1}{4(y^2 - \frac{9}{4})}\,dy$$

$$= \frac{1}{2}\int \frac{1}{y^2 - \frac{3}{2}^2}\,dy$$

$$= \frac{1}{2}\,.\,\frac{1}{2.\frac{3}{2}}\ln\frac{y-\frac{3}{2}}{y+\frac{3}{2}} + C$$

$$= \frac{1}{6}\ln\frac{2\sqrt{2y+3}-3}{2\sqrt{2y-3}+3} + C$$

q. $\displaystyle I = \int\sqrt{\frac{1+x}{1-x}}\,dx = \int\sqrt{\frac{1+x}{1-x}\times\frac{1+x}{1+x}}\,dx = \int\sqrt{\frac{(1+x)^2}{1-x^2}}\,dx$

$$= \int\frac{1+x}{\sqrt{1-x^2}}\,dx = \int\frac{1}{\sqrt{1-x^2}}\,dx + \int\frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \int\frac{1}{\sqrt{1-x^2}}\,dx - \frac{1}{2}\int\frac{-2x\,dx}{\sqrt{1-x^2}} = \sin^{-1}x - \frac{1}{2}\,2\sqrt{1-x^2} + c$$

$$= \sin^{-1}x - \sqrt{1-x^2} + c$$

EXERCISE 16.2

1. $\int \sqrt{x^2 - 36}\, dx$

Solution:

$$I = \int \sqrt{x^2 - 36}\, dx = \sqrt{(x) - (6)^2}\, dx = \frac{x\sqrt{x^2 - 36}}{2} - \frac{(6)^2}{2} \ln\left(x + \sqrt{x^2 - 36}\right) + c$$

$$= \frac{1}{2} x\sqrt{x^2 - 36} - 18 \ln\left(x + \sqrt{x^2 - 36}\right) + c$$

2. $\int \sqrt{1 - 4x^2}\, dx$

Solution:

$$I = \int \sqrt{1 - 4x^2\, dx} = \int \sqrt{(1)^2 - (2x)^2\, dx}$$

Put $y = 2x$

$$\frac{dy}{2} = dx$$

$$\text{Now, } I = \frac{1}{2} \int \sqrt{1^2 - y^2}\, dy = \frac{1}{2} \int \sqrt{1^2 - y^2}\, dy = \frac{1}{2} \times \left[\frac{y}{2} \sqrt{1 - y^2} + \frac{1}{2} \sin^{-1} \frac{y}{1}\right] + C$$

$$= \frac{1}{4} y\sqrt{1^2 - y^2} + \frac{1}{4} \sin^{-1} \frac{y}{1} + c$$

$$= \frac{2x}{4} \sqrt{1^2 - 4x^2} + \frac{1}{4} \sin^{-1} 2x + c = \frac{x\sqrt{1 - 4x^2}}{2} + \frac{1}{4} \sin^{-1} 2x + c$$

3. $I = \int \sqrt{3x^2 + 5}\, dx = \int \sqrt{(\sqrt{3}x)^2 + (\sqrt{5})^2}\, dx$

Put $y = \sqrt{3}\, x$

$$\therefore \quad \frac{dy}{\sqrt{3}} = dx$$

$$\text{Now, } I = \frac{1}{\sqrt{3}} \int \sqrt{y^2 + (\sqrt{5})^2}\, dx$$

$$I = \frac{1}{\sqrt{3}} \int \sqrt{y^2 + (\sqrt{5})^2}\, dx = \frac{1}{\sqrt{3}} \left[\frac{y\sqrt{y^2 + (\sqrt{5})^2}}{2} + \frac{\sqrt{(5)^2}}{2} \ln\left(y + \sqrt{y^2 + (\sqrt{5})^2}\right)\right]$$

$$= \frac{\sqrt{3}x}{2\sqrt{3}} \sqrt{(\sqrt{3}x)^2 + (\sqrt{5^2})} + \frac{5}{2\sqrt{3}} \ln\left(\sqrt{3}x + \sqrt{(\sqrt{3}x^2) + \sqrt{(5)^2}}\right) + c$$

$$= \frac{x\sqrt{3x^2 + 5}}{2} + \frac{5}{2\sqrt{3}} \ln\left(\sqrt{3}\, x + \sqrt{3x^2 + 5}\right) + c$$

4. $I = \int \sqrt{3 - 2x - x^2}\, dx = \int \sqrt{-(x^2 + 2x - 3)}\, dx$

$$= \int \sqrt{-(x^2 + 2.x.1 + 1 - 1 - 3)}\, dx = \int \sqrt{-(\{x + 1)^2 - (2)^2\}}\, dx$$

$$= \int \sqrt{2^2 - (x + 1)^2}\, dx$$

$$= \frac{x + 1}{2} \sqrt{2^2 - (x + 1)^2}) + \frac{2^2}{2} \sin^{-1} \frac{(x + 1)}{2} + c$$

$$= \frac{x+1}{2}\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + c$$

5. $I = \int \sqrt{5-2x+x^2}\,dx = \int \sqrt{4+1-2x+x^2}\,dx$

$$= \int \sqrt{(2)^2 + (x-1)^2}\,dx$$

$$= \frac{(x-1)\sqrt{(2)^2+(x-1)^2}}{2} + \frac{(2)^2}{2}\ln\left(x-1+\sqrt{(2)^2+(x-1)^2}\right) + c$$

$$= \frac{1}{2}(x-1)\sqrt{5-2x+x^2} + 2\ln\left(x-1+\sqrt{5-2x+x^2}\right) + c$$

6. $\int \sqrt{18x-x^2-65}\,dx = \int \sqrt{81-81+2.9x-x^2-65}\,dx$

$$= \int \sqrt{81-65-(x^2-18x+9^2)}\,dx = \int \sqrt{(4)^2-(x-9)^2)}\,dx$$

$$= \frac{1}{2}(x-9)\sqrt{16-(x-9)^2} + \frac{1}{2}16\sin^{-1}\frac{x-9}{4} + c$$

$$= \frac{1}{2}(x-9)\sqrt{18x-x^2-65} + 8\sin^{-1}\frac{x-9}{4} + c$$

7. $\int \sqrt{5x^2+8x+4}\,dx = \int \sqrt{5\left(x^2+\frac{8x}{5}+\frac{4}{5}\right)}\,dx$

$$= \sqrt{5}\int \sqrt{\left(x^2+2.x.\frac{4}{5}+\frac{16}{25}-\frac{16}{25}+\frac{4}{5}\right)}\,dx = \sqrt{5}\int \sqrt{(x^2+4/5)^2+(2/5)^2}\,dx$$

$$= \sqrt{5}\left[\frac{\left(x+\frac{4}{5}\right)}{2}\sqrt{(x+4/5)^2+(2/5)^2} + \frac{\left(\frac{2}{5}\right)^2}{2}\ln\left\{\left(x+\frac{4}{5}\right)+\sqrt{(x+4/5)^2+(2/5)^2}\right\}\right]$$

$$= \frac{(5x+4)\sqrt{5x^2+8x+4}}{10} + \frac{2}{5\sqrt{5}}\ln\left[\frac{5x+4}{5}+\sqrt{x^2+\frac{8x}{5}+\frac{4}{5}}\right]$$

8. $I = \int \sqrt{(x-\alpha)(\beta-x)}\,dx$

Put $x-\alpha = y$

$\therefore\ dx = dy \ \therefore\ x = y+\alpha$

$$I = \int \sqrt{y(\beta-y-\alpha)}\,dy = \int \sqrt{(\beta-\alpha)y-y^2}\,dy = \int \sqrt{\left(\frac{\beta-\alpha}{2}\right)^2-\left(y-\frac{\beta-\alpha}{2}\right)^2}dy$$

$$= \left(y-\frac{\beta-\alpha}{2}\right)\sqrt{\left(\frac{\beta-\alpha}{2}\right)^2-\left(y-\frac{\beta-\alpha}{2}\right)^2} + \frac{1}{2}\frac{(\beta-\alpha)^2}{4}\ \sin^{-1}\frac{y-\left(\frac{\beta-\alpha}{2}\right)}{\left(\frac{\beta-\alpha}{2}\right)} + c$$

$$= \left(x-\alpha-\frac{\beta-\alpha}{2}\right)\sqrt{\left(\frac{\beta-\alpha}{2}\right)^2-\left(x-\alpha-\frac{\beta-\alpha}{2}\right)^2} + \frac{1}{8}(\beta-\alpha)^2\sin^{-1}\frac{x-\alpha-\frac{\beta-\alpha}{2}}{\frac{\beta-\alpha}{2}} + c$$

$$= \frac{1}{2}(2x - 2\alpha - \beta + \alpha)\sqrt{\left(\frac{\beta - \alpha}{2}\right)^2 - \left(x - \frac{\alpha + \beta}{2}\right)^2} + \frac{1}{8}(\beta - \alpha)^2 \sin^{-1}\frac{2x - \alpha - \beta}{\beta - \alpha} + c$$

$$= \frac{1}{2}(2x - \alpha - \beta)\sqrt{(x - \alpha)(\beta - x)} + \frac{1}{8}(\beta - \alpha)^2 \sin^{-1}\frac{2x - \alpha - \beta}{\beta - \alpha} + c$$

9. $I = \int\sqrt{2ax - x^2}\,dx = \int\sqrt{a^2 - (a^2 - 2ax + x^2)}\,dx$

$$= \int\sqrt{a^2 - (x - a)^2}\,dx = \frac{1}{2}(x - a)\sqrt{(a)^2 - (x - a)^2} + \frac{(a)^2}{2}\sin^{-1}\left(\frac{x - a}{a}\right) + c$$

$$= \frac{1}{2}(x - a)\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x - a}{a}\right) + c$$

10. $I = \int(2x - 5)\sqrt{x^2 - 4x + 5}\,dx = \int(2x - 4 - 1)\sqrt{x^2 - 4x + 5}\,dx$

$$= \int(2x - 4)\sqrt{x^2 - 4x + 15}\,dx - \int\sqrt{(x - 2)^2 + 1}\,dx$$

$$= \frac{2}{3}(x^2 - 4x + 5)^{3/2} - \frac{(x - 2)\sqrt{x^2 - 4x + 5}}{2} - \frac{1}{2}\ln[(x - 2) + \sqrt{x^2 - 4x + 5}) + c$$

11. $\int(2 - x)\sqrt{16 - 6x - x^2}\,dx$

$$I = \int(2 - x)\sqrt{16 - 6x - x^2}\,dx = \frac{1}{2}\int(4 - 2x)\sqrt{16 - 6x - x^2}\,dx$$

$$= \frac{1}{2}\int\{10 + (-6 - 2x)\}\sqrt{16 - 6x - x^2}\,dx$$

$$5\int\sqrt{16 - 6x - x^2}\,dx + \frac{1}{2}\int(-6 - 2x)\sqrt{16 - 6x - x^2}$$

$I = I_1 + I_2$

$I_1 = 5\int\sqrt{16 - 6x - x^2}\,dx = 5\int\sqrt{25 - (9 + 6x + x^2)}\,dx$

$$= 5\int\sqrt{(5)^2 - (x + 3)^2}\,dx = 5\left\{\frac{1}{2}(x + 3)\sqrt{(5)^2 - (x + 3)^2} + \frac{52}{2}\sin^{-1}\left(\frac{x + 3}{5}\right)\right\} + c_1$$

$$= \frac{5}{2}(X + 3)\sqrt{16 - 6X - X^2} + \frac{125}{2}\,\mathrm{Sin}^{-1}\left(\frac{x + 3}{5}\right) C_1$$

$$I_2 = \frac{1}{2}\int(-6 - 2x)\sqrt{16 - 6x - x^2}\,dx$$

Put $16 - 6x - x^2 = y$

$\therefore\ dy = (-6 - 2x)\,dx$

$$I_2 = \frac{1}{2}\int y^{1/2}\,dy = \frac{1}{3}y^{3/2} + C_2\ z\ \frac{1}{3}(16 - 6x - x^2)^{3/2} + C_2$$

$$I = I_1 + I_2 = \frac{5}{2}(x + 3)\sqrt{16 - 6x - x^2} + \frac{125}{2}\sin^{-1}\left(\frac{x + 3}{2}\right) + \frac{1}{3}(16 - 6x - x^2)^{3/2} + C$$

12. $\int(2x + 1)\sqrt{4x^2 + 20x + 21}\,dx$

$$I = \int (2x+1)\sqrt{4x^2+20x+21}\,dx = \frac{1}{4}\int (8x+4)\sqrt{4x^2+20x+21}\,dx$$

$$= \frac{1}{4}\int \{(8x+20)-16\}\sqrt{4x^2+20x+21}\,dx$$

$$= \frac{1}{4}\int (8x+20)\sqrt{4x^2+20x+21}\,dx - 4\int \sqrt{4x^2+20x+21}\,dx = I_1 - I_2$$

$$I_1 = \frac{1}{4}\int (8x+20)\sqrt{4x^2+20x+21}\,dx$$

Put $4x^2 + 20x + 21 = y$

$(8x + 20)\,dx = dy$

$$I_1 = \frac{1}{4}\int y^{1/2}\,dy = \frac{1}{4}\frac{y^{3/2}}{3/2} + C_1 = \frac{1}{6}(4x^2+20x+21)^{3/2} + C_1$$

$$I_2 = 4\int\sqrt{4x^2+20x+21}\,dx = 4\int\sqrt{(4x^2+20x+25)-4}\,dx = 4\int\sqrt{(2x+5)^2-(2)^2}\,dx$$

Put $2x + 5 = y$

$\therefore \quad dx = \dfrac{dy}{2}$

$$I_2 = 2\int\sqrt{(y)^2-(2)^2}\,dy = 2\left\{\frac{1}{2}y\sqrt{(y)^2-(2)^2} - \frac{(2)^2}{2}\log\left(y+\sqrt{y^2-4}\right)\right\} + C_2$$

$$= y\sqrt{y^2-4} - 4\ln\left(y+\sqrt{y^2-4}\right) + C_2$$

$$= (2x+5)\sqrt{(2x+5)^2-4} - 4\ln\left(2x+5+\sqrt{(2x+5)^2-4}\right)$$

$$= (2x+5)\sqrt{2x^2+20x+21} - 4\ln\left(2x+5+\sqrt{4x^2+20x+2}\right)$$

$I = I_1 - I_2$

$$= \frac{1}{6}(4x^2+20x+21)^{3/2} - (2x+5)\sqrt{4x^2+20x+21} + 4\ln(2x+5+\sqrt{4x^2+20x+21} + C$$

13. $$I = \int (2x+3)\sqrt{x^2-2x-3}\,dx = \int (2x+3-5+5)\sqrt{x^2-2x-3}\,dx$$

$$= \int (2x-2)\sqrt{x^2-2x-3}\,dx + 5\int\sqrt{x^2-2x-3}\,dx$$

$I = I_1 + I_2$

$$I_1 = \int (2x-2)\sqrt{x^2-2x-3}\,dx$$

Put $y = x^2 - 2x - 3$

$$\therefore \quad dy = (2x-2)\,dx = \int\sqrt{y}\,dy = \frac{y^{3/2}}{3/2} + C = \frac{2}{3}\sqrt{x^2-2x-3} + C$$

$$I_2 = 5\int\sqrt{x^2-2x-3}\,dx = 5\int\sqrt{x^2-2x+1-1-3}\,dx = 5\int\sqrt{(x-1)^2-(2)^2}\,dx$$

$$= 5\left[\frac{1}{2}(x-1)\sqrt{x^2-2x-3} - \frac{4}{2}\log\left(x-1+\sqrt{x^2-2x-3}\right)\right] + C$$

$$= \frac{5}{2}(x-1)\sqrt{x^2-2x-3} - \frac{20}{2}\ln\left(x-1+\sqrt{x^2-2x-3}\right) + C$$

$$= \frac{5}{2}(x-1)\sqrt{x^2-2x-3} - 10\ln\left(x-1+\sqrt{x^2-2x-3}\right) + C$$

$\therefore$ I = $I_1 + I_2$

$$= \frac{2}{3}(x^2 - 2x - 3)^{3/2} + \frac{5}{2}(x - 1)\sqrt{x^2 - 2x - 3} - 10 \ln (x - 1 + \sqrt{(x^2 - 2x - 3)}) + C$$

14. $I = \int e^{3x} . \sin^5 x \, dx$

We suppose $u = \sin 5x$ $v = e^{3x}$

By using formula $\int e^{ax} \sin bx dx = \int \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$

We have,

$$I = \frac{1}{3^2 + 5^2} . e^{3x}(3 \sin 5x - 5 \cos 5x) + C$$

$$\therefore \quad I = \frac{1}{34} e^{3x}(3 \sin 5x - 5 \cos 5x) + C$$

15. $I = \int e^x \cos 3x \, dx$

By using formula $\int e^{ax} . \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$

We have,

$$I = \frac{1}{1^2 + 3^2} e^x (1.\cos 3x + 3 \sin 3x) + C = \frac{1}{10} e^x (\cos 3x + 3 \sin 3x) + C$$

16. $I = \int e^{2x} \sin (x + 1) \, dx$

By using formula $\int e^{ax} . \sin bx = e^{ax} \frac{(a \sin bx - b \cos bx)}{a^2 + b^2}$

We have,

$$I = \frac{1}{(2)^2 + 1^2} e^{2x}[2 \sin (x + 1) - \cos (x + 1)] + C = \frac{1}{5} e^{2x}[2 \sin(x + 1) - \cos(x + 1)] + C$$

EXERCISE 16.3

Evaluate:

1. $\int \frac{dx}{1 + 2 \sin^2 x}$
2. $\int \frac{dx}{5 + 4 \cos x}$
3. $\int \frac{dx}{1 - 3 \sin x}$
4. $\int \frac{dx}{a^2 \sin^2 x - b^2 \cos^2 x}$
5. $\int \frac{dx}{4 \cos x - 1}$
6. $\int \frac{dx}{2 + 3 \cos x}$
7. $\int \frac{\sin x \cos x}{(\sin x + \cos x)^2} dx$
8. $\int \frac{dx}{\sin x + \cos x}$
9. $\int \frac{dx}{1 + \sin x + \cos x}$
10. $\int \frac{dx}{3 + 2 \sin x + \cos x}$

11. $\int \frac{dx}{1 - \sin x + \cos x}$

12. $\int \frac{dx}{2 + \cos x - \sin x}$

13. $\int \frac{dx}{\cos x - \sqrt{3} \sin x}$

14. $\int \frac{1}{1 + 2 \sin x}$

15. $\int \frac{dx}{2 + \sin x}$

16. $\int \frac{dx}{4 + 3 \sinh x}$

17. $\int \frac{dx}{4 + 3 \cosh x}$

18. $\int \frac{\tanh x}{36 \sec hx + \cosh x} dx$

19. $\int \frac{\tanh x dx}{\cosh x + 64 \sec hx}$

20. $\int \frac{\sinh x dx}{4 \tanh x - \operatorname{cosech} x \sec hx}$

Solution:

1. $I = \int \frac{dx}{1 + 2\sin^2 x}$

Dividing numerator and denominator by $\cos^2 x$

$$\int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{2\sin^2 x}{\cos^2 x}} dx = \int \frac{\sec^2 x}{\sec^2 x + 2\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{1 + \tan^2 x + 2\tan^2 x} dx = \int \frac{\sec^2 x}{1 + (\sqrt{3} \tan x)^2} dx$$

Put $y = \sqrt{3} \tan x$

$\Rightarrow$ $dy = \sqrt{3} \sec^2 x \, dx$

$\Rightarrow$ $\frac{dy}{\sqrt{3}} = \sec^2 x \, dx$

$$\therefore \quad I = \frac{1}{\sqrt{3}} \int \frac{dy}{1^2 + y^2} = \frac{1}{\sqrt{3}} \frac{1}{1} \tan^{-1} \frac{y}{1} + C = \frac{1}{\sqrt{3}} \tan^{-1} (\sqrt{3} \tan x) + C$$

2. $I = \int \frac{dx}{5 + 4\cos x}$

$$\int \frac{dx}{5\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}\right) + 4\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)} dx$$

$$= \int \frac{1}{5 \sin^2 \frac{x}{2} + 5\cos^2 \frac{x}{2} + 4\cos^2 \frac{x}{2} - 4\sin^2 \frac{x}{2}} dx$$

$$\int \frac{1}{9\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}} dx$$

Dividing numerator and denominator by $\cos^2 \frac{x}{2}$

$$= \int \frac{\sec^2 \frac{x}{2}}{9 + \tan^2 \frac{x}{2}} dx$$

Put $\tan \frac{x}{2} = y$

or, $\sec^2 \frac{x}{2} \frac{1}{2} = dy$

$\therefore$ $2dy = \sec^2 x \, dx$

Now, $I = 2\int \frac{dy}{(3)^2 + y^2} = \frac{2}{3} \tan^{-1} \frac{\left(\tan \frac{x}{2}\right)}{3} + C = \frac{2}{3} \tan^{-1} \frac{1}{3} \left(\tan \frac{x}{2}\right) + C$

3. $I = \int \frac{dx}{1 - 3\sin x} = \int \frac{dx}{\left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}\right) - 2.3 \sin \frac{x}{2} \cos \frac{x}{2}}$

Dividing by $\cos^2 \frac{x}{2}$ in deno-and num.

$$\int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2} - 6 \tan \frac{x}{2} + 1} dx$$

Put $y = \tan \frac{x}{2}$

$2dy = \sec^2 \frac{x}{2} dx$

Now,

$\therefore$ $I = 2\int \frac{dy}{y^2 - 6y + 1} = 2\int \frac{dy}{y^2 - 2.3y + 9 - 9 + 1}$

$$= 2\int \frac{dy}{(y - 3)^2 - (2\sqrt{2})^2} = 2\frac{1}{2.2\sqrt{2}} \ln \frac{y - 3 - 2\sqrt{2}}{y - 3 + 2\sqrt{2}} + C$$

$$= \frac{1}{2\sqrt{2}} \ln \frac{\tan \frac{x}{2} - 3 - 2\sqrt{2}}{\tan \frac{x}{2} - 3 + 2\sqrt{2}} + C$$

4. $I = \int \frac{dx}{a^2 \sin^2 x - b^2 \cos^2 x}$

Dividing number and deno. by $\cos^2 x$ then

$I = \int \frac{\sec^2 x}{a^2 \tan^2 x - b^2} dx$

Put $y = a \tan x$

$\frac{dy}{a} = \sec^2 x \, . \, dx$

Now,

$$\therefore \quad I = \frac{1}{a}\int \frac{dy}{y^2 - b^2} = \frac{1}{2.ba}\ln\frac{y-b}{y+b} + C = \frac{1}{2ab}\ln\left(\frac{a\tan x - b}{a\tan x + b}\right) + C$$

5. $I = \int \frac{dx}{4\cos x - 1} = \int \frac{dx}{4\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) - \left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right)}$

$$= \int \frac{dx}{4\cos^2\frac{x}{2} - 4\sin^2\frac{x}{2} - \sin^2\frac{x}{2} - \cos^2\frac{x}{2}} = \int \frac{dx}{3\cos^2\frac{x}{2} - 5\sin^2\frac{x}{2}}$$

Dividing numerator and denominator by $\cos^2\frac{x}{2}$

$$I = \int \frac{\sec^2\frac{x}{2}}{3 - (\sqrt{5})^2\tan^2\frac{x}{2}}$$

Put $y = \sqrt{5}\tan\frac{x}{2}$

$\therefore \quad \frac{2dy}{\sqrt{5}} = \sec^2\frac{x}{2}\,dx$

Now, $I = \frac{2}{\sqrt{5}}\int \frac{dy}{(\sqrt{3})^2 - \left(\sqrt{5}\,\text{Tan}\,\frac{x}{2}\right)^2} = \frac{2}{\sqrt{5}}\int \frac{dy}{(\sqrt{3})^2 - y^2}$

$$= \frac{2}{\sqrt{5}}\frac{1}{2\sqrt{3}}\ln\frac{\sqrt{3}+y}{\sqrt{3}-y} + C = \frac{1}{\sqrt{15}}\ln\left(\frac{\sqrt{3}+\sqrt{5}\,\text{Tan}\,\frac{x}{2}}{\sqrt{3}-\sqrt{5}\,\text{Tan}\,\frac{x}{2}}\right) + C$$

6. $I = \int \frac{dx}{2 + 3\cos x} = \int \frac{dx}{2\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) + 3\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)} =$

$$\int \frac{dx}{5\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

Dividing numerator and denominator by $\cos^2\frac{x}{2}$

$$= \int \frac{\sec^2\frac{x}{2}}{(\sqrt{5})^2 - \tan^2\frac{x}{2}}\,dx$$

Put $y = \tan\frac{x}{2}$

$2dy = \sec^2\frac{x}{2}\,dx$

Now, $I = 2\int \frac{dy}{(\sqrt{5})^2 - y^2} = 2\frac{1}{2\sqrt{5}}\ln\frac{\sqrt{5}+y}{\sqrt{5}-y} + C = \frac{1}{\sqrt{5}}\ln\left(\frac{\sqrt{5}+\tan\frac{x}{2}}{\sqrt{5}-\tan\frac{x}{2}}\right) + C$

7. $I = \int \frac{\sin x \cos x}{(\sin x + \cos x)^2} dx = \frac{1}{2} \int \frac{2\sin x \, . \cos x}{(\sin x + \cos x)^2} dx = \frac{1}{2} \int \frac{\sin 2x}{(\sin x + \cos x)^2}$

$= \frac{1}{2} \int \frac{(1 + \sin 2x) - 1}{(\sin x + \cos x)^2} dx = \frac{1}{2} \int \frac{(\sin x + \cos x)^2 - 1}{(\sin x + \cos x)^2} dx$

$= \frac{1}{2} \int \frac{(\sin x + \cos x)^2}{(\sin x + \cos x)^2} dx - \frac{1}{2} \int \frac{dx}{(\sin x + \cos x)^2}$

$I_1 = \frac{1}{2} x - \frac{1}{2} I_2 = \int \frac{dx}{(\sin x + \cos x)^2}$

Dividing numerator and denominator by $\cos^2 x$

$I = I_1 + I_2 = \frac{x}{2} - \frac{1}{2} \int \frac{\sec^2 dx}{(1 + \tan x)^2}$

Put $\tan x + 1 = y$, $dy = \sec^2 x.dx$

$= \frac{x}{2} - \frac{1}{2} \int \frac{dy}{y^2} = \frac{x}{2} + \frac{1}{2} \frac{1}{y} + C = \frac{x}{2} + \frac{1}{2} \frac{1}{\tan x + 1} + C$

8. $I = \int \frac{dx}{\sin x + \cos x}$

Put $1 = r\cos\theta \qquad 1 = r\sin\theta$

So that $r^2 = 2$

$\therefore \quad r = \sqrt{2}$

$\tan\theta = 1$

$\therefore \quad \theta = \frac{\pi}{4}$

$I = \int \frac{dx}{r\cos\theta \, . \sin x + r\sin\theta \, . \cos x} = \frac{1}{r} \int \frac{dx}{\sin (x + \theta)} = \frac{1}{r} \int \operatorname{cosec} (x + \theta) \, dx$

$= \frac{1}{r} \ln \tan \frac{1}{2} (x + \theta) + C = \frac{1}{\sqrt{2}} \ln \tan \frac{1}{2} \left(x + \frac{\pi}{4}\right) + C = \frac{1}{\sqrt{2}} \ln \left[\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right] + C$

9. $I = \int \frac{dx}{1 + \sin x + \cos x} = \int \frac{dx}{(1 + \cos x) + \sin x} = \int \frac{dx}{2\cos^2 \frac{x}{2} + 2\sin x \frac{x}{2} \cos \frac{x}{2}}$

Dividing by $\cos^2 \frac{x}{2}$ in deno. and num.

$= \frac{1}{2} \int \frac{\sec^2 \frac{x}{2}}{1 + \tan \frac{x}{2}} dx$

Put $1 + \tan \frac{x}{2} = y$

$\sec^2 \frac{x}{2} \frac{1}{2} dx = dy$

$\therefore \quad \sec^2 \frac{x}{2} dx = 2dy$

$$I = \int \frac{dy}{y} = \ln y + C = \ln\left(1 + \tan\frac{x}{2}\right) + C$$

10. $I = \int \frac{dx}{3 + 2\sin x + \cos x} = \int \frac{1}{3\cos^2\frac{x}{2} + 3\sin^2\frac{x}{2} + 4\sin\frac{x}{2}.\cos\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$

dx

$$= \int \frac{1}{4\cos^2\frac{x}{2} + 2\sin^2\frac{x}{2} + 4\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$$

Dividing denominator and numerator by $\cos^2\frac{x}{2}$

$$= \int \frac{\sec^2\frac{x}{2}}{4 + 2\tan^2\frac{x}{2} + 4\tan\frac{x}{2}}\,dx = \frac{1}{4}\int \frac{\sec^2\frac{x}{2}}{1 + \frac{1}{2}\tan^2\frac{x}{2} + \tan\frac{x}{2}}$$

Put $\tan\frac{x}{2} = y$

$\Rightarrow \sec^2\frac{x}{2}\,dx = 2dy$

from (i)

$$I = \frac{1}{4}\int \frac{2dy}{1 + \frac{1}{2}y^2 + y} = \int \frac{dy}{y^2 + 2y + 2} = \int \frac{dy}{(y+1)^2 + 1^2}$$

$$= \frac{1}{1}\tan^{-1}(y+1) + C = \tan^{-1}\left(\tan\frac{x}{2} + 1\right) + C$$

11. $I = \int \frac{1}{1 - \sin x + \cos x} = \int \frac{1}{(1 + \cos x) - \sin x} = \int \frac{1}{2\cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$

$$= \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2} - \sin\frac{x}{2}\cos\frac{x}{2}}$$

Dividing numerator and denominator by $\cos^2\frac{x}{2}$

$$I = \frac{1}{2}\int \frac{\sec^2\frac{x}{2}}{1 - \tan\frac{x}{2}}\,dx$$

Put $1 - \tan\frac{x}{2} = y$

$-\frac{1}{2}\sec^2\frac{x}{2}\,dx = dy$

$\sec^2\frac{x}{2}\,dx = -2dy = -\frac{2}{2}\int\frac{dy}{y} = -\ln y + C = -\ln\left(1 - \tan\frac{x}{2}\right) + C$

12. $\int \frac{dx}{2 + \cos x - \sin x} = \int \frac{dx}{2\sin^2 \frac{x}{2} + 2\cos^2\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}$

$$= \int \frac{1}{\sin^2 \frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2}} dx$$

$$= \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2} - 2\tan\frac{x}{2} + 1} dx$$

Put $\tan\frac{x}{2} = y$

$\Rightarrow \frac{1}{2}\sec^2 x \frac{x}{2} dx = dy$

$\therefore \sec^2\frac{x}{2} dx = 2dy$

$$= \int \frac{2dy}{y^2 - 2y + 1} = 2\int \frac{1}{(y-1)^2} dy = -\frac{2}{(y-1)} + c = -\frac{2}{(\tan\frac{x}{2} - 1)}$$

13. $\int \frac{dx}{\cos x - \sqrt{3}\sin x}$

Put $1 = r\sin\theta$, $\sqrt{3} = r\cos\theta$

So that $r^2 = 3 + 1 = 4$

$\therefore r = 2$

Also, $\tan\theta = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$

$\therefore \theta = \frac{\pi}{6}$

Now, $\int \frac{dx}{r\sin\theta . \cos x - r\cos\theta \sin x} = \int \frac{dx}{r[\sin(\theta - x)]} = \frac{1}{2}\int \operatorname{cosec}(\theta - x)\, dx$

$$= \frac{1}{2}\left|\ln\tan\left(\frac{\theta - x}{2}\right)\right| + c$$

$$= \frac{1}{2}\ln\tan\left(\frac{\pi}{12} - \frac{x}{2}\right) + c$$

14. $I = \int \frac{1}{1 + 2\sin x} dx = \int \frac{dx}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2.2\sin\frac{x}{2}\cos\frac{x}{2}}$

Dividing by $\cos^2\frac{x}{2}$ in deno and num

$$= \int \frac{\sec^2\frac{x}{2}}{1 + \tan^2\frac{x}{2} + 4\tan\frac{x}{2}} = \int \frac{\sec^2\frac{x}{2}}{\left(\tan\frac{x}{2} + 2\right)^2 - 3}$$

Put $\tan\frac{x}{2} + 2 = y$

$$\text{Sec}^2\frac{x}{2}\,dx = 2dy$$

$$\therefore \quad I = 2\int\frac{dy}{(y)^2-(\sqrt{3})^2} = 2\frac{1}{2\sqrt{3}}\ln\frac{y-\sqrt{3}}{y+\sqrt{3}} + C = \frac{1}{\sqrt{3}}\ln\frac{\tan\frac{x}{2}+2-\sqrt{3}}{\tan\frac{x}{2}+2+\sqrt{3}} + C$$

15. $$I = \int\frac{dx}{2+\sin x} = \int\frac{dx}{2\cos^2\frac{x}{2}+2\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}$$

Dividing numerator and denominator by $\cos^2\frac{x}{2}$

$$= \frac{1}{2}\int\frac{\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2}+\tan\frac{x}{2}}\,dx$$

Put $\tan\frac{x}{2} = y$

$$\text{Sec}^2\frac{x}{2}\,dx = 2dy$$

$$I = \frac{2}{2}\int\frac{dy}{1+y^2+y} = \int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2y+1}{\sqrt{3}} + C = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\tan\frac{x}{2}+1}{\sqrt{3}} + C$$

16. $$I = \int\frac{dx}{4+3\sinh x}$$

or, $$I = \int\frac{dx}{4\left(\cosh^2\frac{x}{2}-\sinh^2\frac{x}{2}\right)+3.2\,\text{sinx}\frac{x}{2}\cosh\frac{x}{2}} =$$

$$\int\frac{\text{sech}^2\frac{x}{2}}{4\left(1-\tanh^2\frac{x}{2}\right)+6\tanh\frac{x}{2}}\,dx$$

Let $\text{tanhx}\,\frac{x}{2} = y$

$$\text{Secx}^2\frac{x}{2} = 2dy$$

$$= \frac{2}{4}\int\frac{dy}{1-y^2+\frac{3}{2}y}$$

$$= \frac{1}{2}\int\frac{1}{-(y^2-\frac{3y}{2}-1)}dy$$

$$= \frac{1}{2} \int \frac{1}{-\left[y^2 - 2.y\,.\,\frac{3}{4} + \frac{9}{16} - \frac{9}{16} - 1\right]} dy$$

$$= \frac{1}{2} \int \frac{1}{-[(y - 3/4)^2 - 25/16]} dy$$

$$= \frac{1}{2} \int \frac{1}{(5/4)^2 - (y - 3/4)^2} dy$$

$$= \frac{1}{2} \times \frac{1}{2 \times 5/4} \ln \frac{5/4 + y - 3/4}{5/4 - y + 3/4}$$

$$= \frac{1}{5} \ln \left(\frac{5 + 4y - 3}{5 - 4y + 3}\right) + c$$

$$= \frac{1}{5} \ln \left(\frac{4y + 2}{8 - 4y}\right) + c$$

$$= \frac{1}{5} \ln \left(\frac{2y + 1}{4 - 2y}\right) + c$$

$$= \frac{1}{5} \ln \left(\frac{2 \tanh\frac{x}{2} + 1}{4 - 2 \tanh \frac{x}{2}}\right) + c$$

17. $I = \int \frac{dx}{4 + 3\cosh x}$

$$I = \int \frac{dx}{4\left(\cos h^2 \frac{x}{2} - \sin h^2 \frac{x}{2}\right) + 3\left(\cos h^2 \frac{x}{2} + \sin h^2 \frac{x}{2}\right)} = \int \frac{dx}{7\cos h^2 \frac{x}{2} - \sin h^2 \frac{x}{2}}$$

Dividing numerator and denominator by $\cos^2 \frac{x}{2}$

$$= \int \frac{\text{sech}^2 \frac{x}{2}}{7 - \tanh^2 \frac{x}{2}}$$

Let $\tanh \frac{x}{2} = y$

$\text{sech}^2 \frac{x}{2} = 2dy$

$$\therefore\ I = 2\int \frac{dy}{7 - y^2} = 2\int \frac{dy}{((\sqrt{7})^2 - (y)^2)} = 2.\frac{1}{2.\sqrt{7}} \ln \left(\frac{\sqrt{7} + y}{\sqrt{7} - y}\right) + C$$

$$= \frac{1}{\sqrt{7}} \ln \left[\frac{\sqrt{7} + \tan h \frac{x}{2}}{\sqrt{7} - \tan h \frac{x}{2}}\right] + C$$

18. $I = \int \frac{\tan hx}{36 \sec hx + \cos hx} dx$

Multiplying by cosx deno. and num.

$$= \int \frac{\cos hx.\, \dfrac{\sin hx}{\cos hx}}{36\, \dfrac{\cos hx}{\cos hx} + \cos hx\, .\, \cos hx} = \int \frac{\sin hx}{(6)^2 + \cos h^2 x}\, dx$$

Put cos hx = y

$\therefore$ sin hx . dx = dy

$$\therefore \int \frac{dy}{(6)^2 + (y)^2} = \frac{1}{6} \tan^{-1} \frac{y}{6} + C = \frac{1}{6} \tan^{-1} \left(\frac{\cos h\, x}{6}\right) + C$$

19. $I = \int \dfrac{\tan hx\, dx}{\cosh x + \operatorname{sech} x\; 64}$

Multiplying cosxh in demoninator and numerator

$$= \int \frac{\cos hx\, \dfrac{\sin hx}{\cos hx}}{\cos hx\, .\, \cos hx + 64\, .\, \cos hx\, \dfrac{1}{\cos hx}} = \int \frac{\sin hx}{\cos h^2 x + 64} dx$$

Let cos hx = y

$\sin h\, x = \dfrac{dy}{dx}$

sin h x . dx = dy

$$\therefore \quad I = \int \frac{dy}{y^2 + 64} = \int \frac{dy}{y^2 + (8)^2} = \frac{1}{8} \tan^{-1} \frac{y}{8} + C = \frac{1}{8} \tan^{-1} \left(\frac{\cosh x}{8}\right) + C$$

20. $I = \int \dfrac{\sin h\, x}{4\tan hx - \operatorname{cosec} hx\, .\, \operatorname{sech}}\, dx$

$$= \int \frac{\sin x\, x}{4\, \dfrac{\sin hx}{\cos hx} - \dfrac{1}{\sinh x\, .\, \cosh x}} dx = \int \frac{\sin h^2 x \cos hx}{4\sin h^2 x - 1}$$

Let sin hx = y

cos hx.dx = dy

$$\therefore \quad I = \int \frac{y^2\, dy}{4y^2 - 1} = \frac{1}{4} \int \frac{4y^2\, dy}{4y^2 - 1} = \frac{1}{4} \int \frac{(4y^2 - 1) + 1}{(4y^2 - 1)}\, dy$$

$$= \frac{1}{4} \int \left(1 + \frac{1}{4y^2 - 1}\right) dy$$

$$= \frac{1}{4} \left[y + \int \frac{1}{4(y^2 - \frac{1}{4})} \right] dy$$

$$= \frac{1}{4} \left[y + \frac{1}{4} \times \int \frac{1}{y^2 - \left(\frac{1}{2}\right)^2} \right] dy$$

$$= \frac{1}{4} \left[y + \frac{1}{4} \times \frac{1}{2 \times \frac{1}{2}} \ln \frac{y - \frac{1}{2}}{y + \frac{1}{2}} \right] + c$$

$= \frac{1}{4}\left[\sinh x + \frac{1}{4}\ln\frac{2\sinh x - 1}{2\sinh x + 1}\right] + c$

$= \frac{1}{4}\sinh x + \frac{1}{16}\ln\frac{2\sinh x - 1}{2\sinh x + 1} + c$

EXERCISE 16.4

Evaluate

1. $\int \frac{2x}{(2x + 3)(3x + 5)} dx$
2. $\int \frac{3x}{(x - a)(x - b)} dx$
3. $\int \frac{1}{(x + 2)(x + 3)^2} dx$
4. $\int \frac{x^2\, dx}{(x - a)(x - b)(x - c)}$
5. $\int \frac{x^2 + 1}{x - 1} dx$
6. $\int \frac{dx}{1 + x + x^2 + x^3}$
7. $\int \frac{7x^2 - 18x + 13}{(x - 3)(x^2 + 2)} dx$
8. $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$
9. $\int \frac{1}{x^4 - 1} dx$
10. $\int \frac{x^3\, dx}{(x^2 + a^2)(x^2 + b^2)}$
11. $\int \frac{x^2 + 4}{x^4 + 16} dx$
12. $\int \frac{x^3\, dx}{(x - a)(x - b)(x - c)}$
13. $\int \frac{x^3\, dx}{2x^4 - x^2 - 10}$
14. $\int \frac{dx}{(x - 1)^2 (x - 3)^2}$
15. $\int \frac{dx}{(x - 1)^2 (x - 4)^3}$

Solution:

1. $\int \frac{2x}{(2x + 3)(3x + 5)} dx$

Let $\frac{2x}{(2x + 3)(3x + 5)} = \frac{A}{2x + 3} + \frac{B}{3x + 5}$

$2x = A(3x + 5) + B(2x + 3)$

Equating the coefficient of x and constant terms we to get

$3A + 2B = 2$

$5A + 3B = 0$

$A = -6$ and $B = 10$

$\therefore \quad \frac{2x}{(2x + 3)(3x + 5)} = \frac{-6}{2x + 3} + \frac{10}{3x + 5}$

So, we have by integration

$$\int \frac{dx}{(2x + 3)(3x + 5)} = -3 \ln (2x + 3) + \frac{10}{3} \ln (3x + 5) + C$$

2. $\int \frac{3x}{(x-a)(x-b)}\,dx$

Let $\frac{3x}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$

$\Rightarrow \frac{3x}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$

$\Rightarrow 3x = A(x-b) + B(x-a)$ … (i)

Put $x = a$

$3a = A(a-b)$

$\therefore A = \frac{3a}{a-b}$

$x = b,\ 3b = B(b-a)$

$B = \frac{-3b}{a-b}$

$\therefore \frac{3x}{(x-a)(x-b)} = \frac{3a}{(x-a)(a-b)} - \frac{3b}{(a-b)(x-b)}$

or, $\int \frac{3x}{(x-a)(x-b)}\,dx = \frac{3}{a-b}\int\left\{\frac{a}{x-a} - \frac{b}{x-b}\right\}dx$

$$= \frac{3}{a-b}\ [a \ln(x-a) - b\ln(x-b)] + C$$

3. $\int \frac{1}{(x+2)(x+3)^2}\,dx$

Let $\frac{1}{(x+x\,2)(x+3)^2} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

$1 = A(x+3)^2 + B(x+2)(x+3) + C(x+2)$

Put $x = -2$

$1 = A.1$

$\therefore A = 1$

Put $x = -3$

$I = C(-1)$

$\therefore C = -1$

Put $x = 0,\ 1 - 9A + B.2.3 + C.2$

$6B = -6$

$\therefore B = -1$

$\therefore \frac{1}{(x+2)(x+3)^2} = \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{(x+3)^2}$

$\int \frac{1}{(x+2)(x+3)^2}\,dx = \int\left\{\frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{(x+3)^2}\right\}dx$

$\text{Ln}\,(x+2) - \ln(x+3) + \frac{1}{x+3} + C = \ln\frac{x+2}{x+3} + \frac{1}{x+3} + C$

4. $\int \frac{x^2\,dx}{(x-a)(x-b)(x-c)}$

Let $\frac{x^2}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$

$x^2 = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$

$\therefore$ Put $x = b$

$$b^2 = B(b-a)(b-c) + c(b-c)(b-b)$$

$$\Rightarrow B = \frac{b^2}{(b-a)(b-c)}$$

Similarly, Again put $x = a$

$$\frac{a^2}{(a-b)(a-c)} = A$$

$$\frac{c^2}{(c-b)(c-a)} = C$$

Then,

$$= \frac{a^2}{(a-b)(a-c)} \cdot \int \frac{1}{x-a}\,dx + \frac{b^2}{(b-a)(b-c)} \int \frac{1}{x-b}\,dx + \frac{c^2}{(c-b)(c-a)} \int \frac{1}{x-c}\,dx$$

$$= \frac{a^2}{(a-b)(a-c)} \ln(x-a) + \frac{b^2}{(b-a)(b-c)} \ln(x-c) + \frac{c^2}{(c-b)(c-a)} \ln(x-c) + c$$

5. $\displaystyle\int \frac{x^2+1}{x-1}\,dx$

$$\int \frac{x^2}{x-1}\,dx + \int \frac{1}{x-1}\,dx = \int \frac{x^2-1+1}{x-1}\,dx + \int \frac{1}{x-1}\,dx$$

$$= \int (x+1)dx + \int \frac{1}{x-1} + \int \frac{1}{x-1}\,dx = \frac{x^2}{2} + x + 2\ln(x-1) + C$$

6. $\displaystyle\int \frac{dx}{1+x+x^2+x^3} = \int \frac{dx}{x^3+x^2+x+1} = \int \frac{1}{x^2(x+1)+(x+1)}\,dx =$

$$\int \frac{1}{(x^2+1)(x+1)}\,dx$$

Let, $\displaystyle\frac{1}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{x^2+1}$

$1 = A(x^2+1) + (Bx+C)(x+1)$

Put $x = 0$

$1 = A + C$ ... ... (i)

Put the value of $x = -1$

$x = -1$

$1 = 2A$

$\therefore\ \frac{1}{2} = A$ and from (i), $C = \frac{1}{2}$

Putting $x = 1$

$1 = 2A + (B + C)\,2$

$1 = 2 \times \frac{1}{2} + \left(B + \frac{1}{2}\right) 2$

$\Rightarrow 1 = 1 + 2B + 1$

$\Rightarrow B = \frac{-1}{2}$

$$\therefore I = \frac{1}{2}\int \frac{1}{(x+1)} + \int \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1}\,dx = \frac{1}{2}\ln(x+1) - \frac{1}{2} \times \frac{1}{2}\int \frac{2x+2}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln(x+1) - \frac{1}{4}\ln(x^2+1) + 2\int \frac{1}{x^2+1}\,dx = \frac{1}{2}\ln(x+1) - \frac{1}{4}\ln(x^2+1) + 2\tan^{-1}x +$$

c

7. $\int \frac{7x^2 - 18x + 13}{(x-3)(x^2+2)} dx$

Let, $\frac{7x^2 - 18x + 13}{(x-3)(x^2+2)} = \frac{A}{(x-3)} + \frac{Bx + C}{(x^2+2)}$

$7x^2 - 18x + 13 = (x^2 + 2)A + (Bx + C)(x - 3)$

Put $x = 0$

$13 = 2A - 3C$ ... ... (i)

Again put $x = 3$

$7 \times 3^2 - 18 \times 3 + 13 = 11A$

or, $63 - 54 + 13 = 11A$

or, $22 = 11A$

$\therefore$ $A = 2$

Put the value of $A = 2$ in equation (i)

$13 = 2 \times 2 - 3C$

or, $\frac{13 - 4}{3} = -C$

$\therefore$ $C = -3$

Put $x = 1$

$2 = 3A - 2(B + C)$

$2 = 6 - 2(B - 3)$

$-4 = -2(B - 3)$

or, $2 = B - 3$

$\therefore$ $B = 5$

$= \int \frac{7x^2 - 18x + 13}{(x-3)(x^2+2)} dx = 2\int \frac{1}{x-3} dx + \int \frac{5x - 3}{x^2 + 2} dx$

$= 2\ln(x-3) + \frac{5}{2}\int \frac{2x - \frac{3}{5} \times 2}{x^2 + 2} dx = 2\ln(x-3) + \frac{5}{2}\int \frac{2x}{x^2+2} dx - \frac{6}{5} \times \frac{5}{2}\int \frac{1}{x^2+2} dx$

$= 2\ln(x-3) + \frac{5}{2}\ln(x^2+2) - \frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$

8. $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$

Let $I = \int \frac{x^2 - 1}{x^4 + x^2 + 1} dx = \int \frac{1 - \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx = \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)^2 - 1} dx$

Put $x + \frac{1}{x} = y$

$\therefore$ $\left(1 - \frac{1}{x^2}\right) dx = dy$

$I = \int \frac{dy}{y^2 - 1} = \frac{1}{2.1}\ln\frac{y-1}{y+1} + C = \frac{1}{2}\ln\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1} + C = \frac{1}{2}\ln\frac{x^2 - x + 1}{x^2 + x + 1} + C$

9. $\int \frac{1}{x^4 - 1} dx$

Let $\frac{1}{x^4 - 1}$ dx = $\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx + D}{x^2 + 1}$

Put $1 = A(x + 1)(x^2 + 1) + B(x - 1)(x^2 + 1) + (cx + 1)(x^2 - 1)$

$x = 1 \qquad 1 = A.2.2 \qquad \therefore A = \frac{1}{4}$

$x = -1 \qquad 1 = B.(-2).2 \qquad \therefore B = \frac{-1}{4}$

$x = 0 \qquad 1 = A + B(-1) + (-1) D$

$1 = \frac{1}{4} + \frac{1}{4} - D$

$\therefore \quad D = -\frac{1}{2}$

Equating the coefficients of x^3

$0 = A + B + C$

or, $\frac{1}{4} - \frac{1}{4} + C$

$\therefore \quad C = 0$

$\therefore \quad \frac{1}{x^4 - 1} = \frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x^2+1}$

$$\int \frac{1}{x^4 - 1} dx = \int \left\{\frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x^2+1}\right\} dx$$

$$= \frac{1}{4} \ln (x - 1) - \frac{1}{4} \ln (x + 1) - \frac{1}{2} \tan^{-1} x + c = \frac{1}{4} \ln \frac{x-1}{x+1} - \frac{1}{2} \tan^{-1} x + c$$

10. $\int \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx$

$\frac{x^2}{(x^2 + a^2)(x^2 + b^2)} = \frac{y}{(y + a^2)(y + b^2)} = \frac{A}{y + a^2} + \frac{B}{y + b^2}$

where $y = x^2$

$y = A(y + b^2) + B(y + a^2)$

When $y = -a^2$

$-a^2 = A(-a^2 + b^2)$

$\therefore \quad A = \frac{a^2}{a^2 - b^2}$

When $y = -b^2$

$-b^2 = B(-b^2 + a^2)$

$\therefore \quad B = -\frac{b^2}{a^2 - b^2}$

$\frac{y}{(y + a^2)(y + b^2)} = \frac{a^2}{a^2 - b^2}\frac{1}{y + a^2} - \frac{b^2}{a^2 - b^2}\frac{1}{y + b^2}$

or, $\frac{x^2}{(x^2 + a^2)(x^2 + b^2)} = \frac{a^2}{a^2 - b^2}\frac{1}{x^2 + a^2} - \frac{b^2}{a^2 - b^2}\frac{1}{x^2 + b^2}$

or, $\int \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx = \int \left[\frac{-a^2}{a^2 - b^2}\frac{1}{x^2 + a^2} - \frac{b^2}{a^2 - b^2}\frac{1}{x^2 + b^2}\right] dx$

$$= \frac{a^2}{a^2 - b^2}\frac{1}{a} \tan^{-1}\frac{x}{a} - \frac{b^2}{a^2 - b^2}\frac{1}{b} \tan^{-1}\frac{x}{b} + c$$

$$= \frac{a}{a^2 - b^2} \tan^{-1}\frac{x}{a} - \frac{b}{a^2 - b^2} \tan^{-1}\frac{x}{b} + C$$

11. $\int \frac{x^2+4}{x^4+16}\,dx = \int \frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}}\,dx = \int \frac{1+\frac{4}{x^2}}{\left(x-\frac{4}{x}\right)^2+8}$

Put $x - \frac{4}{x} = y$

$\Rightarrow 1+\frac{4}{x^2}\,dx = dy$

$I = \int \frac{dy}{y^2+(2\sqrt{2})^2} = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{y}{2\sqrt{2}} + C = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x^2-4}{2\sqrt{2}} + C$

12. $\int \frac{x^3}{(x-a)(x-b)(x-c)}\,dx$

Let $\frac{x^3}{(x-a)(x-b)(x-c)} = 1 + \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{c}{(x-c)}$

$\Rightarrow \quad x^3 = (x-a)(x-b)(x-c) + A(x-b)(x-c) + B(x-a)(x-c) + c(x-a)(x-b)$ ... (i)

Putting x = a, x = b, x = c turn by turn, we get,

$A = \frac{a^2}{(a-b)(a-c)}$, $B = \frac{b^2}{(b-a)(b-c)}$, $c = \frac{c^2}{(c-a)(c-b)}$

Now, $\int \frac{x^3}{(x-a)(x-b)(x-c)}\,dx$

$I = 1 + \frac{a^2}{(a-b)(a-c)}\int \frac{1}{x-a}\,dx + \frac{b^2}{(b-a)(b-c)}\int \frac{1}{x-b}\,dx + \frac{c^2}{(c-a)(c-b)}$

$\int \frac{1}{x-c}\,dx$

$= x + \frac{a^2}{(a-b)(a-c)}\ln(x-a) + \frac{b^2}{(b-a)(b-c)}\ln(x-b) + \frac{c^2}{(c-a)(c-b)}\ln(x-c)$

$+ C$

13. $\int \frac{x^3\,dx}{2x^4-x^2-10}$

Put $x^2 = y$

$2xdx = dy$

$\therefore \quad dx = \frac{dy}{2x}$

Now, $\int \frac{x^3dx}{2x^4-x^2-10} = \int \frac{x^2\,xdx}{2(x^2)^2 = x^2 - 10} = \int \frac{y\,.\,x}{2y^2-y-10}\cdot\frac{dy}{2x}$

$= \frac{1}{2}\int \frac{y\,dy}{2y^2-5y+4y-10} = \frac{1}{2}\int \frac{y\,dy}{y(2y-5)+2(2y-5)} = \frac{1}{2}\int \frac{y\,dy}{(y+2)(2y-5)}$ ... (i)

Let $\frac{y}{(y+2)(2y-5)} = \frac{A}{(y+2)} + \frac{B}{(2y-5)}$

$\Rightarrow \quad y = A(2y-5) + B(y+2)$ ... (ii)

Putting y = –2 in (ii), we get,

$-2 = A(-4-5) + B \times 0 \Rightarrow A = \frac{2}{9}$

Again, putting $y = \frac{5}{2}$ in (ii), we get

$$\frac{5}{2} = A \times 0 + B\left(\frac{5}{2} + 2\right) \Rightarrow B = \frac{5}{9}$$

$$\therefore \quad \frac{y}{(y+2)(2y-5)} = \frac{2}{9(y+2)} + \frac{5}{9(2y-5)}$$

from (i) $\frac{1}{2}\int \frac{y\,dy}{(y+2)(2y-5)} = \frac{1}{9}\int \frac{1}{y+2}\,dy + \frac{5}{18}\int \frac{1}{2y-5}\,dy$

$$= \frac{1}{9}\ln(y+2) + \frac{5}{36}\ln(2y-5) + c = \frac{5}{36}\ln(2x^2-5) + \frac{1}{9}\ln(x^2+2) + c$$

14. $\int \frac{dx}{(x-1)^2(x-3)^2}$

Put $x - 1 = z(x-3)$

$\Rightarrow \quad x - zx = 1 - 3z \Rightarrow x = \frac{1-3z}{1-z}$

$$dx = \frac{(1-z) - 3 - (1-3z)\times(-1)}{(1-z)^2}\,dz$$

$$\Rightarrow dx = \frac{-3 + 3z + 1 - 3z}{(1-z)^2}\,dz = \frac{-2}{(1-z)^2}\,dz$$

Here, $\frac{1}{(x-1)^2(x-3)^2} = \frac{1}{z^2(x-3)^2(x-3)^2} = \frac{1}{z^2\left[\frac{1-3z}{1-z} - 3\right]^4}$

$$= \frac{1}{z^2\left[\frac{1-3z-3+3z}{1-z}\right]^4} = \frac{(1-z)^4}{-16z^2}$$

Now, $\int \frac{dx}{(x-1)^2(x-3)^2} = \int \frac{-2}{(1-z)^2} \times \frac{(1-z)^4}{-16z^2}$

$$= \frac{-1}{8}\int \frac{(1-z)^2}{z^2}\,dz = \frac{-1}{8}\int \frac{1-2z+z^2}{z^2}\,dz = \frac{-1}{8}\int\left(\frac{1}{z^2} - \frac{2}{z} + 1\right)dz$$

$$= \frac{-1}{8}\left[-\frac{1}{z} - 2\ln z + z\right] + c$$

$$= \frac{-1}{8}\left[-\left(\frac{x-3}{x-1}\right) - 2\ln\left(\frac{x-1}{x-3}\right) + \left(\frac{x-1}{x-3}\right)\right] + c$$

15. $I = \int \frac{dx}{(x-1)^2(x-4)^3}$

Put $x - 1 = z(x-3)$

$\therefore \quad x = \frac{3z-1}{z-1}$

or, $dx = \frac{3(z-1) - (3z-1)}{(z-1)^2}\,dz = \frac{3z - 3 - 3z + 1}{(z-1)^2}dz = \frac{-2}{(z-1)^2}\,dz$

Also, $\frac{1}{(x-1)^2(x-3)^3} = \frac{1}{z^2(x-3)^5} = \frac{1}{z^2\left(\frac{3z-1}{z-1} - 3\right)^5} = \frac{(z-1)^5}{32z^2}$

So, $I = \int \frac{(z-1)^5}{32z^2} \cdot \frac{-2}{(z-1)^2}\, dz = \frac{-1}{16} \int \frac{(z-1)^3}{z^2}\, dz$

$$= \frac{-1}{16} \int \frac{z^3 - 3z^2 + 3z - 1}{z^2} dz = \frac{-1}{16} \int \left(z - 3 + \frac{3}{z} - \frac{1}{z^2}\right) dz$$

$$= \frac{-1}{16} \left(\frac{z^2}{2} - 3z + 3\ln z + \frac{1}{z}\right) + C = \frac{-1}{32} z^2 + \frac{3}{16} z - \frac{3}{16} \ln z - \frac{1}{16} z + C$$

$$= \frac{-1}{32} \left(\frac{x-1}{x-3}\right)^2 + \frac{3}{16} \frac{x-1}{x-3} - \frac{3}{16} \ln \left|\frac{x-1}{x-3}\right| - \frac{1}{16} \left(\frac{x-1}{x-3}\right) + C$$

■

CHAPTER 17

DIFFERENTIAL EQUATIONS

EXERCISE 17.1

1. Determine the order and degree of the following differential equations.

a. $\frac{dy}{dx} = 4x$

b. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = 0$

c. $\frac{d^2y}{dx^2} = xe^x$

d. $x\frac{dy}{dx} + \frac{3}{\frac{dy}{dx}} = y^2$

e. $\frac{dy}{dx} = \sqrt{\frac{1-x^2}{1-y^2}}$

f. $\left[\frac{dy}{dx}\right] + 3y\left[\frac{d^2y}{dx^2}\right]^3 = 0$

Solution

a. Given, $\frac{dy}{dx} = 4x$

Here, $\frac{dy}{dx}$ is the first order derivative, so its order is 1

Here, the power of $\frac{dy}{dx}$ is 1. So its degree is 1

b. Given, $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = 0$

Here, $\frac{d^2y}{dx^2}$ is the second order derivative, so it's order is 2

Here, the power of $\frac{d^2y}{dx^2}$ is 1. So, it's degree is 1.

c. Given, $\frac{d^2y}{dx^2} = xe^x$

Here, $\frac{d^2y}{dx^2}$ is the second order derivative. So, its' order is 2

Here, the power of $\frac{d^2y}{dx^2}$ is 1, so its degree is 1.

d. Given, $x\frac{dy}{dx} + \frac{3}{\frac{dy}{dx}} = y^2 \Rightarrow x\left(\frac{dy}{dx}\right)^2 + 3 = y^2\frac{dy}{dx}$

Here, $\frac{dy}{dx}$ is the first order derivative, so it's order is 1, and power of $\frac{dy}{dx}$ is 2, so degree is 2.

e. $\frac{dy}{dx} = \sqrt{\frac{1-x^2}{1-y^2}}$

Here, $\frac{dy}{dx}$ is the first order derivative. So its order is 1.

Here, the power of $\frac{dy}{dx}$ is 1. So, its degree is 1.

f. Given, $\frac{dy}{dx} + 3y\left(\frac{d^2y}{dx^2}\right)^3 = 0$

Here, $\frac{d^2y}{dx^2}$ is the second order derivative. So, its order is 2.

Here, the power of $\frac{d^2y}{dx^2}$ is 3. So, its' degree is 3.

EXERCISE 17.2

1. Solve the following differential equations

a. $\frac{dy}{dx} = \frac{x+4}{y+2}$ b. $x^2dx+y^2dy = 0$ c. $y.\frac{dy}{dx} = \cos x$

d. $\frac{dy}{dx} = e^{x+y}$ e. $e^{x-y}dx+e^{y-x}dy = 0$ f. $\frac{dy}{dx} = e^{x-y}+x^3e^{-y}$

Solution:

a. $\frac{dy}{dx} = \frac{x+4}{y+2}$

$(y + 2)\, dy = (x + 4)\, dx$

Integrating both sides

$\int(y + 2)\, dy = \int(x + 4)\, dx$

or, $\frac{y^2}{2} + 2y = \frac{x^2}{2} + 4x + c$

or, $y^2 + 4y = x^2 + 8x + c$

b. $x^2dx + y^2dy = 0$

Integrating

$\int x^2dx + \int y^2dy = \int 0$

$\frac{x^3}{3} + \frac{y^3}{3} = \frac{c}{3}$

$\therefore\ x^3 + y^3 = c$

c. $y\frac{dy}{dx} = \cos x$

$ydy = \cos x\, dx$

Integrating both sides

$\int ydy = \int \cos x\, dx$

or, $\frac{y^2}{2} = \sin x + c'$ where c' is constant

$\therefore\ y^2 = 2\sin x + 2c'$

$\therefore\ y^2 = 2\sin x + c$

d. $\frac{dy}{dx} = e^{x+y}$

$\frac{dy}{dx} = e^x . e^y$

$e^{-y}\, dy = e^x\, dx$

Integrating

$\int e^{-y}\, dy = \int e^x\, dx$

$-e^{-y} = e^x + c$

or, $\frac{-1}{e^y} = e^x + c$

or, $-1 = e^y (e^x + c)$

$\therefore\ e^y(e^x + c) = -1$

e. $e^{x-y}\, dx + e^{y-x}\, dy = 0$

or, $e^x .e^{-y}\, dx + e^y . e^{-x}\, dy = 0$

or, $\frac{e^x}{e^y}dx + \frac{e^y}{e^x}\, dy = 0$

$e^{2x}\, dx + e^{2y}\, dy = 0$

Integrating

$\int e^{2x}\, dx + \int e^{2y}\, dy = \int 0$

$\frac{e^{2x}}{2} + \frac{e^{2y}}{2} = \frac{c}{2}$

$\therefore\ e^{2x} + e^{2y} = c$

f. $\frac{dy}{dx} = e^{x-y} + x^3e^{-y}$

$\frac{dy}{dx} = \frac{(e^x + x^3)}{e^y}$

$e^y\, dy = e^xdx + x^3dx$

Integrating

$\int e^ydy = \int e^xdx + \int x^3dx$

$e^y = e^x + \frac{x^4}{4} + c$

$\therefore\ 4e^x - 4e^y + x^4 + c = 0$

2. Solve the differential equations.

a. $(x+2)\frac{dy}{dx} = (y+2)$

b. $x\frac{dy}{dx} + y - 1 = 0$

c. $\cos x \cos y \frac{dy}{dx} = -\sin x \sin y$

d. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

e. $(e^y + 1)\cos x\, dx + e^y . \sin x . dy = 0$

f. $(xy^2+x)dx+(x^2y+y)dy = 0$

g. $\sqrt{1 + x^2}\, dy + \sqrt{1 + y^2}\, dx = 0$

h. $x\sqrt{1 - y^2}\, dx + y\sqrt{1 - x^2}\, dy = 0$

i. $(1-x^2)dy - xydx = xy^2dx$

Solution:

a. $(x + 2)\frac{dy}{dx} = y + 2$

or, $\left(\frac{1}{y+2}\right) dy = \frac{dx}{x+2}$

Integrating both sides

$\int \frac{1}{x+2} dx = \int \frac{1}{y+2} dy$

$\Rightarrow \ln(x + 2) = \ln(y + 2) + \ln c$

$\Rightarrow x + 2 = c(y + 2)$

b. $x\frac{dy}{dx} + y - 1 = 0$

$x\frac{dy}{dx} = (1 - y)$

or, $\frac{1}{1-y} dy = \frac{dx}{x}$

Integrating

$-\ln (l - y) = \ln x + \ln c$

$\ln(1 - y)^{-1} = \ln cx$

$\therefore\ cx = \frac{1}{1-y}$

or, $x(1 - y) = \frac{1}{c}$ $\therefore x(1 - y) = c$

c. $\text{Cos}x . \cos y \frac{dy}{dx} = -\sin x . \sin y$

$\frac{\cos y}{\sin y} dy = -\frac{\sin x}{\cos x} dx$

$\cot y\, dy = -\tan x\, dx$

Integrating both sides

$\int \cot y\, dy = -\int \tan x\, dx$

$\ln \sin y = \ln \cos x + \ln c$

$\sin y = c \cos x$

d. $\text{Sec}^2 x . \tan y dx + \sec^2 y \tan x\, dy = 0$

or, $\frac{\sec^2 x}{\tan x} = -\frac{\sec^2 y}{\tan y} dy$

Integrating both sides we get

$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = \int 0$

$\ln(\tan x) + \ln(\tan y) = \ln c$

$\ln(\tan x . \tan y) = \ln c$

$\therefore\ \tan x . \tan y = c$

e. $(e^y + 1) \cos x dx + e^y . \sin x\, dy = 0$

$\frac{\cos x\, dx}{\sin x} + \frac{e^y\, dy}{e^y + 1} = 0$

Integrating both sides

$\int \frac{\cos x}{\sin x} dx + \int \frac{e^y}{1+e^y} dy = \int 0$

$\ln \sin x + \ln (e^y + 1) = \ln c$

$\ln \sin x(1 + e^y) = \ln c$

$\therefore\ \sin x (1 + e^y) = c$

f. $(xy^2 + x)\, dx + (x^2y + y)\, dy = 0$

or, $x(1 + y^2)\, dx + y(1 + x^2)\, dy = 0$

or, $\frac{x\, dx}{1+x^2} + \frac{y}{1+y^2} dy = 0$

or, $\left(\frac{2x}{1+x^2}\right) dx + \left(\frac{2y}{1+y^2}\right) dy = 0$

Integrating both sides

$\int \frac{2x}{1+x^2} + \int \frac{2y}{1+y^2} dy = \int 0$

$\ln(1 + x^2) + \ln(1 + y^2) = \ln c$

$\therefore\ \ln(1 + x^2)(1 + y^2) = \ln c$

$\therefore\ (1 + x^2)(1 + y^2) = c$

g. $\sqrt{1+x^2}\, dy + \sqrt{1 + y^2}\, dx = 0$

or, $\frac{dy}{\sqrt{1 + y^2}} + \frac{dx}{\sqrt{1 + x^2}} = 0$

Integrating

h. $x\sqrt{1 - y^2}\, dx + y\sqrt{1 - x^2}\, dy = 0$

$\frac{x}{\sqrt{1 - x^2}} dx + \frac{y}{\sqrt{1 - y^2}} dy = 0$

Put $1-x^2 = u$ and $1-y^2 = v$

$$\int \frac{1}{\sqrt{1+y^2}}\,dy + \int \frac{1}{\sqrt{1+x^2}}\,dx = \int 0$$

or, $\ln(y + \sqrt{1+y^2}) + \ln(x + \sqrt{1+x^2}) = \ln c$

or, $\ln\{(x + \sqrt{1+x^2})(y + \sqrt{1+y^2})\} = \ln c$

or, $(x + \sqrt{1+x^2})(y + \sqrt{1+y^2}) = c$

Then, $-2xdx = du$, $-2ydy = dv$

So, $\dfrac{(-2x)dx}{\sqrt{1-x^2}} + \dfrac{(-2y)\,dy}{\sqrt{1-y^2}} = 0$

$\dfrac{dy}{\sqrt{u}} + \dfrac{dv}{\sqrt{v}} = 0$

$u^{-1/2}\,du + v^{-1/2}\,dv = 0$

Integrating

$2u^{1/2} + 2v^{1/2} = 2c$

$\therefore\ \sqrt{u} + \sqrt{v} = c$

$\therefore\ \sqrt{1-x^2} + \sqrt{1-y^2} = c$

i. $(1-x^2)dy + xydx = x^2ydx$

or, $(1-x^2)dy = xy(1+y)dx$

or, $\dfrac{dy}{y(1+y)} = \dfrac{x\,dx}{1-x^2}$

or, $\left(\dfrac{1}{y} - \dfrac{1}{y+1}\right) dy = \dfrac{1}{-2}\left(\dfrac{-2x}{1-x^2}\right) dx$

Integrating

$\ln y - \ln(y+1) = -\dfrac{1}{2}\ln(1-x^2) + \ln c$

$\ln\left(\dfrac{y}{1+y}\right) = \ln(1-x^2)^{-1/2} + \ln c \Rightarrow \dfrac{y}{1+y} = c(1-x^2)^{-1/2}$

3. Solve the followings:

a. $\sec^2y\,(1+x^2)\,.dy + 2x\tan y\,dx = 0$ and $y(1) = \dfrac{\pi}{4}$.

b. $\cos y dy + \cos x \sin y dx = 0$, $y\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2}$.

Solution:

a. $\sec^2y\,(1+x^2)dy + 2x.\tan y\,dx = 0$

or, $\dfrac{\sec^2y}{\tan y}\,dy + \dfrac{2x}{1+x^2}\,dx = 0$

Integrating, we get,

$\ln(\tan y) + \ln(1+x^2) = \ln c$

$(1+x^2)\tan y = c$

When, $x = 1$, there $y = \dfrac{\pi}{4}$

$\therefore\ (1+1)\tan\dfrac{\pi}{4} = c$

$2.1 = c \ \therefore\ c = 2$

Hence, $(1+x^2)\tan y = 2$

b. $\cos y\,dy + \cos x \sin y\,dx = 0$

$\cos y\,dy = -\cos x.\sin y\,dx$

$\dfrac{\cos y}{\sin y}\,dy = -\cos x\,dx$

Integrating, we get

$\ln\sin y = -\sin x + c$

$\sin x + \ln\sin y = c$

When $x = \dfrac{\pi}{2}$ then $y = \dfrac{\pi}{2}$

then $c = 1$

$\therefore\ \sin x + \ln\sin y = 1$

EXERCISE 17.3

1. Solve the following differential equations.

a. $x\dfrac{dy}{dx} = y + x$.

b. $\dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{y^2}{x^2}$

c. $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$.

d. $x\left(\dfrac{dy}{dx} + \tan\dfrac{y}{x}\right) = y$.

e. $\dfrac{dy}{dx} - \dfrac{y}{x} - \sin\dfrac{y}{x} = 0$

f. $\dfrac{dy}{dx} = \dfrac{y}{x} + \cos^2\left(\dfrac{y}{x}\right)$

Solution:

a. $x\dfrac{dy}{dx} = y + x$

$\frac{dy}{dx} = \frac{y}{x} + 1$ ... ... (i) It is a homogenous

Thus, differential equation

Put y = vx

Then $\frac{dy}{dx} = v + x\frac{dy}{dx}$

(i) becomes

$v + x\frac{dv}{dx} = v + 1$

$\therefore \quad x\frac{dv}{dx} = 1$

or, $dv = \frac{dx}{x}$

Integrating both sides

$\int dv = \int \frac{1}{x}\,dx$

$v = \ln x + c$

$\frac{y}{x} = \ln x + c$

b. $\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$

It is homogeneous differential equation.

Put y = vx

$\frac{dy}{dx} = v + x.\frac{dv}{dx}$

Then $v + x.\frac{dv}{dx} = v^2 + v$

$x\frac{dv}{dx} = v^2$

$v^{-2}\,dv = \frac{1}{x}\,dx$

Integrating both sides

$\int v^{-2}dv = \int \frac{1}{x}\,dx$

$-v^{-1} = \ln x + \ln c$

$-\frac{1}{v} = \ln(cx)$

$-\frac{x}{y} = \ln(cx)$

$\Rightarrow \quad y\ln(cx) + x = 0$

c. $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ ... ... (i)

It is homogenous differential equation,

Put y = vx then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(i) becomes, $v + x\frac{dv}{dx} = v + \tan v$

$x\frac{dv}{dx} = \tan v$

$\cot v\,dv = \frac{1}{x}\,dx$

Integrating $\int \cot v\,dv = \int \frac{1}{x}\,dx$

$\ln \sin v = \ln cx$

$\sin\left(\frac{y}{x}\right) = cx$

d. $x\left(\frac{dy}{dx} + \tan\frac{y}{x}\right) = y$

or, $\frac{dy}{dx} + \tan\frac{y}{x} = \frac{y}{x}$ ... ... (i)

Put y = vx then $\frac{dy}{dx} = v + x\,.\,\frac{dv}{dx}$

$v + x\frac{dv}{dx} + \tan v = v$

e. $\frac{dy}{dx} - \frac{y}{x} - \sin\frac{y}{x} = 0$ ... ... (i)

Put y = vx then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(i) becomes,

$v + x\,.\,\frac{dv}{dx} - v - \sin v = 0$

$x\frac{dv}{dx} = -\tan v$

$-\cot v\, dv = \frac{dx}{x}$

Integrating

$\int -\cot v\, dv = \int \frac{1}{x} dx$

$\Rightarrow$ $-\ln(\sin v) = \ln x + \ln c$

$\Rightarrow$ $\ln(\sin v)^{-1} = \ln cx$

$\Rightarrow$ $\frac{1}{\sin v} = cx$

$\Rightarrow$ $\operatorname{cosec} v = cx$

$\therefore$ $\operatorname{cosec}\left(\frac{y}{x}\right) = cx$

$x\frac{dv}{dx} = \sin v$

$\frac{dv}{\sin v} = \frac{dx}{x}$

$\operatorname{cosec} v\, dv = \frac{1}{x} dx$

Integrating

$\int \operatorname{cosec} v\, dv = \int \frac{1}{x} dx$

$\ln(\operatorname{cosec} v - \cot v) = \ln x + \ln c$

$\operatorname{cosec} v - \cot v = cx$

or, $\frac{1}{\sin v} - \frac{\cos v}{\sin v} = cx$

or, $\frac{1 - \cos v}{\sin v} = cx$

or, $\tan\frac{v}{2} = cx \therefore \tan\frac{y}{2x} = cx$

f. $\frac{dy}{dx} = \frac{y}{x} + \cos^2\left(\frac{y}{x}\right)$

Put $y = vx$

$\frac{dy}{dx} = v + x.\frac{dv}{dx}$

$v + x.\frac{dv}{dx} = v + \cos^2 v$

$x\frac{dv}{dx} = \cos^2 v$

$\sec^2 v\, dv = \frac{1}{x} dx$

Integrating

$\int \sec^2 v\, dx = \int \frac{1}{x} dx$

$\tan v = \ln(x) + c$

$\tan\left(\frac{y}{x}\right) = \ln(x) + c$

$\tan\left(\frac{y}{x}\right) = \ln(x) + c$

2. Solve the followings.

a. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

b. $x^2y dx - (x^3+y^3)dy = 0$

c. $(x^2 + y^2)\, dx - 2xydy = 0$

d. $(x + y)\, dx + (y - x)\, dy = 0$

e. $(x + y)^2 dx = xy\, dy$

f. $x^2\frac{dy}{dx} = \frac{y(x + y)}{2}$

Solution:

a. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$y^2 = (xy - x^2)\frac{dy}{dx}$

b. $x^2ydx - (x^3 + y^3)\, dy = c$

$\frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}$ ... ... (i)

Put $y = vx$

$\therefore \ \dfrac{dy}{dx} = \dfrac{y^2}{xy - x^2}$ ... ... (i)

Put $y = vx$

Then, $v + x\dfrac{dv}{dx} = \dfrac{v^2x^2}{x^2v - x^2}$

$v + x\dfrac{dv}{dx} = \dfrac{v^2}{v-1}$

$x\dfrac{dv}{dx} = \dfrac{v^2}{v-1} - v$

$x\dfrac{dv}{dx} = \dfrac{v^2 - v^2 + v}{v-1}$

$x\dfrac{dv}{dx} = \dfrac{v}{v-1}$

$\dfrac{(v-1)}{v}\,dv = \dfrac{1}{x}\,dx$

Integrating

$v - \ln v = \ln x + \ln c$

$v = \ln(cxv)$

$\dfrac{y}{x} = \ln(cy)$

c. $(x^2 + y^2)\,dx - 2xy\,dy = 0$

$\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$ ... ... (i)

$v + x\dfrac{dv}{dx} = \dfrac{1 + v^2}{2v}$

$x\dfrac{dv}{dx} = \dfrac{1 + v^2 - 2v^2}{2v}$

$x\dfrac{dv}{dx} = \dfrac{1 - v^2}{2v}$

$\left(\dfrac{2v}{1-v^2}\right) dv = \dfrac{dx}{x}$

$-\ln(1-v^2) = \ln x + \ln c$

$\ln(1-v^2)^{-1} = \ln(cx)$

$(1 - v^2)^{-1} = cx$

$\dfrac{1}{1 - v^2} = cx$

$\dfrac{x^2}{x^2 - y^2} = cx$

$\dfrac{x}{x^2 - y^2} = c$

$\therefore \ x^2 - y^2 = \dfrac{x}{c}$

$\therefore \ x^2 - y^2 = cx$

e. $(x + y)^2\,dx = xy\,dy$

$\dfrac{dy}{dx} = v + x.\dfrac{dv}{dx}$

(i) becomes

$v + x.\dfrac{dv}{dx} = \dfrac{v}{1 + v^3}$

$x\dfrac{dv}{dx} = \dfrac{v - v - v^4}{1 + v^3}$

$x\dfrac{dv}{dx} = \dfrac{-v^4}{1+v^3}$

$\dfrac{(1+v^3)}{v^4}\,dv = -\dfrac{dx}{x}$

$v^{-4}\,dv + \dfrac{1}{v}\,dv = -\dfrac{1}{x}\,dx$

Integrating

$\dfrac{v^{-3}}{-3} + \ln v = -\ln x - \ln c$

$\dfrac{1}{3v^3} = \ln(cxv)$

$\dfrac{x^3}{3y^3} = \ln(cy)$

d. $(x + y)\,dx + (y - x)\,dy = 0$

$(y - x)\,dy = -(x + y)\,dx$

or, $\dfrac{dy}{dx} = -\dfrac{(x + y)}{y - x} \ \therefore \ \dfrac{dy}{dx} = \dfrac{x + y}{x - y}$ ... (i)

Put $y = vx$, then $\dfrac{dy}{dx} = v + x.\dfrac{dv}{dx}$

Then (i) reduces to

or, $v + x\,.\dfrac{dv}{dx} = \dfrac{x + vx}{x - vx}$

or, $v + x\dfrac{dv}{dx} = \dfrac{1 + v}{1 - v}$

or, $x\dfrac{dv}{dx} = \dfrac{1 + v}{1 - v} - v$

or, $x\dfrac{dv}{dx} = \dfrac{1 + v - v + v^2}{1 - v}$

or, $x\dfrac{dv}{dx} = \dfrac{1 + v^2}{1-v}$

or, $\left(\dfrac{1 - v}{1 + v^2}\right) dv = \dfrac{dx}{x}$

or, $\dfrac{1}{1+v^2}\,dv - \dfrac{1}{2}\left(\dfrac{2v}{1 + v^2}\right) dv = \dfrac{1}{x}\,dx$

Integrating: $\tan^{-1} v - \dfrac{1}{2}\ln(1 + v^2) = \ln x + c$

$\tan^{-1}\left(\dfrac{y}{x}\right) = \ln x + \ln\sqrt{1 + v^2} + c$

$\tan^{-1}\left(\dfrac{y}{x}\right) = \ln\sqrt{x^2 + y^2} + c$

f. $x^2\dfrac{dy}{dx} = \dfrac{y(x + y)}{2}$ or, $\dfrac{dy}{dx} = \dfrac{y(x + y)}{2x^2}$ ... (i)

$$\frac{dy}{dx} = \frac{(x+y)^2}{xy}$$

$$v + x\frac{dv}{dx} = \frac{(x+vx)^2}{x^2v}$$

$$v + x\frac{dv}{dx} = \frac{1+v^2}{v}$$

$$x\frac{dv}{dx} = \frac{(1+v)^2 - v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1+2v}{v}$$

$$\frac{v}{1+2v}\,dv = \frac{dx}{x}$$

$$\frac{1}{2}\frac{(1+2v-1)}{1+2v}\,dv = \frac{1}{x}\,dx$$

$$\frac{1}{2}\left(1 - \frac{1}{1+2v}\right)dv = \frac{1}{x}\,dx$$

$$\frac{1}{2}\,dv - \frac{1}{4}\left(\frac{2}{1+2v}\right)dv = \frac{1}{x}\,dx$$

Integrating

$$\frac{1}{2}v - \frac{1}{4}\ln(1+2v) = \ln(cx)$$

$$\frac{1}{2}\left(\frac{y}{x}\right) - \frac{1}{4}\ln\left(1 + \frac{2y}{x}\right) = \ln(cx)$$

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(i) becomes,

or, $v + x\,.\,\frac{dv}{dx} = \frac{vx(x+vx)}{2x^2}$

or, $v + x\,.\,\frac{dv}{dx} = \frac{v(1+v)}{2}$

or, $x\frac{dv}{dx} = \frac{v+v^2}{2} - v$

or, $x\frac{dv}{dx} = \frac{v+v^2-2v}{2}$

or, $x\frac{dv}{dx} = \frac{v^2-v}{2}$

or, $\frac{1}{v(v-1)}\,dv = \frac{dx}{2x}$

or, $\left(\frac{1}{v-1} - \frac{1}{v}\right)dv = \frac{1}{2}\frac{1}{x}\,dx$

Integrating: $\ln(v-1) - \ln(v) = \frac{1}{2}\ln x + \ln c$

$\ln\left(\frac{v-1}{v}\right)\ \ln(c\sqrt{x})$

$\therefore\ \frac{v-1}{v} = c\sqrt{x}$

$1 - \frac{x}{y} = c\sqrt{x} \Rightarrow y - x = cy\sqrt{x}$

$\therefore\ (y-x)^2 = cxy^2$

3. Solve the followings by reducing them into homogenous form.

a. $\frac{dy}{dx} = \frac{y+1}{x+y+1}$ b. $\frac{dy}{dx} = \frac{y+x+1}{x+1}$

Solution:

a. $\frac{dy}{dx} = \frac{y+1}{x+y+1}$

Put $y + 1 = vx$

Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\,.\,\frac{dv}{dx} = \frac{vx}{x+vx}$$

$$v + x\frac{dv}{dx} = \frac{v}{1+v}$$

$$x\frac{dv}{dx} = \frac{v}{1+v} - v$$

$$x\frac{dv}{dx} = \frac{v-v-v^2}{1+v}$$

$$\frac{(1+v)}{v^2}\,dv = -\frac{dx}{x}$$

$$\left(v^{-2} + \frac{1}{v}\right)dv = -\frac{1}{x}\,dx$$

Integrating

$$-v^{-1} + \ln v = -\ln x - \ln C$$

b. $\frac{dy}{dx} = \frac{y+x+1}{x+1}$

Put $y = v(x+1)$

$$\frac{dy}{dx} = v + (x+1)\frac{dv}{dx}$$

So, $v + (x+1)\frac{dv}{dx} = v + 1$

$$(x+1)\frac{dv}{dx} = 1$$

$$dv = \frac{dx}{x+1}$$

Integrating

$$\int dv = \int\frac{1}{x+1}\,dx$$

$$v = \ln(x+1) + c$$

$$\frac{y}{x+1} = \ln(x+1) + c$$

$\therefore\ y = (x+1)\{\ln(x+1) + c\}$

$\frac{1}{v} - \text{lhv} = \ln Cx$

$\Rightarrow \frac{x}{y+1} - \ln\left(\frac{y+1}{x}\right) = \ln Cx$

$\Rightarrow \frac{x}{y+1} = \ln\left(\frac{y+1}{x} . Cx\right)$

$\Rightarrow \frac{x}{y+1} = \ln[C(y+1)]$

EXERCISE 17.4

1. Solve the following equations by reducing them into exact form.

a. $x\,dx - y dy = 0$

b. $x\,dy - y dx = 0$

c. $(x + y^2)\ dx = 2xydy.$

d. $y\,dx - \frac{x}{2}\,dy = 0$

e. $\frac{1}{x+1}\,dx + \frac{1}{y+1}\,dy = 0$

f. $\frac{1}{1+x^2}\,dx + \frac{1}{1+y^2}\,dy = 0$

g. $\frac{x}{1+x^2}\,dx + \frac{y}{1+y^2}\,dy = 0$

h. $(x+y)\,dy + (y - x)\,dx = 0.$

i. $2xy\,dx + (x^2 - y^2)\,dy = 0$

j. $(x^2 + xy^2)\,dx + (x^2y + y^2)\,dy = 0$

Solution:

a. $xdx - ydy = 0$

$\Rightarrow d\left(\frac{x^2}{2} - \frac{y^2}{2}\right) = 0$

Integrating both sides

$\int xdx - \int ydy = \int 0$

$\frac{x^2}{2} - \frac{y^2}{2} = \frac{c}{2}$

$x^2 - y^2 = c$

b. $xdy - ydx = 0$

$\frac{xdy - ydx}{x^2} = 0$

$d\left(\frac{y}{x}\right) = 0$

Integrating

$\int d\left(\frac{y}{x}\right) = \int 0$

$\frac{y}{x} = c$

$\therefore\ y = cx$

c. $(x + y^2)dx = 2xydy$

$\frac{xdy}{x^2} = \frac{2xy\,dy - y^2dx}{x^2}$

$\frac{1}{x}\,dx = \frac{2xy\,dy - y^2dx}{x^2}$

$\frac{1}{x}\,dx = d\left(\frac{y^2}{x}\right) \Rightarrow d\ln x = d\left(\frac{y^2}{x}\right)$

Integrating we get

$\ln x = \frac{y^2}{x} + c$

$x\ln x = y^2 + cx$

d. $ydx - \frac{x}{2}\,dy = 0$

$ydx = \frac{x}{2}\,dy$

or, $2\frac{dx}{x} = \frac{dy}{y}$

$\int 2d\ln x = \int d\ln y$

$\ln y = 2\ln x + \ln c$

$\ln y = \ln x^2c$

$y = cx^2$

e. $\frac{1}{x+1}\,dx + \frac{1}{y+1}\,dy = 0$

Integrating

f. $\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0$

Integrating

$$\int \frac{1}{x+1}\,dx \times \int \frac{1}{y+1}\,dy = \int 0$$

or, $\ln(x + 1) + \ln(y + 1) = \ln c$

or, $\ln(x + 1)(y + 1) = \ln c$

$\therefore$ $(x + 1)(y + 1) = c$

$$\int \frac{dy}{1+x^2} + \int \frac{dy}{1+y^2} = \int 0$$

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}c$

or, $\tan^{-1}\left(\frac{x + y}{1 - xy}\right) = \tan^{-1}c$

$\therefore$ $\frac{x + y}{1 - xy} = c$

$\therefore$ $x + y = c(1 - xy)$

g. $\frac{x}{1+x^2}\,dx + \frac{y}{1+y^2}\,dy = 0$

or, $\frac{2x}{1+x^2}\,dx + \frac{2y}{1+y^2}\,dy = 0$

Integrating

$\ln(1 + x^2) + \ln(1 + y^2) = \ln c$

$\therefore$ $\ln(1 + x^2)(1 + y^2) = \ln c$

$\therefore$ $(1 + x^2)(1 + y^2) = c$

h. $(x + y)\,dy + (y - x)\,dx = 0$

$xdy + ydy + ydx - xdx = 0$

$(xdy + ydx) + ydy - xdx = 0$

$d(xy) + ydy - xdx = 0$

Integrating

$$\int d(xy) + \int ydy - \int xdx = \int 0$$

$xy + \frac{y^2}{2} - \frac{x^2}{2} = \frac{c}{2}$

$2xy + y^2 - x^2 = c$

i. $2xy\,dx + (x^2 - y^2)\,dy = 0$

$2xydx + x^2dy - y^2dy = 0$

$d(x^2y) - y^2dy = 0$

Integrating, we get

$x^2y - \frac{y^3}{3} = c$

$3x^2y - y^3 = c$

j. $(x^2 + xy^2)\,dx + (x^2y + y^2)\,dy = 0$

or, $x^2dx + xy^2dx + x^2y\,dy + y^2dy$

or, $x^2dx + (xy^2dx + x^2ydy) + y^2dy$

or, $x^2dx + \frac{1}{2}\,d(x^2y^2) + y^2dy = 0$

Integrating

$\frac{x^3}{3} + \frac{1}{2}(x^2y^2) + \frac{y^3}{3} = c$

$2x^3 + 3x^2y^2 + 2y^3 = c$

2. Solve the following equations by reducing them into exact form.

a. $\cos x \cos y\,dy - \sin x \sin y\,dx = 0$

b. $\sin x \cos x\,dx - \sin y \cos y dy = 0.$

c. $\frac{dy}{dx} = \frac{1 - \cos y}{1 + \cos x}$

d. $\frac{dy}{dx} = \frac{4x - y + 5}{x + 5y + 4}$

e. $\frac{dy}{dx} = \frac{x + y + 1}{2y - x + 2}$

f. $\frac{dy}{dx} = \frac{x - 3(y + 1)}{y + 3(x + 1)}$

g. $(\sin x \tan y - 1)dx - \cos x \sec^2 y\,dy = 0$

Solution:

a. $\cos x.\cos y\,dy - \sin x.\sin y\,dx = 0$

$\cos x.\cos y dy = \sin x.\sin y\,dx$

$\frac{\cos y}{\sin y}\,dy - \frac{\sin x}{\cos x}\,dx = 0$

$\frac{\cos y}{\sin y}\,dy - \frac{\sin x}{\cos x}\,dx = 0$

Integrating

$$\int \frac{\cos y}{\sin y}\,dy - \int \frac{\sin x}{\cos x}\,dx = \int 0$$

$\ln(\sin y) + \ln(\cos x) = \ln c$

b. $\sin x \cos x\,dx - \sin y.\cos y\,dy = 0$

$\frac{1}{2}\sin 2x\,dx - \frac{1}{2}\sin 2y\,dy = 0$

or, $\sin 2x\,dx - \sin 2y dy = 0$

Integrating

$$\int \sin 2x\,dx - \int \sin 2y dy = \int 0$$

$\frac{-\cos 2x}{2} + \frac{\cos 2y}{2} = \frac{c}{2}$

$\therefore$ $\cos 2y - \cos 2x = c$

$\ln(\sin y.\cos x) = \ln c$

$\sin y.\cos x = c$

c. $\dfrac{dy}{dx} = \dfrac{1-\cos y}{1+\cos x}$

$\dfrac{dy}{dx} = \dfrac{2\sin^2 y/2}{2\cos^2 x/2}$

or, $\operatorname{cosec}^2 \dfrac{y}{2}\, dy = \sec^2 \dfrac{x}{2}\, dx$

Integrating

$\int \operatorname{cosec}^2 \dfrac{y}{2}\, dy = \int \sec^2 \dfrac{x}{2}\, dx$

or, $\dfrac{-\cot \frac{y}{2}}{\frac{1}{2}} = \dfrac{\tan \frac{x}{2}}{\frac{1}{2}} + \dfrac{c}{\frac{1}{2}}$

$\therefore$ $-\cos \dfrac{y}{2} = \tan \dfrac{x}{2} + c$

d. $\dfrac{dy}{dx} = \dfrac{4x - y + 5}{x + 5y + 4}$

or, $4xdx - ydx + 5dx = xdy + 5ydy + 4dy$

or, $4xdx + 5dx = (xdy + ydx) + 5ydy + 4dy$

$4xdx + 5dx = d(xy) + 5ydy + 4dy$

Integrating both sides

$\int 4xdx + 5\int dx = \int d(xy) + 5\int ydy + 4\int dy$

$\dfrac{4x^2}{2} + 5x = xy + \dfrac{5y^2}{2} + 4y + c$

$4x^2 + 10x = 2xy + 5y^2 + 8y + c$

$4x^2 - 5y^2 + 10x - 8y - 2xy = c$

e. $\dfrac{dy}{dx} = \dfrac{x + y + 1}{2y - x + 2}$

or, $2ydy - xdy + 2dy = xdx + ydx + dx$

$2ydy + 2dy = xdx + ydx + xdy + dx$

$2ydy + 2dy = xdx + d(xy) + dx$

Integrating

$\int 2ydy + \int 2dy = \int xdx + \int d(xy) + \int dx$

$y^2 + 2y = \dfrac{x^2}{2} + xy + x + c$

$2y^2 + 4y = x^2 + 2xy + 2x + c$

$2y^2 - x^2 - 2xy + 4y - 2x = c$

f. $\dfrac{dy}{dx} = \dfrac{x - 3(y+1)}{y + 3(x+1)}$

or, $ydy + 3(x + 1)\, dy = xdx - 3\,(y + 1)\, dx$

or, $ydy + 3xdy + 3dy = xdx - 3ydx - 3dx$

or, $ydy + 3(xdy + ydx) + 3dy = xdx - 3dy$

$ydy + 3d(xy) + 3dy = (x - 3)\, dx$

Integrating we get

$\dfrac{y^2}{2} + 3xy + 3y = \dfrac{x^2}{2} - 3x + c$

$\therefore$ $y^2 + 6xy + 6y = x^2 - 6x + c$

g. $(\sin x.\ \tan y - 1)\, dx = \cos x.\sec^2 y\, dy = 0$

$\sin x.\tan y\, dx - \cos x.\sec^2 y\, dy = dx$

$-d(\cos x.\tan y) = dx$

$d(\cos x.\tan y) + dx = 0$

Integrating both sides

$\int d(\cos x.\tan y) + \int dx = \int 0$

$\cos x.\tan y + x = c$

EXERCISE 17.5

1. Solve the following differential equations.

a. $(1-x^2)\dfrac{dy}{dx} - xy = 1.$

b. $\sec x \dfrac{dy}{dx} - y = \sin x.$

c. $\cos^2 x \dfrac{dy}{dx} + y = 1.$

d. $\sin x \dfrac{dy}{dx} + y \cos x = \sin 2x$

Solution:

a. $(1-x^2)\frac{dy}{dx} - x.y = 1$

$\therefore \quad \frac{dy}{dX} + \left(\frac{-x}{1-x^2}\right) y = \frac{1}{1-x^2}$ ... ... (i)

Comparing (i) with $\frac{dy}{dx} + py = Q$, we get

$p = \frac{-x}{1-x^2}$ and $Q = \frac{1}{1-x^2}$

$\int pdx = \frac{1}{2}\int \frac{-2x}{1-x^2}\,dx$

$= \frac{1}{2}\ln(1-x^2) = \ln\sqrt{1-x^2}$

I.F. $= e^{\int pdx} = e^{\ln\sqrt{(1-x^2)}} = \sqrt{1-x^2}$

Multiplying (i) both sides by I.F.

$\left[\frac{dy}{dx} + \left(\frac{-x}{1-x^2}\right)y\right]\sqrt{1-x^2} = \frac{1}{(1-x^2)}\sqrt{1-x^2}$

$d(y\,.\sqrt{1-x^2}) = \frac{1}{\sqrt{1-x^2}}$

Integrating

$\int d(y.\sqrt{1-x^2}) = \int \frac{1}{\sqrt{1-x^2}}\,dx$

$y.\sqrt{1-x^2} = \sin^{-1}x + c$

b. $\sec x\,.\frac{dy}{dx} - y = \sin x$

$\frac{dy}{dx} - \cos x\,.\,y = \sin x\,.\,\cos x$ ... ... (i)

Comparing (i) with $\frac{dy}{dx} + p.y = Q$, we get

$p = -\cos x$ and $Q = \sin x.\cos x$

$\int pdx = -\int \cos x dx = -\sin x$

I.F. $= e^{\int pdx} = e^{-\sin x}$

Multiplying (i) both sides by I.F.

$d(y.e^{\sin x}) = e^{-\sin x}\,.\,\sin x\,.\,\cos x\,dx$

Integrating

$\int d(y.e^{-\sin x}) = \int e^{-\sin x}\,.\,\sin x.\cos x\,dx$

$y.e^{-\sin x} = \int e^{-u}\,.\,u\,du$ where $\sin x = u$

$y.e^{-\sin x} = -u\,e^{-u} - e^{-u} + c$

$y.e^{-\sin x} = (-1-u)\,e^{-u} = + c$

$y.e^{-\sin x} = (-1 - \sin x)\,e^{-\sin x} + c$

$\therefore \quad y = (1 - \sin x) + c\,e^{\sin x}$

$\therefore \quad y + 1 + \sin x = c\,e^{\sin x}$

c. $\cos^2x\frac{dy}{dx} + y = 1$

$\frac{dy}{dx} + \sec^2x.y = \sec^2x$ ... ... (i)

Comparing (i) with $\frac{dy}{dx} + p.y = Q$, we get

$\sec^2x$ and $Q = \sec^2x$

$\int pdx = \int \sec^2x\,dx = \tan x$

Integrating factor (I.F.) $= e^{\int pdx} = e^{\tan x}$

Multiplying (i) both sides by I.F.

$\left(\frac{dy}{dx} + \sec^2 x.y\right) e^{\tan x} = \sec^2x\,e^{\tan x}$

$d(ye^{\tan x}) = e^{\tan x}\sec^2x$

Integrating both sides,

$y.e^{\tan x} = \int e^{\tan x}\,.\,\sec^2x\,dx$

d. $\frac{dy}{dx} + \cos x\,y = 2\cos x$

Here, $p = \cos x$ and $q = 2\cos x$

$\int pdx = \int \cos x\,dx = \ln\sin x$

I.F. $= e^{\int pdx} = e^{\ln\sin x} = \sin x$

Multiplying (i) by I.F. we get

$d(y.\sin x) = 2\cos x\,.\,\sin x$

Integrating

$\int d(y.\sin x) = \int \sin 2x\,dx$

$y.\sin x = -\frac{\cos 2x}{2} + c$

$y = -\frac{1}{2}\cos 2x\,.\,\mathrm{cosec}\,x + c.\mathrm{cosec}\,x$

$y.e^{\tan x} = e^{\tan x} + c$
$y = 1 + c\, e^{-\tan x}$

2. Solve the following differential equations.

a. $\dfrac{dy}{dx} + y = 1.$

b. $\dfrac{dy}{dx} + y = e^x.$

c. $\dfrac{dy}{dx} + 2y = \dfrac{1}{2}(x^2 - x)$

d. $\text{Sec}3x \dfrac{dy}{dx} - 2y\sec 3x = 1.$

e. $\cos^2 x \dfrac{dy}{dx} + y = \tan x.$

f. $x \dfrac{dy}{dx} + 2y = x^2 \ln x.$

g. $x \dfrac{dy}{dx} - x = 1 + y$

h. $\dfrac{dy}{dx} + \dfrac{y}{x} = e^x.$

i. $(1 + x^2)\dfrac{dy}{dx} + y = \tan^{-1}x.$

Solution:

a. Given, differential equation is $\dfrac{dy}{dx} + y = 1$ ... ... (i)

It is a linear differentiate equation of the type $\dfrac{dy}{dx} + py = Q$

Here, $p = 1$, $Q = 1$

$\int pdx = \int 1dx = x$

Integrating factor (I.F.) = $e^{spdx} = e^x$
Integrating equation (i) both sides by I.F.

$\left[\dfrac{dy}{dx} + y\right] e^x = 1 \times e^x$

Comparing both sides, we get $\int d(y.e^x) = \int e^x\, dx$

$\therefore\ y.e^x = e^x + c$
$\therefore\ y = 1 + ce^{-x}$

b. Given,
$\dfrac{dy}{dx} + y = e^x$ ... ... (i)

Comparing (i) with $\dfrac{dy}{dx} + py = Q$, we get

$p = 1$, $Q = e^x$

$\int pdx = \int 1dx = x$

Integrating factor (I.F.) is given by $e^{\int pdx}$
I.F. = e^x
Multiplying (i) both sides by I.F., we get

$\left(\dfrac{dy}{dx} + y\right) e^x = e^x . e^x$

or, $d(y.e^x) = e^{2x}$
Integrating both sides

$\int d(y.e^x) = \int e^{2x}\, dx$

$y.e^x = \frac{e^{2x}}{2} + c$

$\therefore \quad y = \frac{e^x}{2} + ce^{-x}$

c. $\frac{dy}{dx} + 2y = \frac{1}{2}(x^2 - x)$ ... ... (i)

Here, $p = 2$ and $Q = \frac{1}{2}(x^2 - x)$

$\int pdx = 2x$

I.F. $= e^{\int pdx} = e^{2x}$

Multiplying (i) both sides by I.F.

$\left(\frac{dy}{dx} + 2y\right) e^{2x} = \frac{1}{2}(x^2 - x)e^{2x}$

$d(y.e^{2x}) = \frac{1}{2}(x^2 - x)\, e^{2x}$

Integrating both sides

$\int d(y.e^{2x}) = \frac{1}{2}\int (x^2 - x)\, e^{2x}$

$y.e^{2x} = \frac{1}{2}\left[(x^2 - x)\frac{e^{2x}}{2} - \int (2x - 1)\frac{e^{2x}}{2}\, dx\right]$

$y.e^{2x} = \frac{1}{2}(x^2 - x)\, e^{2x} - \frac{1}{4}\int (2x-1)\, e^{2x} dx$

$y.e^{2x} = \frac{1}{4}(x^2 - x)\, e^{2x} - \frac{1}{4}\left[(2x - 1)\frac{e^{2x}}{2} - \int \frac{2e^{2x}}{2}\, dx\right]$

$y.e^{2x} = \frac{1}{4}(x^2 - x)\, e^{2x} - \frac{1}{8}(2x - 1)\, e^{2x} + \frac{1}{8}e^{2x} + c$

$y.e^{2x} = \frac{1}{8}e^{2x}(2x^2 - 2x - 2x + 1 + 1) + c$

$\Rightarrow y.e^{2x} = \frac{1}{4}(x - 1)^2\, e^{2x} + c$

d. $\text{Sec}3x \frac{dy}{dx} - 2y \sec 3x = 1$

$\therefore \quad \frac{dy}{dx} - 2y = \cos 3x$ ... ... (i) It is a linear differential equation of the form $\frac{dy}{dx} + py = Q$

Here, $p = -2$ and $Q = \cos 3x$

Now, $\int pdx = -\int 2dx = -2x$

I.F. $= e^{\int pdx} = e^{-2x}$

Multiplying (i) by I.F.

$\left(\frac{dy}{dx} - 2y\right) e^{-2x} = \cos 3x \,.\, e^{-2x}$

or, $d(y.e^{-2x}) = \text{co}3x \,.\, e^{-2x}$

Integrating both sides,

$\int d(y.e^{-2x}) = \int \cos 3x \,.\, e^{-2x}\, dx$

$y.e^{-2x} = \int \cos 3x . e^{-2x}\, dx$ ... ... (ii)

Let $I = \int \cos 3x . e^{-2x}\, dx$

or, $I = \cos 3x \int e^{-2x}\, dx - \int \left\{ \frac{d}{dx}(\cos 3x) . \int e^{-2x}\, dx \right\} dx = \frac{\cos 3x.\ e^{-2x}}{-2} - \int$

$\frac{-3}{-2} \sin 3x . e^{-2x}\, dx)$

$I = -\frac{1}{2}\cos 3x . e^{-2x} - \frac{3}{2}\int \sin 3x . e^{-2x}\, dx$

or, $I = -\frac{1}{2}\cos 3x . e^{-2x} - \frac{3}{2}\left[\sin 3x \int e^{-2x}\, dx - \int \left\{ \frac{d}{dx}(\sin 3x) . \int e^{-3x}\, dx \right\} \right]$

$I = -\frac{1}{2}\cos 3x . e^{-2x} - \frac{3}{2}\left[\sin 3x . \frac{e^{-2x}}{-2} - \int \frac{3}{-2} \cos 3x . e^{-2x}\, dx \right]$

$I = -\frac{1}{2}\cos 3x . e^{-2x} + \frac{3}{4}\sin 3x\, e^{-2x} - \frac{9}{4} I$

or, $\frac{13}{4} I = -\frac{1}{2}\cos 3x . e^{-2x} + \frac{3}{4}\sin 3x\, e^{-2x}$

$I = \frac{1}{13}(3 \sin 3x - 2\cos 3x)\, e^{-2x}$

from (ii),

$y . e^{-2x} = \frac{1}{13}(3 \sin 3x - 2\cos 3x)\, e^{-2x} + c$

$\therefore\ y = \frac{1}{13}(3 \sin 3x - 2\cos 3x) + ce^{2x}$

e. $\cos^2 x \frac{dy}{dx} + y = \tan x$

$\Rightarrow \frac{dy}{dx} + \sec^2 x . y = \tan x . \sec^2 x$ ... ... (i)

$p = \sec^2 x$ and $Q = \sec^2 x . \tan x$

$\int p dx = \int \sec^2 x dx = \tan x$

I.F. $= e^{\int p dx} = e^{\tan x}$

Multiplying (i) by I.F. we get,

$d(y.\ e^{\tan x}) = \tan x.\ \sec^2 x . e^{\tan x}$

Integrating

$\int dy . e^{\tan x} = \int e^{\tan x} \tan x . \sec^2 x\, dx$

$y . e^{\tan x} = \int e^u . u du$ where $u = \tan x$

$y . e^{\tan x} = u . e^u - e^u$

Integrating by parts

$y . e^{\tan x} = (u - 1)\, e^u$

$y . e^{\tan x} = (\tan x - 1)\, e^{\tan x} + c$

$\therefore\ y = (\tan x - 1) + c . e^{-\tan x}$

f. $x\frac{dy}{dx} + 2y = x^2 \ln x$

$\frac{dy}{dx} + \frac{2}{x} y = x \ln x$ ... ... (i)

Comparing (i) with $\frac{dy}{dx} + py = Q$, we get

$p = \frac{2}{x}$ and $Q = x \ln x$

$\int p dx = \int \frac{2}{x} dx = 2 \ln x = \ln x^2$

I.F. $= e^{\int p dx} = e^{\ln x^2} = x^2$

Multiplying (i) by I.F.

$\left(\frac{dy}{dx} + \frac{2}{x} y \right) x^2 = x^3 \ln x$

$d(y.x^2) = x^3 . \ln x$

Integrating both sides, we get

$\int d(y.x^2) = \int x^3 \ln x\, dx$

$y.x^2 = \ln x. \int x^3 dx - \int \left\{ \frac{d}{dx} (\log x) . \int x^3\, dx \right\}$

dx

$y.x^2 = \ln x . \frac{x^4}{4} - \int \frac{1}{x} . \frac{x^4}{4}\, dx$

$y.x^2 = \frac{x^4}{4} \ln x - \frac{1}{4} \int x^3\, dx$

$y.x^2 = \frac{x^4}{4} \ln x - \frac{1}{16} x^4 + c$

$\therefore \quad y = \frac{x^2}{4} \ln x - \frac{x^2}{16} + cx^{-2}$

g. $x \frac{dy}{dx} - x = 1+y$

$x \frac{dy}{dx} - y = 1 + x$

$\frac{dy}{dx} - \frac{1}{x} y = \left(\frac{1+x}{x}\right)$ ... ... (i)

Comparing (i) with $\frac{dy}{dx}$ + py = Q, we get

$p = -\frac{1}{x}$ and $Q = \frac{1 + x}{x}$

$\int pdx = \int -\frac{1}{x} dx = -\ln x = \ln x^{-1}$

I.F. $= e^{\int pdx} = e^{\int pdx} = e^{\ln x^{-1}} = x^{-1}$

Multiplying (i) both sides by I.F.

$\left(\frac{dy}{dx} - \frac{x}{1} y\right) x^{-1} = \left(\frac{1+ x}{x}\right) x^{-1}$

$d(y.x^{-1}) = \frac{1 + x}{x^2}$

$d(y.x^{-1}) = x^{-2} + \frac{1}{x}$

Integrating both sides,

$\int (y.x^{-1}) = \int x^{-2}\, dx + \int \frac{1}{x}\, dx$

$y.x^{-1} = -x^{-1} + \ln x + c$

$y = -1 + x\ln x + cx = x\ln x - 1 + cx$

h. $\frac{dy}{dx} + \frac{y}{x} = e^x$ ... ... (i)

Comparing (i) with $\frac{dy}{dx}$ + p.y = Q we get

$p = \frac{1}{x}$ and $Q = e^x$

$\int pdx = \ln x$

I.F. $= e^{\int pdx} = e^{\ln x} = x$

Multiplying (i) both sides by I.F.

$\left(\frac{dy}{dx} + \frac{y}{x}\right) x = x.e^x$

$d(y.x) = x.e^x$

Integrating both sides,

$\int d(y.x) = \int x.e^x\, dx$

$y.x = xe^x - e^x + c$

$y = \frac{(x - 1)}{x} e^x + \frac{c}{x}$

i. $(1 + x^2) \frac{dy}{dx} + y = \tan^{-1}x$

$\frac{dy}{dx} + \left(\frac{1}{1+x^2}\right) y = \frac{\tan^{-1}x}{(1 + x^2)}$ ... . (i)

Comparing (i) with $\frac{dy}{dx}$ + py = Q, we get

$p = \frac{1}{1+x^2}$, $Q = \frac{\tan^{-1}x}{1+x^2}$

$\int p dx = \int \frac{1}{1+x^2} dx = \tan^{-1}x$

I.F. $= e^{\int pdx} = e^{\tan^{-1}}x$

Multiplying (i) by I.F. both sides

We get,

$d(y.e^{\tan^{-1}}x) = \frac{\tan^{-1} x}{1+x^2} . e^{\tan^{-1}}x$

Integrating both sides

$\int d(y.\ e^{\tan^{-1}}x) = \int e^{\tan^{-1}}x \frac{\tan^{-1}x}{1+x^2} dx$

$y.e^{\tan^{-1}}x = \int e^{\tan^{-1}}x \frac{\tan^{-1}x}{1+x^2} dx$

Put $e^{\tan^{-1}}x = u$ in RHS

Then $\frac{1}{1+x^2} dx = du$

Then,

$y.e^{\tan^{-1}}x = \int e^u.u\ du$

$y.e^{\tan^{-1}}x = \int u.e^u\ dx$

$y.\ e^{\tan^{-1}}x = u \int e^u\ du - \int \left\{ \frac{du}{du} . \int e^u\ du \right\} du$

$y.\ e^{\tan^{-1}}x = u.e^u - \int 1.e^u\ du$

$y.\ e^{\tan^{-1}}x = ue^u - e^u + c$

$y.e^{\tan^{-1}}x = \tan^{-1}x\ e^{\tan^{-1}x} - e^{\tan^{-1}e} + c$

$y = \tan^{-1}x - 1 + \frac{c}{e^{\tan^{-1}x}}$

3. Solve the following differential equations.

a. $(1+x)\frac{dy}{dx} - xy = 1-x.$

b. $\frac{dy}{dx} + \frac{4x}{x^2+1} y = -\frac{1}{(x^2+1)^2}.$

c. $(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$

d. $\frac{dy}{dx} + \frac{y}{x \ln x} = \frac{1}{x}$

e. $\frac{dy}{dx} + \frac{y}{x} = y^2$

f. $\frac{dy}{dx} + xy = xy^3$

g. $(1-x^2)\frac{dy}{dx} + xy = xy^2$

Solution:

a. $(1+x)\frac{dy}{dx} - xy = 1 - x$

$\frac{dy}{dx} - \frac{x}{1+x} .y = \frac{1-x}{1+x}$ ... ... (i)

$p = -\frac{x}{1+x}$ and $Q = \frac{1-x}{1+x}$

$\int pdx = -\int \frac{x}{x+1}\,dx = -\int \frac{x+1-1}{x+1}\,dx = -\int 1dx + \int \frac{1}{x+1} = -x + \ln(x+1)$

I.F. $= e^{\int pdx} = e^{-x+\ln(x+1)} = e^{-x}(x+1)$

Multiplying (i) by by I.F. we get

$d(y.e^{x}(x+1)) = (1-x)\,e^{-x}$

Integrating $\int d(y(x+1)\,e^{-x}) = \int e^{-x}(1-x)\,dx$

or, $y(x+1)\,e^{-x} = \frac{(1-x)\,e^{-x}}{-1} + e^{-x} + c$

or, $ye^{-x}(x+1) = -(1-x)\,e^{-x} + e^{-x} + c$

or, $ye^{-x}(x+1) = e^{-x}(1-1+x) + c$

or, $y(x+1)\,e^{-x} = ex^{-x} + c$

$\therefore$ $y(x+1) = x + ce^{x}$

b. $\frac{dy}{dx} + \frac{4x}{1+x^2}.y = -\frac{1}{(x^2+1)^2}$ ... ... (i)

Here, $p = \frac{4x}{x^2+1}$ and $Q = \frac{-1}{(x^2+1)^2}$

$\int pdx = 2\int \frac{2x}{x^2+1}\,dx = 2\ln(x^2+1)$

$= \ln(x^2+1)^2$

I.F. $= e^{\int pdx} = (x^2+1)^2$

Multiplying (i) by I.F. we get

$d\{y.(x^2+1)^2\} = -1$

$(x^2+1)^2 \times (x^2+1)^2$

$\Rightarrow \int d\{y(x^2+1)^2\} = -x + c$

$y(x^2+1)^2 = -x + c$

c. $(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$

$\therefore$ $\frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{2}{(x^2-1)}$ ... ... (i)

Comparing (i) with $\frac{dy}{dx} + p.y = Q$, we get

$p = \frac{2x}{x^2-1}$ and $Q = \frac{2}{(x^2-1)^2}$

Now, $\int pdx = \int \frac{2x}{x^2-1}\,dx = \ln(x^2-1)$

I.F. $= e^{\int pdx} = e^{\ln(x2-1)} = (x^2-1)$

Multiplying (i) both sides by I.F.

$\left(\frac{dy}{dx} + \frac{2x}{x^2-1}y\right)(x^2-1) = \frac{2}{(x^2-1)^2}(x^2-1)$

$d\{y(x^2-1)\} = \frac{2}{x^2-1}$

Integrating $\int d(y.(x^2-1)) = 2\int \frac{1}{x^2-1}\,dx$

$y.(x^2-1) = 2.\ln\left|\frac{x-1}{x+1}\right| + c$

d. $\frac{dy}{dx} + \frac{y}{x.\ln x} = \frac{1}{x}$ ... ... (i)

Comparing (i) with $\frac{dy}{dx} + p.y = Q$

$p = \frac{1}{x.\ln x}$ and $Q = \frac{1}{x}$

$\int pdx = \int \frac{1}{x.\ln x}\,dx$

Put $\ln x = u$ then $\frac{1}{x}dx = du$

e. $\frac{dy}{dx} + \frac{y}{x} = y^2$ or, $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x.y} = 1$ ... (i)

Put $\frac{1}{y} = z$ then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

(i) becomes

$-\frac{dz}{dx} + \frac{1}{x}.z = 1$ or, $\frac{dz}{dx} - \frac{1}{x}.z = -1$ ... (ii)

Comparing (ii) with $\frac{dy}{dx} + p.y = Q$ we get

$p = -\frac{1}{x}$ and $Q = -1$

$\int pdx = \int \frac{du}{u} = \ln u$

I.F $= e^{\int pdx} = e^{\ln u} = u = \ln x$

Multiplying (i) by I.F. we get

$d(y.\ln x) = \frac{1}{x}\ln x$

Integrating both sides

$\int d(y.\ln x) = \int \frac{1}{x}\ln x\, dx$

$y.\ln x = \int v dv$ where $\ln x = v$

$y.\ln x = \frac{v^2}{2} + c$

$y.\ln x = \frac{(\ln x)^2}{2} + c$

$\therefore \quad y = \frac{1}{2}\ \ln x + \frac{c}{\ln x}$

f. $\frac{dy}{dx} + xy = xy^3$

$\frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x$ ... ... (i)

Put $\frac{1}{y^2} = z$

Then $-\frac{1}{y^3}\frac{dy}{dx} = \frac{dz}{dx}$

(i) becomes

$\frac{dz}{dx} - xz = -x$ ... ... (ii)

Comparing (ii) with $\frac{dy}{dx} + p.y = Q$, we get

$p = -x$ and $Q = -x$

$\int pdx = -\int xdx = e^{-x^2/2}$

I.F. $= e^{\int pdx} = e^{-\frac{x^2}{2}}$

Multiplying (ii) by I.F.

$d(z,e^{-x^2/2}) = -x.e^{-\frac{x^2}{2}}$

Integrating

$z.e^{-\frac{x^2}{2}} = \int -x\,.\,e^{-\frac{x^2}{2}}\,dx$

$z.e^{-\frac{x^2}{2}} = \int e^u\,du$ where $u = -\frac{x^2}{2}$

$\int pdx = -\int \frac{1}{x}\,dx = -\ln x = \ln x^{-1}$

I.F. $= e^{\int pdx} = e^{\ln x-1} = x^{-1} = \frac{1}{x}$

Multiplying (ii) by I.F.

$\left(\frac{dz}{dx} - \frac{1}{x}\,.\,z\right)\frac{1}{x} = -1\,.\,\frac{1}{x}$

$d\left(z\,.\,\frac{1}{x}\right) = -\frac{1}{x}$

Integrating both sides, we get

$\int d\left(z\,.\,\frac{1}{x}\right) = -\frac{1}{x}$

Integrating both sides, we get

$\int d\left(z\,.\,\frac{1}{x}\right) = -\int \frac{1}{x}\,dx$

$z.\frac{1}{x} = -\ln x + c \Rightarrow \frac{1}{y} = -x\ln x + cx$

$\therefore \quad \frac{1}{xy} + \ln x = c$

g. $(1 - x^2)\frac{dy}{dx} + x.y = xy^2$

$\frac{1}{y^2}\frac{dy}{dx} + \left(\frac{x}{1-x^2}\right)\frac{1}{y} = \frac{x}{1-x^2}$ ... ... (i)

Put $\frac{1}{y} = z$ then $\frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$

Then, $-\frac{dz}{dx} + \left(\frac{x}{1-x^2}\right)\,.\,z = \frac{x}{1-x^2}$

$\frac{dz}{dx} - \left(\frac{x}{1-x^2}\right)z = -\frac{x}{1-x^2}$ ... ... (ii)

Here, $p = \frac{-x}{1-x^2}$ and $Q = \frac{-x}{1-x^2}$

$\int pdx = \frac{1}{2}\int -\frac{2x}{1-x^2}$

$= \frac{1}{2}\ln(1-x^2) = \ln\sqrt{1-x^2}$

I.F. $= e^{\int pdx} = e^{\ln\sqrt{1-x^2}} = \sqrt{1-x^2}$

Multiplying (ii) by I.F. we get

$d(z.\sqrt{1-x^2} = -\frac{x}{1-x^2}\sqrt{1-x^2}$

Integrating

$z\sqrt{1-x^2} = \int \frac{x}{\sqrt{1-x^2}}\,dx$

$z.\sqrt{1-x^2} = \frac{1}{2}\int \frac{du}{\sqrt{u}}$ where $1-x^2 = u$

$$\frac{1}{y^2}\ e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} + c$$

$$\frac{1}{y^2} = 1 + c\,e^{\frac{x^2}{2}}$$

$$\frac{1}{y}\cdot\sqrt{1-x^2} = \sqrt{1-x^2} + c$$

EXERCISE 17.6

1. 40% of a radioactive substance disappears in 100 years.
   a. What is its half life?
   b. After how many years will 90% be gone?

Solution:

Let y = y(t) be the amount of radioactive substances at time t.

Then $y(t) = ke^{ct}$ … (i)

At t = 0, $y(0) = ke^{c\times 0} \Rightarrow y(0) = k$

$\therefore$ $y(t) = y(0)e^{ct}$ … (ii)

At t = 100, 40% of radioactive substance disappears.

$\therefore$ 60% remaining i.e.

$\therefore$ Remaining radioactive substance after 100 years is

$$y(100) = \frac{60}{100} \times y(0)$$

Now from (ii) at t = 100, we get

$y(100) = y(0)\ e^{100c}$

or, $\frac{60}{100} \times y(0) = y(0)\ e^{100c}$

or, $\frac{6}{10} = e^{100c}$

or, $100c = \ln \frac{6}{10}$

or, $100c = \ln 0.6$

or, $c = \frac{\log 0.6}{100} = -0.0051082$

Substituting the value of c on (ii) we get

$y(t) = y(0)\ e^{-0.0051082t}$ … (iii)

a. Half life:

Now, the half life is the time duration after which half of its initial value disappears, let the half life period be t = T.

Then, $y(T) = y(0)\ e^{-0.0051082T}$

or, $\frac{y(0)}{2} = y(0)\ e^{-0.0051082T}$

or, $e^{-0.0051082T} = 0.5$

or, $-0.0051082T = \ln 0.5$

$\therefore$ $T = \frac{\log 0.5}{0.0051082} = \frac{-0.693171}{-0.0051082} = 135.69$

$\therefore$ Half life, T = 135.69 years

b. Since 90% of the value is gone means 10% of the original value remains. Let T_1 be the required years.

Then, $y(T_1) = y(0)\ e^{-0.0051082T_1}$

or, $\frac{1}{10}$ y(0) = y(0) $e^{-0.0051082\ T_1}$

or, $0.1 = e^{-0.0051082T_1}$

or, –0.0051082T = ln 0.1

or, $T_1 = \frac{\log 0.1}{-0.0051082} = \frac{-2.3025851}{-0.0051082} = 450.76$

$\therefore$ T_1 = 450.76 years

2. Radioactive carbon (Carbon – 14) has a half life of 5568 years (At the end of that period of time one half of the original amount remains). Derive the formula for the amount remaining after t years.

Solution:

Let y(t) be the amount of radioactive carbon at time t.

Then $y(t) = ke^{ct}$ … (i)

At time t = 0, $y(0) = ke^{o} \Rightarrow y(0) = k$

$\therefore$ $y(t) = y(0)\ e^{ct}$ … (ii)

At time t = 5568, half of the original amount remains

$\therefore$ $y(5568) = \frac{y(0)}{2}$ $\therefore$ From (ii), $y(5568) = y(0)\ e^{5568c}$

$\Rightarrow$ $\frac{y(0)}{2} = y(0)\ e^{5568c}$

or, $e^{5568c} = 0.5$

or, 5568c = ln 0.5

$\therefore$ $c = \frac{\log 0.5}{5568}$

Using value of c on (i) we get

$y(t) = ke^{\frac{\log 0.5}{5568} \times t}$

Which is required formula.

3. The half life of isotopic radium is approximately 2000 years. Beginning with 200 grams of this radium, find a formula that will give the amount remaining after t years. After how many years will one third of the original 200 grams remains?

Solution:

Let y(t) be the amount of isotopic radium remaining after t years

Then $y(t) = ke^{ct}$ … (i)

At time t = 0, $y(0) = ke^{0} \Rightarrow 200 = k.1$ $\therefore$ k = 200

$\therefore$ $y(t) = 200\ e^{ct}$ … (ii)

Again at t = 2000, $y(2000) = \frac{1}{2} \times 200$

$\therefore$ $y(2000) = 200e^{20000c}$

or, $\frac{1}{2} \times 200 = 200e^{2000c} \Rightarrow e^{2000c} = 0.5$

or, 2000c = ln 0.5

or, $c = \frac{\log 0.5}{2000} = -0.00034657.$

Substituting the value of c on (ii) we get

$y(t) = 200e^{-0.00034657\ t}$

Again, let at t = T, $y(T) = \frac{1}{3} \times 200$ grams.

Then $y(T) = 200\ e^{-0.00034657T}$

$\Rightarrow\ \frac{1}{3} \times 200 = 200\ e^{-0.00034657T}$

or, $-0.00034657T = \ln 1/3$ i.e. $-\ln 3$

$\therefore\ T = \frac{\log 3}{0.0003457} = \frac{1.0986123}{0.00034657} = 3170.6$

$\therefore\ T = 3170.6$ years

4. At time t = 0, five million bacteria were living in George's lung. Three hours later the number had increased to nine million. Assuming that the conditions for growth do not change over the next few hours, find the number of bacteria at the end of (a) 12 hours. (b) 36 hours.

Solution:

Let y(t) denotes the amount of bacteria at time t.

Then $y(t) = ke^{ct}$ … (i)

At time t = 0, $y(0) = ke^{o}$ $\Rightarrow 5 = ke^{o} \Rightarrow k = 5$

$\therefore\ y(t) = 5e^{ct}$ … (ii)

Again at t = 3, y(3) = 9

$\therefore\ y(3) = 5e^{c\times 3} \Rightarrow 9 = 5e^{3c}$ or, $e^{3c} = 9/5 = 1.8$

$\therefore\ 3c = \ln 1.8$ $\therefore\ c = \frac{\log 1.8}{3} = 0.1959277$

Substituting the value of c on (ii) we get

$y(t) = 5e^{0.1959288t}$

a. At t = 12 hours:

$y(12) = 5 \times e0.1959288 \times 12 = 5\times 10.4976 = 52.488$ millions

b. At t = 36 hours:

$y(36) = 5 \times e0.1959288 \times 36 = 5 \times 1156.8314 = 5784.416$ millions

5. Assume that the population of a certain city increases at a rate proportional to the number of its inhabitants at any time. If the population doubles in 40 years, in how many years it triple?

Solution:

Let x be the number of individuals in at time t. We know that the population of a certain city increases at a rate proportional to the number of its inhabitants at any time. Hence, we are led to the differential equation.

$\frac{dx}{dt} = kx$ … (i)

where, k is a constant of proportionality. The population x is positive and is increasing and hence dx/dt > 0. Therefore, from (i), we must have k > 0. Now, suppose that at time $t_0 = 0$ the population is x_0. Then, in addition to the differential equation (i), we have the initial condition.

$x(t_0) = x(0) = x_0$ … (ii)

The differential equation(i)is separable. Separating variables, integrating, and simplifying, we obtain.

$x = Ce^{kt}$

Applying the initial condition, $x = x_0$ at $t = t_0 = 0$, to this, we have

$x_0 = Ce^{kt0} = C$

From this we at once find $C = x^0 e^{-kt_0}$ and hence we obtain the unique solution $x = x_0 e^{k(t-t_0)}$ of the differential equation (i), which satisfies the initial condition (ii). Now, when t = 40, we have $x = 2x_0$. Hence, we obtain

$$2x_0 = x_0 e^{40k} \Leftrightarrow 2 = e^{40k} \Leftrightarrow \frac{\ln 2}{40} = k$$

If we let $x = 3x_0$, $x_0 e^{(\ln 2/40)t} \Leftrightarrow 3 = e^{(\ln 2/40)t} \Leftrightarrow \ln 3 = \frac{\ln 2}{40} t \Leftrightarrow t = 40\left(\frac{\ln 3}{\ln 2}\right) \approx 63.40$

Therefore, the population will triple in about 63.40 years.

6. The population of the city of Bingville increases at a rate proportional to the numbers of its inhabitants present ant any time t. If the population of Bingville was 30,000 in 1970 and 35,000 in 1980, what will be the population of Bingville in 1990?

Solution:

According to the formula in the exercise 5, we have

$x = x_0 e^{k(t-t_0)}$

Hence we obtain

$$x(1980) = 30{,}000\ e^{k(1980-1970)} \Leftrightarrow \frac{35{,}000}{30{,}000} = e^{10k} \Leftrightarrow \frac{1}{10} \ln\left(\frac{7}{6}\right) = k$$

Therefore, the population of Bingville in 1990 is

$$x(1990) = 30{,}000 e^{2\ln\left(\frac{7}{6}\right)} \approx 40{,}833$$

■

CHAPTER 18

LINEAR PROGRAMMING

EXERCISE 18

1. Find the basic solution of the following system of equations.
   a. $x + 2y + z = 6$
   $4x + 3y + z = 12$
   b. $x + 2y + z = 4$
   $2x + y + 5z = 5$

Solution:

a. Here, given equations are $x + y + z = 6$ and $4x + 3y + z = 12$
There are 3 variables and 2 equations so, there are two basic solution and one non-basic.
Case – I: if $z = 0$, then,
$x + y = 6$ ... ... (i)
$4x + 3y = 12$ ... ... (ii)
Solving equation (i) and (ii)
$\therefore$ $y = \frac{12}{5}$ (basic)
$\therefore$ $x = \frac{6}{5}$ (basic)
$\therefore$ $z = 0$ (non–basic)
Case – II: if $y = 0$
$x + z = 6$ ... ... (iii)
$4x + z = 12$ ... ... (iv)
Solving (iii) and (iv0
$\therefore$ $x = 2$ (basic)
$\therefore$ $z = 4$ (basic)
$\therefore$ $y = 0$ (non–basic)
Case–III: if $x = 0$
$y + z = 6$ ... ... (v)
$3y + z = 12$ ... ... (vi)
Solving (v) and (vi)
$\therefore$ $y = 6$ (basic)
$\therefore$ $z = –6$ (basic)
$\therefore$ $x = 0$ (non basic)

b. Here, Given equations are
$x + 2y + z = 4$
$2x + y + 5z = 5$
There are 3 variables in 2 equations among them 2 are basic and 1 is non-basic.
Case–I: if $z = 0$
$x + 2y = 4$ ... ... (i)
$2x + y = 5$ ... ... (ii)
Solving equation (i) and equation (ii)
$\therefore$ $y = 1$ (basic)
$\therefore$ $x = 2$ (basic)
$\therefore$ $z = 0$ (non–basic)
Case–II: if $y = 0$
$x + z = 4$ ... ... (iii)
$2x + 5z = 5$ ... ... (v)

Solving (iii) and (iv)

$\therefore$ z = –1 (basic)

$\therefore$ y = 0 (non–basic)

$\therefore$ x = 3 (basic)

Case–III: if x = 0

2y + z = 4 ... ... (v)

y + 5z = 5 ... ... (vi)

Solving (v) and (vi)

$\therefore$ $z = \frac{2}{3}$ (basic)

$\therefore$ $y = \frac{5}{3}$ (basic)

$\therefore$ x = 0 (non–basic)

2. Find all basic feasible solutions of the following system of equations.

a. x + 2y – z = 3
   x – y + z = 5

b. 2x + 3y + z = 12
   x + 2y – 3z = 5

Solution:

a. Given equations are

x + 2y – z = 3

x – y + z = 5

There are 3 variables and 2 equations. So, among them 2 are basic and 1 is non-basic.

Case–I: if z = 0

x + 2y = 3 ... ... (i)

x – y = 5 ... ... (ii)

Solving (i) and (ii)

$\therefore$ $x = \frac{13}{3}$ (basic)

$\therefore$ $y = \frac{-2}{3}$ (basic)

$\therefore$ z = 0 (non–basic)

Case–II: if y = 0

x – z = 3 ... ... (iii)

x + z = 5 ... ... (iv)

Solving (ii) and (i)

$\therefore$ x = 4 (basic)

$\therefore$ z = 1 (basic)

$\therefore$ y = 0 (non–basic)

Case–III: if x = 0

2y – z = 3 ... ... (v)

–y + z = 5 ... ... (vi)

Solving (v) and (vi)

$\therefore$ y = 8 (basic)

$\therefore$ z = 13 (basic)

$\therefore$ x = 0 (non-basic)

Since, the case II and III are non-negative, so they give basic feasible solution.

$\therefore$ The basic feasible solution are (4, 0, 1) and (0, 8, 13)

b. Here, the given equations are

2x + 3y + z = 12

x + 2y – 3z = 5

There are 3 variables and 2 equations. Among them 2 are basic and 1 is non-basic.

Case–I: if z = 0

$2x + 3y = 12$ ... ... (i)

$x + 2y = 5$ ... ... (ii)

Solving (i) and (ii)

$\therefore$ y = –2 (basic)

$\therefore$ x = 9 (basic)

$\therefore$ z = 0 (non–basic)

Case–II: if y = 0

$2x + z = 12$ ... ... (iii)

$x - 3z = 5$ ... ... (iv)

Solving (iii) and (iv)

$\therefore$ $z = \frac{2}{7}$ (basic)

$\therefore$ $x = \frac{41}{7}$ (basic)

$\therefore$ y = 0 (non–basic)

Case III: If x = 0,

$3y + z = 12$ .... (v)

$2y - 3z = 5$ .... (vi)

solving (v) and (vi), we get

$y = \frac{41}{11}$ and $z = \frac{9}{11}$

Since, the cases II and III are non-negative, so the basic feasible solution are

$\left(\frac{41}{7}, 0, \frac{3}{7}\right)$ and $\left(0, \frac{41}{11}, \frac{9}{11}\right)$

3. Express the following LP in standard form. Also find the optimal solution using simplex method.

a. Max. $Z = 10x + 15y$
Subject to
$x + 2y \leq 20$
$x + y \leq 16$
$x, y \geq 0$

b. Max. $Z = 3x + 5y$
Subject to the constraints
$x + 2y \leq 5$
$-2x + 3y \geq -7$
$x, y \geq 0$

Solution:

a. Introducting the slack–variable s_1, s_2, the given LPP can be written as

$x + 2y + s_1 = 20$

$x + y + s_2 = 16$

$z - 10x - 15y - 0.s_1 - 0.s_2 = 0$

$x, y, s_1, s_2 \leq 0$

Basic variables	x	y	s_1	s_2	RHS
s_1	1	$\boxed{2}$	1	0	20
s_2	1	1	0	1	16
	–10	–15	0	0	0

Pivot element = 2

Applying $R_1 \to \frac{1}{2} R_1$ we get,

Basic variables	x	y	s_1	s_2	RHS
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	10
s_2	1	1	0	1	16
	–10	–15	0	0	0

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + 15R_2$ we get,

Basic variables	x	y	s_1	s_2	RHS
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	10
s_2	$\boxed{\frac{1}{2}}$	0	$-\frac{1}{2}$	1	6
	$-\frac{5}{2}$	0	$\frac{15}{2}$	0	150

Pivot element = $\frac{1}{2}$

Basic variables	x	y	s_1	s_2	RHS
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	10
x	$\boxed{1}$	0	–1	2	12
	$-\frac{5}{21}$	0	$\frac{15}{2}$	0	150

Applying $R_1 \to R_1 - \frac{1}{2} R_2$, $R_3 \to R_3 + \frac{5}{2} R_2$ we get

Basic variables	x	y	s_1	s_2	RHS
y	0	1	1	1	4
x	1	0	–1	2	12
	0	0	5	5	180

Here, all the entries in the last row are non–negative so, we get a optimal solution as,

$x = 12$, $y = 4$ and

Max (z) = $10x + 15y$

$= 10 \times 12 + 15 \times 4$

$= 120 + 60$

$= 180$

b. Introducing the slack–variable s_1, s_2, the given LPP can be written as

$x + 2y + s_1 = 5$

$2x - 3y + s_2 = 7$

$x, y, s_1, s_2 \geq 0$

and $z - 3x - 5y - 0.s_1 - 0.s_2 = 0$

The initial simplex tableau

Basic variables	x	y	s_1	s_2	RHS
s_1	1	$\boxed{2}$	1	0	5
s_2	2	–3	0	1	7

	–3	–5	0	0	0

Here, poivot element = 2

To make pivot element 1, applying $R_1 \rightarrow \frac{1}{2} R_1$, we get

Basic variables	x	y	s_1	s_2	RHS
y	$\frac{1}{2}$	$\boxed{1}$	$\frac{1}{2}$	0	$\frac{5}{2}$
s_2	2	–3	0	1	7
	–3	–5	0	0	0

Applying $R_2 \rightarrow R_2 + 3R_1$ and $R_3 \rightarrow R_3 + 5R_1$, we get

Basic variables	x	y	s_1	s_2	RHS
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$
s_2	$\boxed{\frac{7}{2}}$	0	$\frac{3}{2}$	1	$\frac{31}{2}$
	$-\frac{1}{2}$	0	$\frac{5}{2}$	0	$\frac{25}{2}$

Pivot element = $\frac{7}{2}$

Applying $R_2 \rightarrow \frac{2}{7} R_2$ we get,

Basic variables	x	y	s_1	s_2	RHS
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$
X	$\boxed{1}$	0	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{31}{7}$
	$-\frac{1}{2}$	0	$\frac{5}{2}$	0	$\frac{25}{2}$

Applying $R_1 \rightarrow R_1 - \frac{1}{2} R_2$ and $R_3 \rightarrow R_3 + \frac{1}{2} R_2$ we get,

Basic variables	x	y	s_1	s_2	RHS
y	0	1	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{2}{7}$
x	1	0	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{31}{7}$
	0	0	$\frac{19}{7}$	$\frac{1}{7}$	$\frac{103}{7}$

Here, all the entries in the last row are non–negative. So, we get an optimal solution as,

$x = \frac{31}{7}$, $y = \frac{2}{7}$ and

Max (z) = 3x + 5y

$$= 3 \times \frac{31}{7} + 5 \times \frac{2}{7}$$

$$= \frac{93}{7} + \frac{10}{7} = \frac{103}{7}$$

4. By using simplex method find the maximum value of

a. Max. $Z = 7x_1 + 5x_2$
Subject to
$x_1 + 2x_2 \le 6$
$4x_1 + 3x_2 \le 12$
$x_1, x_2 \ge 0$

b. Max. $Z = 9x + y$
Subject to
$2x + y \le 8$
$4x + 3y \le 18$
$x, y \ge 0$

c. Max. $F = 6x_1 - 9x_2$
Subject to $2x_1 - 3x_2 \le 6,$
$x_1 + x_2 \le 20$
$x_1, x_2 \ge 0$

d. Max. $Z = 2x + 3y$
Subject to $x + 2y \le 10$
$2x + y \le 14$
$x, y \ge 0$

Solution:

a. Here, max. $z = 7x_1 + 5x_2$
Subject to $x_1 + 2x_2 = 6$
$4x_1 + 3x_2 \le 12$
$x_1, x_2 \ge 0$
Introducting the slack – variable s_1, s_2 given LPP can be written as,
$x_1 + 2x_2 + s_1 = 6$
$4x_1 + 3x_2 + s_2 = 12$ and
$z - 7x_1 - 5x_2 - 0.s_1 - 0.s_2 = 0$
The simplex tableau is;

Basic variables	x_1	x_2	s_1	s_2	RHS
r	1	2	1	0	6
s	4	3	0	1	12
	–7	–5	0	0	0

The most negativity entry is –7 so, x_1 column is pivot column. Then,
$\frac{6}{1} = 6, \frac{12}{4} = 3$
Here, 3 < 6 so, 4 is pivot element.
$R_2 \to \frac{1}{4} R_2$

Basic variables	x_1	x_2	s_1	s_2	RHS
s_1	1	2	1	0	6
x_1	1	$\frac{3}{4}$	0	$\frac{1}{4}$	3
	–7	–5	0	0	0

$R_1 \to R_1 - R_2$, $R_3 \to R_3 + 7R_2$

Basic variables	x_1	x_2	s_1	s_2	RHS
s_1	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	3
x_1	1	$\frac{3}{4}$	0	$\frac{1}{4}$	3
	0	$\frac{1}{4}$	0	$\frac{7}{4}$	21

Here, all the elements in last row are positive so, it is optimal solution.
Max. (Z) = 21 at $x_1 = 3$, $x_2 = 0$.

b. Here, max. $z = 9x + y$
Subject to $2x + y \le 8$
$4x + 3y \le 18$

$x, y \geq 0$
Introducting the slack – variable s_1, s_2 given LPP can be written as,
$2x + y + s_1 = 8$
$4x + 3y + s_2 = 18$
$z - 9x - y - 0.s_1 - 0.s_2 = 0$
The simplex tableau is

Basic variables	**x**	**y**	**s_1**	**s_2**	**RHS**
s_1	②	1	1	0	8
s_2	4	3	0	1	18
	–9	–1	0	0	0

The most negativity entry is –9 so, x column is pivot column. Then, $\frac{8}{2} = 4$, $\frac{18}{4} = 4.5$
Here, 2 is pivot element.
$R_1 \to \frac{1}{2} R_1$

Basic variables	**x**	**y**	**s_1**	**s_2**	**RHS**
x	1	2	$\frac{1}{2}$	0	4
s_2	4	3	0	1	18
	–9	–1	0	0	0

$R_2 \to R_2 - 4R_1$

Basic variables	**x**	**y**	**s_1**	**s_2**	**RHS**
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	4
s_2	0	1	–2	1	2
	–9	–1	0	0	0

$R_3 \to R_3 + 9R_1$

Basic variables	**x**	**y**	**s_1**	**s_2**	**RHS**
x	1 0	$\frac{1}{2}$	$\frac{1}{2}$	0 1	4 2
s_2		1	–2		
	0	$\frac{7}{2}$	$\frac{9}{2}$	0	36

Here, all the element in R_3 are positive so, it is optimal solution.
Max. (z) = 36 at x = 4, y = 0.

c. Here, max. $F = 6x_1 - 9x_2$
Subject to $2x_1 - 3x_2 \leq 6$
$x_1 + x_2 \leq 20$
$x_1, x_2 \geq 0$
Introducting the slack – variable s_1, s_2 given LPP can be written as,
$2x_1 - 3x_2 + s_1 = 6$
$x_1 + x_2 + s_2 = 20$ and
$F - 6x_1 + 9x_2 - 0.s_1 - 0.s_2 = 0$
The initial simplex tableau is;

Basic variables	**x**	**y**	**s_1**	**s_2**	**RHS**
r s	② 1	–3 1	1 0	0 1	6 20
	–6	9	0	0	0

The most negativity entry is –6 so, x column is pivot column. Then, $\frac{6}{2} = 3$, $\frac{20}{1} =$ 20.

Here, 3 < 20 so, 2 is pivot column.

$R_1 \to \frac{1}{2} R_1$

Basic variables	**x**	**y**	**s_1**	**s_2**	**RHS**
x	1 1	$\frac{-3}{2}$	$\frac{1}{2}$	0 1	3 20
s_2		1	0		
	–6	9	0	0	0

Now, $R_2 \to R_2 - R_1$, $R_3 \to R_3 + 6R_1$

Basic variables	**x**	**y**	**s_1**	**s_2**	**RHS**
x	1 0	$\frac{-3}{2}$	$\frac{1}{2}$	0 1	3 17
s_2		$\frac{5}{2}$	$\frac{-1}{2}$		
	0	0	3	0	18

Here, all the elements in last row are positive so, the

Max. (F) = 18, at x = 3, y = 0.

d. Here, max. z = 2x + 3y

Subject to $x + 2y \le 10$

$2x + y = \le 14$

$x, y \ge 0$

Introducting the slack – variable s_1, s_2 given LPP can be written as,

$x + 2y + s_1 = 10$

$2x + y + s_2 = 14$ and

$z - 2x - 3y - 0.s_1 - 0.s_2 = 0$

The initial simplex tableau;

Basic variables	**x**	**y**	**s_1**	**s_2**	**RHS**
s_1	1	2	1	0	10
s_2	2	1	0	1	14
	–2	–3	0	0	0

The most negativity entry is –3 so, y column is pivot column. Then, $\frac{10}{2} = 5$, $\frac{14}{1}$ = 14

Here, 5 < 14 so 2 is pivot element.

$R_1 \to \frac{1}{2} R_1$

Basic variables	**x**	**y**	**s_1**	**s_2**	**RHS**
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	5
x	2	1	0	1	14
	–2	–3	0	0	0

$R_2 \to R_1$, $R_3 \to 3R_1$

Basic variables	**x**	**y**	**s_1**	**s_2**	**RHS**
y	$\frac{1}{2}$	1 0	$\frac{1}{2}$	0 1	5 9

s	$\frac{3}{2}$		$\frac{-1}{2}$		
	$-\frac{1}{2}$	0	$\frac{3}{2}$	0	15

All the values in last row is not positive. So, it is not optimal solution.

Here, the most negativity entry is $\frac{-1}{2}$ so x column is pivot column. Then,

$$\frac{5}{\frac{1}{2}} = \frac{5\times2}{10}, \frac{9}{\frac{3}{2}} = \frac{9\times2}{3} = 6$$

Here, 6 < 10 so, $\frac{3}{2}$ is pivot element.

$R_2 \to \frac{2}{3} R_2$

Basic variables	x	y	s_1	s_2	RHS
y	$\frac{1}{2}$	1 0	$\frac{1}{2}$	0 $\frac{2}{3}$	5 6
x	1		$\frac{-1}{3}$		
	$\frac{-1}{2}$	0	$\frac{3}{2}$	0	15

$R_1 \to R_1 \frac{-1}{2} R_2$, $R_3 \to \frac{1}{2} R_2 + R_3$

Basic variables	x	y	s_1	s_2	RHS
y	0 1	1 0	$\frac{5}{6}$	$\frac{-1}{3}$	2 6
x			$\frac{-1}{3}$	$\frac{2}{3}$	
	0	0	$\frac{4}{3}$	$\frac{1}{3}$	18

Here, all the elements in R_3 are positive so, it is optimal solution.
Max. (z) = 18 at {x = 6, y = 2}

5. One kind of cake takes 200 gm of flour and 25 gm of fat and another kind of cake takes 100 gm of flour and 50 gm of fat. Suppose we want to make as many cakes as possible but have only 4 kg of flour and 1.175 kg of fat available, although there is no shortage of the various other ingredients. If the profit on selling first and second kinds of cakes are Rs. 3 and Rs. 2 respectively.
   a. Formulate the LP Problem
   b. Express it into standard form
   c. Solve the LP Problem by simplex method and find the maximum profit.

Solution:

Types of cakes	Quality of		Profit (Rs.)
	Flour (gm)	Fat (gm)	
First	200	25	3
Second	100	50	2
Total	4000	1175	

Let the number of first types of cake = x and the number of second type of cake = y
Then the total quantity of flour = 200x + 100y
By given condition,

$200x + 100y \le 4000 \Rightarrow 2x + y \le 40$ …. (i)
Again the total quantity of fat = $25x + 50y$
By given conditions
$25x + 50y \le 1175 \Rightarrow x + 2y \le 47$ …. (ii)
Since x and y cannot be negative, so $x \ge 0, y \ge 0$.
Total profit = $3x + 2y$
$\therefore$Objective function is
Maximize $Z = 3x + 2y$ subject to
$2x + y \le 40$
$x + 2y \le 47$
$x \ge 0, y \ge 0$
Introducing the slack variable S_1 and S_2, the given LPP can be written as,
$Z - 3x - 2y = 0$ such that
$2x + y + s_1 = 40$
$x + 2y + s_2 = 47$
$x, y, s_1, s_2 \ge 0$
The initial simplex tableau

Basic Variable	x	y	s_1	s_2	RHS
s_1	$\boxed{2}$	1	1	0	40
s_2	1	2	0	1	47
	–3	–2	0	0	0

Here, Pivot element = 2

To make the pivot element 1, applying $R_1 \to \frac{1}{2} R_1$, we get

Basic Variable	x	y	s_1	s_2	RHS
x	$\boxed{1}$	$\frac{1}{2}$	$\frac{1}{2}$	0	20
s_2	1	2	0	1	47
	–3	–2	0	0	0

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 3R_1$, we get

Basic Variable	**x**	**y**	**s_1**	**s_2**	**RHS**
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	20
s_2	0	$\boxed{\frac{3}{2}}$	$-\frac{1}{2}$	1	27
	0	$-\frac{1}{2}$	$\frac{3}{2}$	0	60

Pivot element = $\frac{3}{2}$

To make the pivot element 1, applying $R_2 \to \frac{2}{3} R_2$, we get

Basic Variable	**x**	**y**	**s_1**	**s_2**	**RHS**
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	20
	0	$\boxed{1}$	$-\frac{1}{3}$	$\frac{2}{3}$	18
	0	$-\frac{1}{2}$	$\frac{3}{2}$	0	60

Applying $R_1 \to R_1 - \frac{1}{2} R_2$ and $R_3 \to R_3 + \frac{1}{2} R_2$ we get,

Basic Variable	x	y	s_1	s_2	RHS
x	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	11
y	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	18
	0	0	$\frac{4}{3}$	$\frac{1}{3}$	69

Since all the entries in the last row is non-negative. So, we get an optimal solution as $x = 11$, $y = 18$ and

Max. $(z) = 3x + 2y = 3 \times 11 + 2 \times 18 = 33 + 36 = 69$

6. Two spare parts X and Y are to be produced. Each one has to go through two processes A and B. Each of X has to spend 3 hours in A and 9 hours in B. Also each of Y has to spend 4 hours in A and 4 hours in B. The time available for A and B are at most 36 hours and 60 hours respectively. If the profit per unit of X and Y are Rs. 50 and Rs. 60 respectively.
   a. Formulate the LP Problem
   b. Express it into standard form
   c. Solve the LP Problem by simplex method and find the maximum profit.

Solution:

Process	Spare parts		Total time available
	X	Y	
A	3	4	36
B	9	4	60
Profit per unit	Rs. 50	Rs. 60	

Let the number of parts of the type X and Y respectively. Then the formulation of the above LP problem is given below.

Max profit $(Z) = 50x + 60y$ subject to

$3x + 4y \leq 36$

$9x + 4y \leq 60$

$x, \; y \geq 0$

Introducing the slack variable S_1 and S_2, the given LPP can be written as,

$z - 50x - 60y$ subject to

$3x + 4y + s_1 = 36$

$9x + 4y + s_2 = 60$

$x, y, s_1, s_2 \geq 0$

The initial simplex tableau

Basic Variable	x	y	s_1	s_2	RHS
s_1	3	$\boxed{4}$	1	0	36
s_2	9	4	0	1	60
	–50	–60	0	0	0

Since, –60 is the most negative entry, so y-column is the pivot column. Again since $\frac{36}{4} = 9$, $\frac{60}{4} = 15$ and $9 < 15$, so r-row is the pivot row, thus getting 4 (interesting of y-column and r-row) is the pivot entry.

Applying $R_1 \to \frac{1}{4} R_1$, we get

Basic Variable	x	y	s_1	s_2	RHS
y	$\frac{3}{4}$	1	$\frac{1}{4}$	0	9
s_2	9	4	0	1	60
	–50	–60	0	0	0

Applying $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 + 60R_1$, we get

Basic Variable	**x**	**y**	$\mathbf{s_1}$	$\mathbf{s_2}$	**RHS**
y	$\frac{3}{4}$	1	$\frac{1}{4}$	0	9
s_2	6	0	–1	1	24
	–5	0	15	0	540

–5 is the only negative entry, so x-column is the pivot column. Again since, $\frac{9}{3/4} =$ 12,

$\frac{24}{6} = 4$ and $4 < 12$, so r-row is the pivot row, thus getting 6 is the pivot entry.

Applying $R_2 \to \frac{1}{6} R_2$, we get

Basic Variable	x	y	s_1	s_2	RHS
y	$\frac{3}{4}$	1	$\frac{1}{4}$	0	9
x	$\boxed{1}$	0	$-\frac{1}{6}$	$\frac{1}{6}$	4
	–5	0	15	0	540

Applying $R_1 \to R_1 - \frac{3}{4} R_2$ and $R_3 \to R_3 + 5R_2$ we get,

Basic Variable	**x**	**y**	$\mathbf{s_1}$	$\mathbf{s_2}$	**RHS**
y	0	1	$\frac{3}{8}$	$-\frac{1}{8}$	6
x	1	0	$-\frac{1}{6}$	$\frac{1}{6}$	4
	0	0	$\frac{85}{6}$	$\frac{5}{6}$	560

Since all the entries in the last row is non-negative. So, we get an optimal solution is obtained.

Maximum value of $Z = 560$ when $x = 4$, $y = 6$

Checking: Max $Z = 50x + 60y = 50 \times 4 + 60 \times 6 = 560$

■

CHAPTER 19

SYSTEM OF LINEAR EQUATION

EXERCISE 19.1

1. Solve the following system of linear equations by Gauss elimination method:

a. $4x + 5y = 12$
$3x + 2y = 9$

b. $5x + 2y = 4$
$7x + 3y = 5$

c. $5x - 3y = 8$
$2x + 5y = 59$

d. $2x - 3y = 7$
$3x + y = 5$

Solution:

a. Given equations are

$4x + 5y = 12$ ... (i)

$3x + 2y = 9$ ... (ii)

Multiplying by 3 in (i) & 4 in eq. (ii) and subtracting eq. (ii) from eq. (i)

Forward elimination

$12x + 15y = 36$
$12x + 8y = 36$
$- \quad - \quad\quad -$

$7y = 0$

$\therefore$ y = 0

Backward substitution

Put the value of y in eq. (i), we get

$4x + 5 \times 0 = 12$

or, $4x = 12$

$x = 3$

$\therefore$ x = 3, y = 0

b. Given equation are

$5x + 2y = 4$ ... (i)

$7x + 3y = 5$ ... (ii)

Multiplying by 7 in eq. (i) & 5 in eq. (ii) and subtracting eq. (ii) from eq. (i)

Forward elimination

$35x + 14y = 28$
$35x + 15y = 25$
$- \quad - \quad\quad -$

$-y = 3$

$\therefore y = -3$

Backward substitution

$5x + 2y = 4$

or, $5x + 2 \times (-3) = 4$

or, $5x = 4 + 6$

or, $5x = 10$

$\therefore$ x = 2

Hence, the value of x & y are 2 and –3 respectively.

c. $5x - 3y = 8$ ... (i)

$2x + 5y = 59$ ... (ii)

Forward elimination

Multiplying by 2 in eq. (ii) & 5 in eq. (ii) and subtracting eq. (ii) from eq. (i)

$10x - 6y = 16$
$10x + 25y = 295$
$- \quad - \qquad -$

$-31y = -279$
$\therefore\ y = 9$
Backward substitution
Put the value of y in eq. (i)
$5x - 3 \times 9 = 8$
or, $5x = 8 + 27$
or, $5x = 35$
$\therefore\ x = 7$
Hence, $x = 7$ & $y = 9$

d. $2x - 3y = 7$ ... (i)
$3x + y = 5$ ... (ii)
Forward elimination
Multiplying by 3 in eq. (i) & 2 in eq. (ii) and subtracting eq. (ii) from eq. (i)
$6x - 9y = 21$
$6x + 2y = 10$
$- \quad - \qquad -$

$-11y = 11$
$\therefore\ y = -1$
Backward substitution,
Put the value of y in eq. (ii)
or, $2x - 3y = 7$
or, $2x - 3 \times (-1) = 7$
or, $2x = 7 - 3$
$\therefore\ x = 2$
Hence, the required value of x & y are 2 & –1 respectively.

2. Solve the following system of linear equations by Gauss elimination method:

a. $5x - y + 4z = 5$
$2x + 3y + 5z = 2$
$5x - 2y + 6z = -1$

b. $x - y + 2z = 7$
$3x + 4y - 5z = -5$
$2x - y + 3z = 12$

c. $2x + 3y + 3z = 5$
$x - 2y + z = -4$
$3x - y - 2z = 3$

d. $x + 2y + 3z = 14$
$3x + 4y + 2z = 17$
$2x + 3y + z = 11$

Solution:

a. Here, given equation are
$5x - y + 4z = 5$ ... (i)
$2x + 3y + 5z = 2$ ... (ii)
$5x - 2y + 6z = -1$ ... (iii)
Multiplying by 2 in eq. (i) & 5 in eq. (ii) and subtracting eq. (ii) from eq. (i)
$10x + 15y + 25z = 10$
$10x - 2y + 8z = 10$
$- \quad - \quad - \quad -$

$-17y + 17z = 0$ ... (iv)
Again,
Subtracting eq. (iii) from eq. (i)

$5x - y + 4z = 5$
$5x - 2y + 6z = -1$
$- \quad + \quad - \quad +$

$y - 2z = 6$ ... (v)
Multiplying by 17 in eq. (v) & subtracting eq. (v) from eq. (iv)
$17y + 17z = 0$
$17y - 34z = 102$
$- \quad + \quad -$

$51z = -102$
$\therefore\ z = -2$
The system of linear equations becomes
$5x - y + yz = 5$
$y - 2z = 6$
$2z = -2$
Put the value of z in eq. (v)
$y - 2 \times (-2) = 6$
or, $y = 6 - y$
$\therefore\ y = 2$
Again, put the values of x & y in eq. (i)
$5x - y + yz = 5$
or, $5x - 2 + 4 \times (-2) = 5$
or, $5x = 5 + 7 + 8$
or, $5x = 15$
$\therefore\ x = 3$
Hence, x = 3, y = 2, z = –2

b. Here, given equations are
$x - y + 2z = 7$ ... (i)
$3x + 4y - 5z = -5$ ... (ii)
$2x - y + 3z = 12$ ... (iii)
Multiplying by 3 in eq. (i) & subtracting eq. (ii) from eq. (i)
$3x - 3y + 6z = 21$
$3x + 4y - 5z = -5$
$- \quad - \quad + \quad +$

$-7y + 11z = 26$ ... (iv)
Multiplying by 2 in eq. (i) and subtracting eq. (iii) from eq. (i)
$2x - 2y + 4z = 14$
$2x - y + 3z = 12$
$- \quad + \quad - \quad -$

$-y + 2 = 2$ ... (v)
Multiplying by 7 in eq. (v) and subtracting eq. (v) from eq. (iv)
$-7y + 11z = 26$
$-7y + 7z = 14$
$+ \quad - \quad -$

$-y + z = 2$ ... (iv)
Multiplying by 7 in eq. (v) and subtracting eq. (v) from eq. (iv)
$-7y + 11z = 26$
$-7y + 7z = 14$

$+ \quad - \quad -$

---

or, $4z = 12$

$\therefore z = 3$

The system of linear equations become

$x - y + 2z = 7$

$-y + z = 2$

$z = 3$

Put the value of z in eq. (v)

$-y + 3 = 2$

or, $-y = -3 + 2$

or, $y = 1$

Again, Put the value of y & z in eq. (i)

$x - 1 + 2 \times 3 = 7$

or, $x = 7 - 6 + 1$

or, $x = 2$

Hence, $x = 2$, $y = 1$, $z = 3$.

c. We have,

$2x + 3y + 3z = 5$ ... (i)

$x - 2y + z = -4$ ...(ii)

$3x - y - 2z = 3$ ...(iii)

from (i) and (ii), we get

$7y + z = 13$ ...(iv)

from (ii) and (iii) we get

$5y - 5z = 15$

$\therefore \; y - z = 3$ ...(v)

Adding (iv) and (v), we get

$8y = 16$

$\therefore \; y = 2$

from (v), $2 - z = 3$

$\therefore \; z = -1$

from (ii), $x - 4 - 1 = -4$

$\Rightarrow \; x = -4 + 5 = 1$

Hence, $x = 1$, $y = 2$, $z = -1$

d. Here, given equations are

$x + 2y + 3z = 14$ ... (i)

$3x + 4y + 2z = 17$ ... (ii)

$2x + 3y + z = 11$ ... (iii)

Multiplying by 3 in eq. (i) and subtracting eq. (ii) from eq. (i)

$3x + 6y + 9z = 42$

$3x + 4y + 2z = 17$

$- \quad - \quad - \quad -$

$2y + 7z = 25$. (iv)

Multiplying by 2 in eq. (i) and subtracting eq. (iii) from (ii)

$2x + 4y + 6z = 28$

$2x + 3y + z = 11$

$- \quad - \quad - \quad -$

$y + 5z = 17$ ... (v)

Multiplying by 2 in eq. (v) & subtracting eq. (v) from (iv)

$2y + 7z = 25$
$2y + 10z = 34$
$- \quad - \quad +$

$+3z = +9$

$\therefore z = 3$

The system of linear equations becomes

$x + 2y + 3z = 14$
$y + 5z = 17$
$z = 3$

Put the value of z in eq. (v)

or, $y + 5 \times 3 = 17$
or, $y = 17 - 15 = 2$
$\therefore y = 2$

Put the value of x & y in eq. (i)

$x + 2y + 3z = 14$
or, $x + 2 \times 2 + 3 \times 3 = 14$
or, $x = 14 - 9 - 4$
or, $x = 14 - 13$
$\therefore \quad x = 1$

Hence, $x = 1$, $y = 2$ & $z = 3$

3. Test the consistency of the following system of equations by Gaussian elimination method.

a. $x + 3y = 5$
$3x + y = 4$

b. $3x - 2y = 3$
$3x - 2y = 6$

c. $-2x + 5y = 3$
$6x - 15y = -9$

d. $x - 2y - 5z = -12$
$2x - y = 7$
$-4x + 5y + 6z = 1$

e. $2x - y + 4z = 4$
$x + 2y - 3z = 1$
$3x + 3z = 6$

f. $x + 3y + 4z = 8$
$2x + y + 2z = 5$
$5x + 2z = 7$

Solution:

a. $x + 3y = 5$ ... (i)
$3x + y = 4$ ... (ii)

Multiplying by 3 in eq. (i) and subtracting eq. (ii) from eq. (i) Forward elimination

$3x + 9y = 15$
$3x + y = 4$
$- \quad - \quad -$

$8y = 11$

$y = \frac{11}{8}$

Backward substitution

Put the value of y in eq. (i)

$x + \frac{3 \times 11}{8} = 5$

or, $x = 5 - \frac{33}{8}$

or, $x = \frac{7}{8}$

$\therefore$ $x = \frac{7}{8}$, $y = \frac{11}{8}$

It is consistent and has unique solution.

b. Here,

$3x - 2y = 3$ ... (i)

$3x - 2y = 6$ ... (ii)

Subtracting eq. (ii) from eq. (i)

$3x - 2y = 3$

$3x - 2y = 6$

$\underline{-\quad +\quad -}$

$0 = -3$

Hence, it is inconsistent and has no solution.

c. $-2x + 5y = 3$ ... (i)

$6x - 15y = -9$ ... (ii)

Multiplying by 3 in eq. (i) & adding eq. (i) and eq. (ii)

$-6x + 15y = 9$

$6x - 15y = -9$

$\underline{-\quad +\quad\quad +}$

$0 = 0$

It is consistent having infinitely many solution.

d. Given equations are:

$x - 2y - 5z = -12$ ... (i)

$2x - y = 7$ ... (ii)

$-4x + 5y + 6z = 1$ ... (iii)

Multiplying by 4 in eq. (i) and adding eq. (i) & eq. (iii)

$4x - 8y - 20z = -$
48

$-4x + 5y + 6z = 1$

$\underline{+\quad\quad +}$

$-3y - 14z = -47$

or, $3y + 14z = 47$ ... (iv)

Multiplying by 2 in eq. (i) and subtracting eq. (ii) from eq. (i)

$2x - 4y - 10z = -24$

$2x - y = 7$

$\underline{+\quad -}$

$-3y - 10z = -31$...
(iv)

Adding eq. (iv) and eq. (v)

$3y + 14z = 47$

$\underline{-3y - 10z = -31}$

$4z = 16$

$\therefore z = 4$

The system of linear equations becomes

$x - 2y - 5z = -12$

$3y + 14z = 47$

$z = 4$

Put the value of z in eq. (i)

$3y + 14 \times 4 = 47$

or, $3y = 47 - 56$
or, $3y = -9$
$\therefore$ $y = -3$

e. Here,
$2x - y + 4z = 4$ ... ... (i)
$x + 2y - 3z = 1$ ... ... (ii)
$3x + 3z = 6$ ... ... (iii)
Multiplying by 2 in equation (i) and subtracting equation (ii) from equation (i)
$2x - y + 4z = 4$
$2x + 4y - 6z = 2$
$- \quad - \quad + \quad -$
$-5y + 10z = 2$ ... ... (iv)
Multiplying by 3 in equation (i) and subtracting equation (iii) from equation (ii)
$3x + 6y - 92 = 3$
$3x + 3z = 6$
$- \quad - \quad -$
$6y - 12z = -3$
or, $2y - 4z = -1$ ... ... (v)
Multiplying by 2 in equation (iv) and adding equation (iv) and equation (v)
$-10y + 20z = 4$
$\underline{10y - 20z = -5}$
$0 \; -1$
Here, $0 = -1$
It is inconsistent having no solution.

f. Here, Given equations are
$x + 3y + 4z = 8$ ... ... (i)
$2x + y + 2z = 5$ ... ... (ii)
$5x + 2z = 7$ ... ... (iii)
Multiplying by 2 in equation (i) and subtracting equation (ii) from equation (i)
$2x + 6y + 8z = 16$
$2x + y + 22 = 5$
$- \quad - \quad - \quad -$
$5y + 6z = 11$ ... ... (iv)
Multiplying by 5 in equation (i) and subtracting equation (iii) from equation (i)
$5x + 15y + 20z = 40$
$5x + 0.y + 22 = 7$
$- \quad - \quad - \quad -$
$15y + 18z = 33$ ... ... (v)
Multiplying by 3 in equation (iv) and subtracting equation (v) from equation (iv)
$15y + 18z = 33$
$15y + 18z = 33$
$- \quad - \quad -$
$0 = 0$
It is consistent having infinitely many solution.

■

CHAPTER 20

PARALLEL FORCES

EXERCISE 20

1. Find the resultant of two parallel forces 4 N and 6 N at a distance of 5 m, when they are (a) like parallel, (b) unlike parallel

Solution:

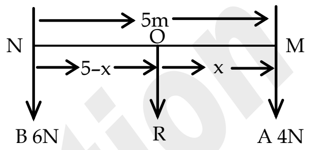
a. Let A and B be two parallel forces acting at points. M and N respectively.

The magnitude of the resultant is given by

R = A + B = 4N + 6N = 10N

The direction of the resultant is same as that of the two forces.

Let the position of the resultant R be at 0, at a at a distance x from M.



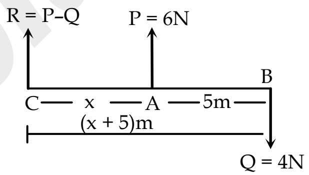
We have, A × Mo = B × No

or, $4 \times x = b\ (45 - x)$

or, $4x + 6x = 30 \Rightarrow x = 3m$

b. When they are unlike parallel

Let the resultant R of unlike parallel force P and Q then R = 6N – 4N = 2N



$$\frac{P}{AC} = \frac{Q}{AB} = \frac{R}{BC}$$

$$\frac{6}{5-x} = \frac{4}{x} = \frac{2}{5}$$

$$\frac{6}{5-x} = \frac{2}{5}$$

$30 = 10 - 2x$

$3x = -10m$

Also, $\frac{4}{x} = \frac{2}{5}$

or, $x = 10m$

∴ Resultant = 2N, 10m away from 6N

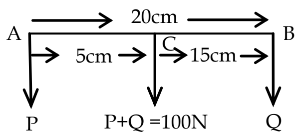
2. Find two like parallel forces at a distance of 20 cm equivalent to 100 N force, the line of action of one of them being at a distance of 5 cm from the given force.

Solution:

Suppose P and Q be two like parallel forces acting at the point A and B such that AB = 20cm. Then the line of action of their resultant is the force P + Q = 100N acting at the point C where AC = 5cm and BC = 15cm. By question, forces are parallel, so we have

$$\frac{P}{BC} = \frac{Q}{AC} = \frac{P+Q}{AB}$$

or, $\frac{P}{15} = \frac{Q}{5} = \frac{100}{20} = 5$



or, $\frac{P}{15} = 5$ or, $\frac{Q}{5} = 5$

or, $P = 15 \times 5$ or, $Q = 5 \times 5$

$= 75N$ or, $Q = 25N$

Hence, the required forces are 75N and 25N

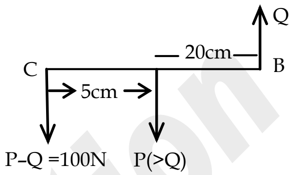
3. Find two unlike parallel forces at a distance of 20 cm equivalent to 100 N force, the line of action of the greater of them being at a distance of 5 cm from the given force.

Solution:

Suppose P and Q be two unlike parallel forces (P > Q) acting at the point A and B such that AB = 20cm. Since forces are unlike, so that their resultant is the force P – Q = 100N acting at the point C, where AC = 5cm. Since forces are parallel. So, we have,

$$\frac{P}{BC} = \frac{Q}{AC} = \frac{P-Q}{BC-AC} = \frac{P-Q}{AB} = \frac{100}{20} = 5$$

$$\frac{P}{25} = \frac{Q}{5} = 5$$



$\therefore \quad \frac{P}{25} = 5$ or, $\frac{Q}{5} = 5$

$\therefore \quad P = 25 \times 5$ or, $Q = 5 \times 5$

$= 125N$ or, $Q = 25N$

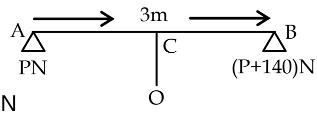
Required forces are 125N and 25N.

4. The extremities of a straight bamboo pole 3 m long rests on two smooth pegs at A and B in the same horizontal line. A heavy load hangs form a point C on the pole. If AC = 3BC and the pressure at B is 140 N more than that at A, find the weight of the load.

Solution:

Let AB be the straight bamboo of 3m long. Let PN and (P + 140)N acting at A and B. Since the heavy load hangs at point C such that AC = 3BC. So, the line of action of the resultant acting at point C. Since forces are parallel,

$$\therefore \quad \frac{P}{BC} = \frac{P+140}{AC} \Rightarrow \frac{P}{BC} = \frac{P+140}{3BC}$$



$\Rightarrow \quad 3P = P + 140$

Hence, P = 70N

Hence, the weight of the load = PV + (P + 140)N

= 70N + (70 + 140)N = 280N

5. A heavy uniform beam 5 m long is supported in a horizontal position by two props, one is at one end and the other is such that the beam projects 1.5 m beyond it. If the weight of the beam is 70 kg wt, find the pressures at the props.

Solution:

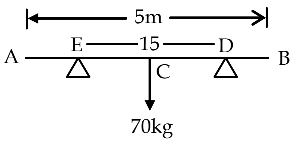
Let AB be the uniform beam AB = 5m

Let C be the midpoint of AB so that AC = CB = 2.5m

Let E and D be two props such that DB = 1.75m, CD = 2.5m – 1.75m = 0.75m, EC = 1.5m – 0.75m = 0.75m

Let R_1 and R_2 be the reactions at E and F.

$$\therefore \quad \frac{R_1}{CD} = \frac{R_2}{EC} = \frac{70}{ED}$$



$$\Rightarrow \quad \frac{R_1}{0.75} = \frac{R_2}{0.75} = \frac{70}{1.5}$$

$$\therefore \quad R_1 = \frac{70 \times 0.75}{1.5} = 35 \text{ kg}$$

$$R_2 = \frac{70 \times 0.75}{1.5} = 35\text{kg}$$

6. P and Q are like parallel forces with the resultant R. If P is moved parallel to itself through a distance x, show that R is displaced by a distance $\frac{Px}{R}$.

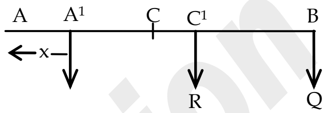
Solution:

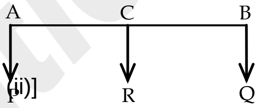
Let R be the resultant of two like parallel forces P and Q acting of A and B respectively. Suppose resultant R acts and P_1 then

$$\frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}$$

from $\frac{P}{CB} = \frac{R}{AB} \Rightarrow CB = \frac{P.AB}{R}$ ... ... (i)

For the second case, let P acts at A^1 such that $AA^1 = x$ then the resultant displace from C to C^1.





So, $\frac{P}{C^1B} = \frac{Q}{A^1C^1} = \frac{R}{A^1B}$

from, $\frac{P}{C^1B} = \frac{R}{A^1B}$

or, $C^1B = \frac{P.\ A^1B}{R}$ ... ... (ii)

Now, $CC^1 = CB - C^1B = \frac{P.AB}{R} - \frac{P.A^1B}{R}$ [$\because$ from (i) and (ii)]

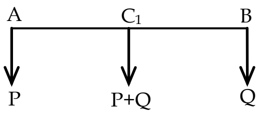
$$= \frac{P}{R}(AB - A^1B)$$

or, $CC^1 = \frac{PX}{R}$ Hence proved.

7. Two like parallel forces of magnitudes P and Q are acting at the end points A and B of a rod AB of length r. If two opposite forces each of magnitude F are added to P and Q, then prove that the line of action of the new resultant will move a distance $\frac{Fx}{P+Q}$.

Solution:

Suppose the two forces P and Q acting at A and B and let their resultant P+Q acting at C_1.



$\therefore \frac{P}{BC_1} = \frac{Q}{AC_1} = \frac{P+Q}{BC_1 + AC_1}$

or, $\frac{P}{BC_1} = \frac{P+Q}{AB}$

$\therefore BC_1 = P.\frac{AB}{P+Q}$

If the force P is moved parallel to itself through a distance x to D then the resultant act at C_2, where AD = x.

Then, $\frac{P}{BC_2} = \frac{Q}{DC_2} = \frac{P+Q}{BC_2+DC_2}$

or, $\frac{P}{BC_2} = \frac{P+Q}{BD}$

$\therefore BC_2 = \frac{P}{P+Q}.BD$

Now, the required distance which the resultant moves $C_1C_2 = BC_2 - BC_1$

$$= \frac{P.BD}{P+Q} - \frac{P.AB}{P+Q} = \frac{P}{P+Q}(BD - AB) = \frac{P}{P+Q}.AD = \frac{Px}{P+Q}$$

8. Straight uniform rod is 3m long when a load of 5N is placed at one end it balances about a point 25cm from that end. Find the weight of the rod.

Solution:

Let W be the weight of the rod AB, acting at centre point C of AB. A load 5N is placed at A, it balances 25cms from that end. i.e. AD = 25cms AB = 3m = 300 cms.

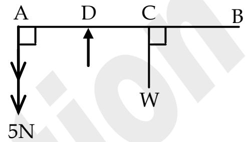
$\therefore$ $AC = BC = \frac{300}{2} = 150$ cms

$\therefore$ DC = AC – AD = 150 cms – 25cms = 125cms

Now, using the like parallel forces theorem.

$$\frac{5}{DC} = \frac{W}{AD}$$

$\therefore$ $W = \frac{5AD}{CD} = \frac{5 \times 25}{125} = 1N$



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CHAPTER 21

DYNAMICS

EXERCISE 21.1

1. a. An average sized onion has a mass 50g. Find the weight of the apple in Newton? ($g = 9.8 m/s^2$)
   b. A bicycle of mass 20 kg is accelerated at $2 m/sec^2$. Find the force acting on it.
   c. Find the acceleration produced when a force of 5hg wt. acts on a mass of 1 kg.

Solution:

a. Here, mass (m) = 50g = $\frac{50}{100}$ kg = 0.05 kg
   $g = 9.8 m/s^2$
   Weight (w) = ?
   ∴ w = mg = 0.05 × 9.8 = 0.49N Ans.

b. Here, mass (m) = 20kg
   Acceleration (a) = $2 m/s^2$
   Force (f) = ?
   ∴ f = ma = 20 × 2 = 40N

c. Here, acceleration (a) = ?
   Force (f) = 5kg = 50N
   Mass (m) = 1kg
   ∴ F = ma
   or, $a = \frac{f}{m} = \frac{50}{1}$
   ∴ $a = 50 m/s^2$

2. a. A bicycle has mass 50kg.If its velocity increases from 2m/sec to 5m/sec in 6 seconds, find the force exerted on it.
   b. A body of mass 10kg falling from a certain height is brought to rest after striking the ground with a speed of 5m/sec. If the resistance force of the ground is 200N, find the duration of contact.
   c. A car is pushed on a frictional smooth plane with an average force of 50N for 10 sec. If the car with mass 500 kg is at rest in the beginning, find the velocity acquired by the car.

Solution:

a. Mass of bicycle (m) = 50 kg
   Initial velocity (u) = 2 m/s
   Final velocity (v) = 5 m/s
   Time taken (t) = 6 sec
   Force exerted (f) = ?
   ∴ $F = \frac{m(v-u)}{t} = \frac{50(5-2)}{6} = \frac{50 \times 3}{6} =$ 25N
   ∴ F = 25N

b. Here, mass of the body (m) = 10kg
   Initial velocity (u) = 5m/s
   Final velocity (v) = 0
   Force on the ground (f) = 200N
   Duration of contact (t) = ?
   Now, applying the formula, $f = \frac{mv - mu}{t}$
   or, $t = \frac{m(v-u)}{f} = \frac{10(0-5)}{200} = -0.25$
   ∴ t = 0.25 sec

c. Here, Average force (f) = 50N Mass of car (m) = 500kg
   Time taken (t) = 10 sec Initial velocity (u) = 0

Final velocity (v) = ?

Now, we have, $f = \dfrac{m(v-u)}{t}$

or, $v = \dfrac{ft}{m} + u$

or, $v = \dfrac{50 \times 10}{500} + 0$

$\therefore$ v = 1 m/s

3. a. A horse directs a horizontal Jet of water, moving with a velocity of 30 m/sec on a vertical wall. If the mass of water per second striking the wall is 3kg/ sec, find the force on the wall.

   b. Sand allowed to fall vertically at a steady rate hits a horizontal floor with a speed $0.04ms^{-1}$. If the force exerted on the floor is 0.004N, find the mass of sand falling per second.

   c. Rain drops falling vertically on ground at the rate of 0.3 kgs^{-1} come to rest after hitting the ground. If the resistance force of the ground is 3N, find the velocity of rain drops just before hitting the ground.

Solution:

a. Mass of water per second $\left(\dfrac{m}{t}\right)$ = 3kg/sec.

Initial velocity (u) = 30 m/s

Final velocity (v) = 0

Force on the wall (f) = ?

Now, apply, $f = \dfrac{mv - mu}{t}$

or, $f = \dfrac{m(v-u)}{t} = \dfrac{m}{t}(v-u) = 3(0-30) = -90$

$\therefore$ f = 90N

b. Here, initial velocity (u) = 0

Fin al velocity (v) = 0.04 m/s

Force exerted on the floor (f) = 0.004N

Mass of sand falling per second (m/t) = ?

Now, applying $f = \dfrac{mv - mu}{t} = \dfrac{m}{t}(v-u)$

or, $\dfrac{m}{t} = \dfrac{f}{v-u} = \dfrac{0.004}{(0.04-0)} = 0.1$

$\therefore$ $\dfrac{m}{t} = 0.1$ kg/s

$\therefore$ Mass of sand falling per second = 0.1kg/sec

c. Quantity of rain falling per second $\left(\dfrac{m}{t}\right)$ = 0.3 kg/s

Force of the ground (f) = 3N

Velocity before hitting the ground (u) = ?

Velocity after hitting the ground (v) = 0

We know, $f = \dfrac{mv - mu}{t} = \dfrac{m}{t}(v-u)$

or, 3 = 0.3 (0 – u)

or, u = –10 m/s

$\therefore$ u = 10 m/s

4. a. A force 1 kg wt. acts on a body continuously for seconds and causes it to describe one metre in that time, find the mass of the body.

   b. A body of, mass 25kg is acted upon by a force of 200N. How long will it take to move the body from rest through 64m?

   c. A force of 520N acting on a body for 30 secs increases its velocity from 290 m/sec to 350 m/sec. Find the mass of the body.

d. A bullet of mass 20g fired into a wall with a velocity of 30m/sec loses its velocity in penetrating into a wall through 3cms. Find the average force exerted by the wall.

e. How large a force required to bring a motorbike of mass 500 kg moving with a velocity of $50ms^{-1}$ to rest at

i. a distance of 50m ii. in 10 secs

f. A constant force of 20N acting on an object reduces if velocity from $30ms^{-1}$ to $10ms^{-1}$ in 3 secs. Find the mass of the object.

g. A car of mass 1000 kg travelling at 36 km/hr is brought to rest over a distance of 20m. Find the average braking force.

h. Find the velocity of a 5kg shot that will just penetrate through a wall 20 cms thick the resistance being 40 tons wt.

Solution:

a. Suppose m be the mass of an object and a be the acceleration. So that

f = ma [∵ f = 1×9.8 = 9.8N]

or, 9.8 = ma

or, $a = \frac{9.8}{m}$

Now, $s = \frac{1}{2}at^2$

or, $l = \frac{1}{2} \times \frac{9.8}{m} \times 10^2$

or, $l = \frac{9.8 \times 50}{m}$

or, m = 490 kg

b. Mass of body (m) = 25kg

Force (f) = 200N

Time taken (t) = ?

Initial velocity (u) = 0

Distance covered (s) = 64m

we have, f = ma

or, $a = \frac{f}{m} = \frac{200}{25} = 8$

∴ $a = 8 \text{ m/s}^2$

Now, $s = ut + \frac{1}{2}at^2$

or, $64 = 0 + \frac{1}{2} \times 8 \times t^2$

or, $4t^2 = 64$

or, $t^2 = 16$

∴ t = 4 sec

c. Here, force (f) = 520N

Time (t) = 30 sec

Initial velocity (u) = 290 m/s

Final velocity (v) = 350 m/s

Mass of body (m) = ?

We have, $f = \frac{m(v - u)}{t}$

or, $m = \frac{ft}{v - u} = \frac{520 \times 30}{350 - 290} = \frac{520 \times 30}{60} = 260$

∴ m = 260 kg

d. Mass of bullet (m) = 20gms = 0.02 kg

Initial velocity (u) = 30 m/s

Distance (s) = 3 cms = 0.03m

Average force (f) = ?

Final velocity (v) = 0

∴ Applying, $v^2 = u^2 + 2as$

or, $0 = 30^2 + 2a \times 0.03$

or, –900 = 0.06a

or, $a = -\frac{900}{0.06} = -15000$

∴ $a = -15,000 \text{ m/s}^2$

Now, force, f = ma = 0.02×(–15,000) = –300

∴ f = 300N

e. Mass of motor bike (m) = 500kg

Initial velocity (u) = 50m/s

Final velocity (v) = 0

f. Force (f) = 20N

Initial velocity (u) = 30 m/s

Final velocity (v) = 10 m/s

Force (f) = ?

(i) A distance of 50m

Since, s = 50m

Now, using, $v^2 = u^2 + 2as$

or, $0^2 = 50^2 + 2a \times 50$

or, $2500 = 100a$

or, $a = -25$ where a is retardation

$\therefore$ $a = 25\ m/s^2$

$\therefore$ f = mass×retardation = 500×25 = 12500N

(ii) In 10 seconds

Now, $v = u + at$

or, $0 = 50 + a\times 10$

or, $10a = -50$

$a = -5\ m/s^2$, where a is retardation

$\therefore$ f = mass × retardation = 500 × 5 = 2500 N

Time taken (t) = 3 sec

Mass (m) = ?

$\therefore$ $F = \dfrac{m(v-u)}{t}$

or, $m = \dfrac{Ft}{v-u} = \dfrac{20 \times 3}{(10-30)} = \dfrac{20 \times 3}{-20}$

$\therefore$ m = 3kg

g. Final velocity (v) = 0

Distance (s) = 20m

Average force (f) = ?

We have,

$\therefore$ $v^2 = u^2 + 2as$

or, $0^2 = 10^2 + 2.a \times 20$

or, $-100 = 40a$

or, $a = -\dfrac{100}{40} = -2.5\ m/s^2$ where a is retardation.

Now, f = mass × retardation = 1000 × 2.5 = 2,500N

h. Mass of shot (m) = 5kg

Penetrating space (S) = 20cms = 0.2m

Resistance (f) = 40 tones = 40×1000×9.8N

If a is the retardation produced by the wall then, f = –ma

or, $a = -\dfrac{f}{m} = -\dfrac{40 \times 1000 \times 9.8}{5} = -78400\ m/s^2$

Let u is initial velocity and v be final velocity then u = 2, v = 0

Using the formula,

$v^2 = u^2 + 2as$

or, $0^2 = u^2 + 2(-78400) \times 0.2$

or, $u^2 = 31,360$

or, $u = \sqrt{31360}$

$\therefore$ u = 177.08 m/s

5. Find the velocity of 4 kg shot that will just penetrate through a wall 16 cms thick, the resistance being 4 metric tons weight.

Solution:

Here, mass of the shot (m) = 4kg

Penetrated space (S) = 16cms = 0.16m

Resistance (F) = 4 metric tons

= 4×1,000×9.8 = 392000N

If a is retardation produced by the wall then, F = –ma or, $a = -\frac{F}{m} = -\frac{39200}{4} = -9800\ m/s^2$

Let u is the initial velocity and v is final velocity then, u = 2, v = 0

Using the formula,

$v^2 = u^2 + 2as$

or, $0^2 = u^2 + 2(-9800) \times 0.16$

or, $u^2 = 3136$

or, $u = \sqrt{3136} \Rightarrow u = 56 m/s$

6. A resultant force of 25N acts on a mass of 0.5 kg starting from rest. Find.
   a. the acceleration
   b. the final velocity after 20 secs
   c. the distance moved (g = 10 m/ sec^2)

Solution:

Here, force acting (f) = 25N

Mass of body (m) = 0.50 kg

Initial velocity (u) = 0

a. The acceleration in ms^{-2}

Now, f = ma

or, $a = \frac{F}{m} = \frac{25}{0.50} = 50\ m/s^2$

b. The final velocity after 20 sec.

$\Rightarrow$ Let v be the velocity after 20 sec.

Then, using v = u + at

or, $v = 0 + 50 \times 20$

$\therefore$ v = 1,000 m/s

Distance of penetration of the target

If a is the retardation of the system, then F = ma

$\Rightarrow a = \frac{F}{m} = \frac{72}{0.006} = 12,000\ m/s^2$

If S is the required distance of penetration of target then, $s = \frac{1}{2}at^2 = \frac{1}{2} \times 12,000 \times (0.01)^2$

$\therefore$ S = 0.6m

c. The distance moved in 20 sec.

If S is required distance moved in 20 sec.

Then, $S = ut + \frac{1}{2}at^2$

or, $S = 0 + \frac{1}{2} \times 50 \times (20)^2$

or, S = 10,000m $\therefore$ S = 10km

7. a. A body of mass 20kg falls 10m form rest and is then brought to rest penetrating 0.5 m into sand. Find the resistance of the sand on it in kg wt.

   b. A mass of 4kg falls 200cms from rest and is then brought to rest by penetrating 20cms into some sand. Find the average thrust of the sand on it.

Solution:

a. Mass of body (m) = 20kg

Distance covered (s) = 10m

Initial velocity (u) = 0

$\therefore$ $v^2 = u^2 + 2gh \Rightarrow v^2 = 20g$ ... ... (i)

The velocity given by (i) is reduced to zero when the body goes to 0.5m into sand. If a is the retardation of the system then,

$(0)^2 = v^2 - 2\times a\times 0.5 \Rightarrow a = v^2 \Rightarrow a = 20g$ m/s^2

Let T be the average thrust of the sand on the body. Now, when the body is penetrating into the sand, then the force acting on the body are

a. A force TN of the sand acting upward
b. The weight 20gN of the body acting downward.

Resultant upward force = (T – 20g)N

Then applying Newton's second law of motion, we have,

or, T – mg = ma

or, T – 20g = 20×ln

or, T = 20g + 200g

$\therefore$ T = 220 kgwt

b. Suppose V is the velocity of the body when it falls 200cms from rest under gravity.

Then u = 0, v = v, h = 2m

or, $v^2 = u^2 + 2gh$

$\therefore$ $v^2 = 0 + 2g \times 2 \Rightarrow v^2 = 4g$ ... ... (i)

The velocity given by (i) is reduced to zero when the body goes to 20cms = 0.2m into sand. If a is the retardation of the system, then

or, $O^2 = v^2 - 2\times a\times 0.2$

or, $a = \dfrac{v^2}{0.4} = \dfrac{4g}{0.4} = 10g$ m/s^2

Let T be the average thrust of the sand on the body.

Now when the body is penetrating into the sand, then the force acting on the body are

a. A force TN of the sand acting upward
b. The weight 4gN of the body acting downward

$\therefore$ Resultant upward thrust = (T – 4g)N

Then apply Newton's second law of motion,

T – mg = ma

or, T = 4g = 4×10g

or, T = 40g $\Rightarrow$ T = 40kg wt

8. A bullet of mass 0.006 kg travelling at 120 m/sec penetrates deeply into a fixed target & is brought to rest in 0.01 secs. Calculate.
The average retarding force exerted on the bullet. (g = 10ms^{-2})

Solution:

Mass of the bullet (m) = 0.006kg Final velocity of the bullet (v) = 120m/s

Time taken (t) = 0.01 sec. Initial velocity (u) = 0

If F be the average retarding force on bullet then,

$$F = \frac{\text{Change in momentum}}{\text{Time taken}} = \frac{m(v-u)}{t} = \frac{0.006 \times (120 - 0)}{0.01} = 72N$$

9. a. A bullet of mass 0.02 kg ejected out of a rifle of mass 10 kg with a speed of 1000m/sec. What will be the speed of the recoil of rifle?
b. A gun weighing 10 kg fires a bullet of 10g with a velocity of 330ms^{-1}. With what velocity does the gun recoil? What is the resultant momentum of the gun and the bullet before firing?
c. A shot of mass 700 kg is fired with a velocity of 600 m/sec from a gun of mass 40 metric tons. If the recoil be resisted by a constant force equal to the

weight of 200 metric tons, through how many metres will the gun recoil?

d. A shot whose mass is 10 kg is discharged by a 5 metric ton gun with a velocity of 245m/sec. Find the constant force which would be required to stop the recoil of the gun in $1\frac{1}{4}$ seconds.
e. A shot of 400 kg is discharged by a gun of 80 metric tons with a velocity of 400 m/sec. Find the constant force which would be required to stop the recoil of the gun in 2 metres.
f. A shot where mass is 40 kg is discharged from a 7,000 kg gun with velocity of 140 ms^{-1}. Find the constant force which acts on the gun would stop it after a recoil of 6.4m.
g. A bullet of mass 2kg is fired from a gun of mass 100 kg with a velocity 250m/sec. Find the recoil velocity of the gun.

Solution:

a. Mass of bullet (M) = 0.02kg
Mass of rifle (m) = 10kg
Muzzle velocity of bullet (v) = 1,000 m/s
Recoil velocity of rifle (v) = ?
We know,
Mass of the bullet × Muzzle velocity = Mass of rifle × Recoil velocity
or, $0.02 \times 1{,}000 = 10 \times v$
or, $v = \dfrac{0.2 \times 1{,}000}{10} = 20$ m/s

b. Here, momentum of the bullet = mv = 330×0.1
Momentum of the gun = Mv = $10 \times v$
∴ Momentum of bullet = Momentum of gun
or, $330 \times 0.1 = 10 \times v$
or, $v = \dfrac{330 \times 0.1}{10} = 3.3$ m/s
Since initial velocity of gun and bullet = 0 m/s
Total momentum before firing = mu + Mu = $0.1 \times 0 + 10 \times 0 = 0$

c. Here, m = Mass of shot = 700kg
v = Velocity of shot = 600 m/s
M = Mass of the gun = 40 metric tons = (40×1000)kg
v = Velocity of the gun = ?
Momentum of the shut = mv = 700×600
Momentum of the gun = Mv = (40×1000) v
By the principal of conservation of linear momentum.
Momentum of shot = momentum of gun (in magnitude)
or, $700 \times 600 = (40 \times 1000)v$
or, $v = \dfrac{700 \times 600}{40 \times 1000} = 10.5$ m/s
∴ Velocity of gun = 109.5 m/s

d. Let v be the recoil velocity of the gun
Then moment of the shut = 10×245
Momentum of the gun = $5000 \times v$
But we know momentum of the shot = Momentum of the gun
or, $10 \times 245 = 5000 \times v$
or, v = 0.49 m/s

The gun recoils with velocity 0.49 m/s. Apply a constant force to the gun so that it will stop after recoiling at time $1\frac{1}{4} = \frac{5}{4}$ seconds.

Let a be the retardation then $0 = v - at \Rightarrow a = \frac{v}{t}$

If f is required constant force to be applied then,

$f = ma = m \times \frac{v}{t} = \frac{5000 \times 0.49}{5/4} = \frac{5000 \times 0.49 \times 4}{5} = 1960N$

e. Let v be the velocity of the gun then momentum of the shot = 400×400
Momentum of the gun = $80,000 \times v$
$\therefore$ Momentum of gun = Momentum of shot
or, $80,000 \times v = 400 \times 400$
or, $v = 2$ m/s
The gun recoils with velocity 2 m/s. Applying a constant force to the gun so that it will stop after recoiling at distance 2 meters.

$\Rightarrow$ Let a be the retardation, then $0^2 = v^2 - 2as \Rightarrow a = \frac{v^2}{2s}$

If f is the required constant force to be applied then,

$f = ma = m \times \frac{v^2}{2s} = 80,000 \times \frac{2^2}{2\times2} = 80,000N$

f. Let v be the recoil velocity at the gun then, momentum of the shot = 40×140
Momentum of the gun = $7,000 \times v$
But, momentum of gun = Momentum of shot $7,000 \times v = 40 \times 140$
or, $v = 0.8$ m/s
The gun recoil with velocity 0.8 m/s. Applying a constant force to the gun so that it wll stop after recoiling at distance 6.4m.

Let a be the retardation, then, $0^2 = v^2 - 2as \Rightarrow a = \frac{v^2}{2s}$

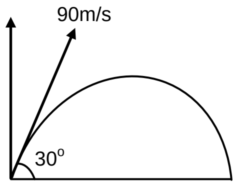
If f be the required constant force to be applied then,

$f = ma = 7,000 \times \frac{(0.8)^2}{2 \times 6.4} = 350N$

g. Let v be the recoil velocity at gun then momentum of bullet = 2×250
Momentum of gun = $100 \times v$
But, momentum of gun = Momentum of bullet
$100 \times v = 2 \times 250$
$v = 5$ m/s

EXERCISE 21.2

1. A ball is thrown with a velocity of 98 m/sec at an elevation of 30°, find
   a. the horizontal range,
   b. time of light
   c. magnitude and direction of the velocity after 2 seconds.
   d. position after 2 seconds.



Solution:

Initial velocity (u) = 98m/s
Angle of elevation (θ) = 30^o

a. Horizontal range (R) = $\frac{u^2 \sin^2 \theta}{g}$

$= \frac{(98)^2 . \sin^2.30}{10} = \frac{9604 \times 50.866}{10} = 831.7m$

b. Time of flight (T) = $\frac{2u \sin\theta}{g} = \frac{2 \times 98 \times \sin30^o}{10}$ = 9.8 sec

c. Let v be the striking velocity of the ball making an angle θ with horizontal.

$\therefore$ v_x = horizontal component = $v\cos\theta = 98.\cos30^o = \frac{98\sqrt{3}}{2} = 49\sqrt{3}$ m

v_y = vertical component = $u \sin\alpha - gt = 98 \sin30^o - 10\times2 = \frac{98}{2} - 20 = 49 - 20 =$ 29m

$\therefore$ Now, $v^2 = v_x^2 + v_y^2 = (49\sqrt{3})^2 - (29)^2 = 7202.58 - 841 = 6361.58$

$\therefore$ v = 79.76 m/s

Direction, $\text{Tan}\theta = \frac{v_y}{v_x} = \frac{29}{49\sqrt{3}} = 0.342$

$\theta = \text{Tan}^{-1}(0.342)$

$\theta = 20^o$

d. If (x, y) be position of projectile after time t = 2 sec

$\therefore$ $x = u\cos\alpha\ t = 98 \times \cos30^o \times 2 = 98 \times \frac{\sqrt{3}}{2} \times 2 = 98\sqrt{3}$ m

$y = u\sin\alpha\ t - \frac{1}{2}gt^2 = 98 \times \sin30^o \times 2 - \frac{1}{2} \times 10 \times 2^2$

$= 98 \times \frac{1}{2} \times 2 - \frac{40}{2} = 98 - 20 = 78$m

$\therefore$ Position (x, y) = $(98\sqrt{3}, 78)$

2. a. Find the velocity and the direction of projection of a shot which passes in a horizontal direction just over the top of a wall which is 250 m off and 125m high (g = 9.8ms^{-2}]

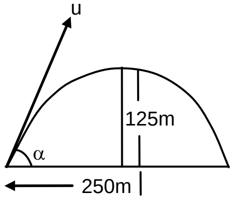
b. A shot is seen to pass horizontally just over a vertical wall 64m high and 96m off. Find the magnitude and direction of the velocity of the shot with which it was fired.

Solution:

a. Let u be the velocity of the projection of a shot making an angle α with the horizon. Since the shot just passes the top of the building, it moves horizontally.

$\therefore$ Max. height (H) = 125m

Horizontal range (R) = 2 × 250m = 500m



$\therefore$ $H = \frac{u^2 \sin^2\alpha}{2g} \Rightarrow 125 = \frac{u^2 \sin^2\alpha}{2g}$ ... ... (i)

and, $R = \frac{u^2 \sin^2\alpha}{g} \Rightarrow 500 = \frac{u^2 \sin^2\alpha}{g}$ ... ... (ii)

Dividing (i) by (ii)

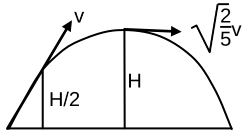
$\frac{1}{4} = \frac{\sin^2\alpha}{2\sin\alpha\ .\ \cos\alpha} \Rightarrow \text{Tan}\alpha = 1 = \text{Tan}45^o \Rightarrow \alpha = 45^o$

Velocity of projection:

Substituting the value of α in (i) $\Rightarrow 125 = \frac{u^2}{2g} \times \left(\frac{1}{\sqrt{2}}\right)^2 \Rightarrow u^2 = 500 \times 9.8 = u =$ 70m/s

b. The velocity of a particle when at its greatest height is $\sqrt{\frac{2}{5}}$ of its velocity when at half its greatest height. Show that the angle of projection is 60^o.

Let u be the velocity and α, angle of projection off a particle. Let H be the greatest height. If v be the velocity at $\frac{H}{2}$, then the velocity at H is $\sqrt{\frac{2}{5}}$ v.



$\therefore \quad \sqrt{\frac{2}{5}}\,v = u\cos\alpha$

or, $v^2 = \frac{5}{2}u^2\cos^2\alpha$ ... ... (i)

Also, $H = \frac{u^2\sin^2\alpha}{2g}$ ... ... (ii)

And, $v^2 = u^2 - 2g\frac{H}{2}$

or, $\frac{5}{2}\;u^2\cos^2\alpha = u^2 - g\frac{u^2\sin^2\alpha}{2g}$ (From (i) and (ii)

or, $\frac{5}{2}\cos^2\alpha = 1 - \frac{\sin^2\alpha}{2}$

or, $5\cos^2\alpha = 2 - \sin^2\alpha$

or, $5 - 5\sin^2\alpha = 2 - \sin^2\alpha$

or, $-4\sin^2\alpha = -3$

or, $\sin^2\alpha = \frac{3}{4}$

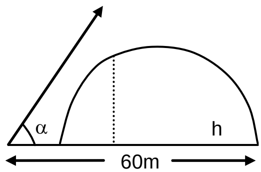
or, $\sin\alpha = \frac{\sqrt{3}}{2}$

$\therefore \quad \alpha = 60^o$

3. A projectile thrown from a point in a horizontal plane come back to the plane in 4 secs at a distance of 58.8m from the point of projection, find the velocity of the projectile.

Solution:

Here, time of projection (t) = 4 sec
Horizontal range (R) = 58.8m
Velocity of projection (u) = ?
Let α be the angle of projection, then
We know,



$t = \frac{2\,u\sin\alpha}{g}$ and $R = \frac{u^2\sin^2\alpha}{g}$

or, $4 = \frac{2u\sin\alpha}{10}$ $\qquad 60R = \frac{v^2\,2\sin\alpha\cos\alpha}{g}$

or, $u\sin\alpha = 20$ ... ... (i) $\qquad u^2\sin\alpha\,.\,\cos\alpha = 300$ ... ... (ii)

from (i) and (ii)

$\left(\frac{20}{\sin\alpha}\right)^2\sin\alpha.\cos\alpha = 300 \Rightarrow \frac{\cos\alpha}{\sin\alpha} = \frac{3}{4}$

or, $\tan\alpha = \frac{4}{3}$, $\qquad \therefore \; \sin\alpha = \frac{4}{5}$

Substituting the value of $\sin\alpha$ in (i)

$u\sin\alpha = 20$

$u\,.\,\frac{4}{5} = 20 \Rightarrow u = 25$ m/s

4. Find the angle of projection when the range on a horizontal plane is 4 times the greatest height attained.

Solution:

Angle of projection (α) = ?

Given, Horizontal range = 4 maximum height

or, $\dfrac{u^2 \sin^2 \alpha}{g} = 4 \cdot \dfrac{u^2 \sin^2\alpha}{2g}$

or, $2\sin\alpha.\cos\alpha = \dfrac{4\sin^2\alpha}{2}$

or, $1 = \dfrac{\sin\alpha}{\cos\alpha} \Rightarrow \text{Tan}\alpha = \text{Tan}45^o$

$\therefore \quad \alpha = 45^o$

5. The horizontal range of a projectile is $4\sqrt{3}$ times its maximum height. Find the angle of projection.

Solution:

Angle of projection (α) = ?

Given, Horizontal range = $4\sqrt{3}$ max. height

$\dfrac{u^2 \sin^2\alpha}{g} = 4\sqrt{3}\ \dfrac{u^2\ \alpha\sin^2\alpha}{2g}$

or, $2\sin\alpha.\cos\alpha = 2\sqrt{3}\ \sin^2\alpha$

or, $\tan\alpha = \dfrac{1}{\sqrt{3}} = \tan30^o$

$\therefore \quad \alpha = 30^o$

6. From the top of a tower 144m high, a particle is projected horizontally with a velocity of 60m/sec. Find its velocity when it reaches the ground.

Solution:

Here, for horizontal projectile, just before hitting the ground,

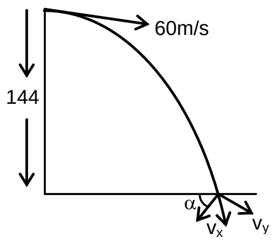
hmax = 144m

u = 60m/s, v = ?, g = 10m/s^2

Let T = time of flight

v = Velocity with which it hits the ground

α = angle made by $\vec{v}$ with positive x–axis



Now,

$u_x = u = 60\text{m/s},\ u_y = 0$

$v_x = ux = 60\text{m/s},\ v_y = uy + gT = 0 + gT$

We have, $T = \sqrt{\dfrac{2H}{g}} = \sqrt{\dfrac{2 \times 144}{10}}$

T = 5.37 sec

Again, we have, $v = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{u^2 + (gT)^2}$

$= \sqrt{(60)^2 + (10 \times 5.37)^2} = \sqrt{3600 + 53.7} = 60.45\ \text{m/s}$

7. A stone is projected from the top of a tower 72.5m high at an angle of 45° which strikes the ground at a distance of 50m from the foot of the tower. Find the velocity of projection.

Solution:

For horizontal projectile, just before hitting the ground hmax = 72.5m, R = 50m, v = ?, g = 10m/s^2, $\alpha = 45^o$.

Let u be the velocity with which body be projected t be the we taken by the body to reach the ground. Now, taking upward direction as positive.

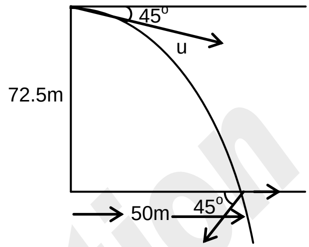
We have,

$-h = u\sin\alpha \,.\, t - \frac{1}{2} \times 10 \times t^2$

or, $-72.5 = \frac{ut}{2} - 5t^2$ ... ... (i)

The particle hits at a distance of 50m from the base of the tower, so that

$s = u\cos\alpha . t \Rightarrow \frac{ut}{\sqrt{2}} = 50$ ... ... (ii)



from (i) and (ii)

$-72.5 = \frac{50\sqrt{2}}{2} = 5t^2$

$-72.5 = 35.46 - 5t^2$

$5t^2 = 107.96$

$t^2 = 21.59$

$t = 4.65$

Again, from (ii)

$\frac{u \times 4.65}{\sqrt{2}} = 50$ $\qquad$ $u = \frac{50\sqrt{2}}{4.65} = 15.21$m/s

Hence, required projected velocity = 15.21 m/s

8. A ball is projected from a point with a velocity 64m/sec from the top of a tower 128m high in direction making an angle 30° with the horizon. Find when and at what distance from the foot of the tower it will strike the ground.

Here, initial velocity (u) = 64 m/s

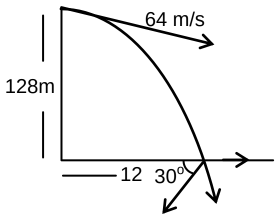
Angle of projection (θ) = 30^o

Height fallen (H) = 128m

Time of flight (T) = ?

Horizontal range (R) = ?

We have,



$H = \frac{1}{2} gT^2 \Rightarrow T = \sqrt{\frac{2H}{g}} = \sqrt{\frac{2 \times 128}{10}} = 5.06$ sec

Horizontal range (R) = uT

= 64×5.06 = 323.8m

9. A canon ball has the same range R on a horizontal plane for two different angles of projection. If H and H' are the greatest heights and t_1 and t_2 are the time of flights in two paths for which this is possible, prove that

a. $R^2 = 16\,HH'$ $\qquad$ b. $R = \frac{1}{2}\,gtt'$

Solution:

a. Let α and α_1 be two different angle of projections having the same range R.

$R = \frac{u^2 \sin^2\alpha}{g} = \frac{u^2 \sin^2\alpha_1}{g}$

or, $\frac{\sin^2\alpha}{2\alpha} = \sin 2\alpha_1$

or, $2\alpha = 180 - 2\alpha_1 \Rightarrow \alpha = 90 - \alpha_1$

So, that, $H = \frac{u^2 \sin\alpha^2}{2g}$ and, $H^1 = \frac{u^2 \sin^2 (90 - \alpha)}{2g} = \frac{u^2 \cos^2\alpha}{2g}$

$t_1 = \frac{2u\sin\alpha}{g}$ and $t_2 = \frac{2u\sin (90 - \alpha)}{g} = \frac{2u \cos\alpha}{g}$

Now, $R^2 = \left(\frac{u^2 \sin^2 \alpha}{g}\right) = \frac{u^2 4 \sin^2\alpha . \cos^2\alpha}{g^2}$

$= 4\frac{u^2\sin^2\alpha}{g} . \frac{u^2 \cos^2\alpha}{g} = 4.4\frac{u^2 \sin^2\alpha}{2g} . \frac{u^2 \cos^2\alpha}{2g}$ = 16HH, Hence proved.

b. Again, $R = \frac{u^2 \sin^2\alpha}{g} = \frac{u^2 . 2\sin\alpha.\cos\alpha}{g} = \frac{1}{2} g \frac{2u^2 \sin\alpha}{g} \times \frac{2u \cos\alpha}{g} = \frac{1}{2} gtt^1$ Hence proved.

10. A particle is projected with a velocity u. If the greatest height attained by the particle be H, prove that the range R on the horizontal plane through the point of projection is

$R = 4\sqrt{H\left(\frac{u^2}{2g} - H\right)}$

Solution:

If α is the angle of projection, then

H = greatest height = $\frac{u^2 \sin\alpha}{2g}$

and, R = horizontal range = $\frac{u^2 \sin^2\alpha}{g}$

Then, $\frac{u^2}{2g} - H = \frac{u^2}{2g} - \frac{u^2 \sin^2\alpha}{2g} = \frac{u^2 (1-\sin^2\alpha)}{2g} = \frac{u^2 \cos^2\alpha}{2g}$

Now, $4\sqrt{H\left(\frac{u^2}{2g} - H\right)} = 4\sqrt{\frac{u^2 \sin^2\alpha}{2g} . \frac{u^2 \cos^2\alpha}{2g}} = \frac{4 . u^2 \sin\alpha. \cos\alpha}{2g} = \frac{u^2 \sin^2\alpha}{g}$

= R

$\therefore$ $R = 4\sqrt{H\left(\frac{u^2}{2g} - H\right)}$

11. If R be the horizontal range and T, the time of flight of a projection, show that $\tan\alpha = \frac{gT^2}{2R}$, where α is the angle of projection.

Solution:

Let u be the velocity of the projection, then

We know,

R = horizontal range = $\frac{u^2 \sin^2\alpha}{g}$ ... ... (i)

T = time of flight = $\frac{2u \sin\alpha}{g}$ ... ... (ii)

from (i) and (ii)

$\frac{gT^2}{2R} = g\left(\frac{2 u\sin\alpha}{g}\right)^2 = \frac{g.4u^2 \sin^2\alpha}{g} = \frac{4u^2 \sin^2\alpha}{4u^2 \sin\alpha.\cos\alpha} = \tan\alpha$

$\therefore$ $\text{Tan}\alpha = \frac{gT^2}{2R}$ Hence proved.

12. A ball is projected with a velocity of 49m/s, find the two directions along which the ball must be projected so as to have arrange of 122.5m.

Solution:

Here, u = 49 m/s, R = 122.5m

Angle of projection (α) = ?

We know that, $R = \dfrac{u^2 \sin^2\alpha}{g}$

or, $\sin^2\alpha = \dfrac{Rg}{u^2} = \dfrac{122.5 \times 9.8}{49 \times 49}$

or, $\sin^2\alpha = \dfrac{1}{2}$

or, $\sin^2\alpha = \sin30^o$ or $\sin150^o$

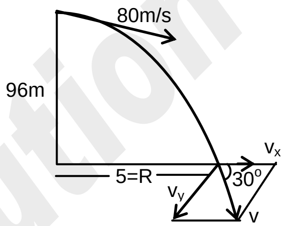
$\therefore$ $\alpha = 15^o$ or, 75^o

13. A body is thrown from the top of a tower 96m high with a velocity 80m/sec at an angle of 30° above the horizon. Find the horizontal distance from the foot of the tower to the point where it hits the ground.

Solution:

Here,

Initial velocity (u) = 80 m/s
Height fallen (H) = 96m
Time of flight (T) = ?
Horizontal range (R) = ?
Angle of projection (Q) = 30^o



We have, $H = \dfrac{1}{2} gT^2$

$T = \sqrt{\dfrac{2H}{g}} = \sqrt{\dfrac{2 \times 96}{10}}$

T = 4.38 sec

Horizontal range (R) = u.T = $80 \times 4.38 = 350.54$m

14. A ball thrown by a player from a height of 2m at an angle of 30° with the horizon with a velocity of 18m/sec is caught by another player at a height of 0.4m from the ground. How far apart were the two players?

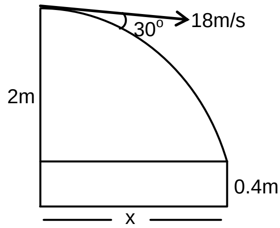
Solution:

Distance between the players (x) = ?
Initial velocity (u) = 18m/s
Angle (θ) = 30^o
Vertical distance to the traveler (y) = 2 – 0.4 = 1.6m

We have, $y = \dfrac{1gT^2}{2}$

$t^2 = \dfrac{2y}{g} = \dfrac{2\times1.6}{10}$, t = 0.57 sec

$\therefore$ x = u.t = $18 \times 0.57 = 10.18$m



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CHAPTER 22

MATHEMATICS FOR ECONOMICS AND FINANCE

EXERCISE 22.1

1. Find the quadratic supply function Qs = f(P) from the information given.

Price (P)	40	50	80
Quantity supplied (Q)	600	3300	15000

Solution:

Let $Q_s = ap^2 + bp + c$ ... ... (i) be a quadratic supply function.
Then according to question, when p = 40 then $Q_s = 600$
$\therefore$ $a \times 40^2 + b \times 40 + c = 600 \Rightarrow 1600a - 40b + c = 600$ ... ... (i)
Similarly, other two points are (50, 3300) and (80, 1500)
Then,
$2500a + 50b + c = 3300$ ... ... (ii)
$6400a + 806 + c = 15000$ ... ... (iii)
From (i) and (ii)
$2500a + 50b + c = 3300$
$1600a + 40b + c = 600$
– – –
$900a + 10b = 2700$
$90a + b = 270$ ... ... (iv)
from (ii) and (iii)
$6400a + 80b + c = 15000$
$2500a + 50b + c = 3300$
– – –
$3900a + 30b = 11700$
$130a + b = 390$ ... ... (v)
from (iv) and (v)
$130a + b = 390$
$90a + b = 270$
– – –
$40a = 120$
$\therefore$ a = 3
from (iv) b = 0
Substituting the value of a and b in (i) we get c = –4200
Hence, required quadratic supply function is $Q_s = 3p^2 - 4200$

2. The supply and demand functions are given by $P = Q^2 + 12Q + 32$, $P = -Q^2 - 4Q + 200$ respectively. Find equilibrium price and quantity.

Solution:

Given, supply function $P_s = Q^2 + 12Q + 32$
Demand function $P_d = Q^2 - 4Q + 200$
For equilibrium, $P_d = P_s$
i.e. $Q^2 + 12Q + 32 = -Q^2 - 4Q + 200$
$2Q^2 + 16Q - 168 = 0$
$Q^2 + 8Q - 84 = 0$
$Q^2 + 14Q - 6Q - 84 = 0$
$\therefore$ Q = 6, Q = –14 (not possible)

When Q = 6

Then $p = 6^2 + 12\times6 + 32 = 36 + 72 + 32 = 140$

$\therefore$ equilibrium price = 140

3. Given the supply and demand functions

$Q_S = (P + 5)\sqrt{P + 10}$

$Q_d = \dfrac{210 - 9P - 3P^2}{\sqrt{P + 10}}$

Calculate the equilibrium price and quantity.

Solution:

Given, $Q_s = (P+5)\sqrt{p + 10}$ and $Q_1 = \dfrac{210 - 9p - 3p^2}{\sqrt{p + 10}}$

For equilibrium condition,

$Qs = Q_d$

i.e. $(P + 5)\sqrt{p + 10} = \dfrac{210 - 9p - 3p^2}{\sqrt{p + 10}}$

or, $(p + 5)(p + 10) = 210 - 9p - 3p^2$

or, $p^2 + 15p + 50 + 3p^2 + 9p - 210 = 0$

or, $4p^2 + 24p - 160 = 0$

or, $p^2 + 6p - 40 = 0$

or, $p^2 + 10p - 4p - 40 = 0$

$\therefore$ p = 4

Then $Q = 9\sqrt{14}$

$\therefore$ equilibrium point $(4, 9\sqrt{14})$

4. The average cost of a product is given as $AC = 15Q - 3600 + \dfrac{486000}{Q}$

Find the quantity for which the total cost is minimum. Also find the minimum cost.

Solution:

Given, average cost (AC) $= 15Q - 3600 + \dfrac{486,000}{Q}$

Total cost function (TS) = AC $\times$ Q

$\therefore$ $TC = 15Q^2 - 3600Q + 486,000$

Comparing it with $y = ax^2 + bx + c$

a = 15, b = –3600 and c = 486,000

Since a > 0, TC represents a parabola concave upward. Being upward, TC has minimum value at $Q = -\dfrac{b}{2a}$ $\left(x = -\dfrac{b}{2a}\right)$

i.e. $Q = +\dfrac{3600}{30} = 120$

$\therefore$ Total cost is minimum at Q = 120 units

Then the min. total cost is $TC = 15\times120^2 - 3600 \times 120 + 486000 = 2,70,000$

5. For the price Rs. P, the quantity demanded is given by Q = 600,000 – 2,500P. Determine the total revenue function R = f(P).

a. What is the concavity of the revenue function?

b. What is the total revenue when price is Rs. 50?

c. Find the price for which the total revenue is maximized.

Solution:

a. Demand function is given by
$Q = 6,00,000 - 2,500P$
Total revenue function (TR) = $P \times Q$
$\therefore$ $R = 600,000p - 2,500p^2$
Comparing it with $y = ax^2 + bx + c$,
We get $a = -2500$, $b = 600,000$ and $c = 0$
Since $a < 0$, the graph is concave downward parabola.

b. When p = Rs. 50, then total revenue is
$R = -2500 \times 50^2 + 600,000 \times 50$ = Rs. 3,62,50,000

c. total revenue is maximized at $P = -\frac{b}{2a}$ i.e. $P = \frac{-600,000}{-5,000} = 120$
$\therefore$ Revenue is maximized at P = Rs. 120

6. Given the fixed cost as 32, variable cost per unit as 5 per unit and the demand function $P = 25 - 2Q$, express the profit function π in terms of Q.
   a. Find the value(s) of Q for break even.
   b. Find the value of Q for which π is maximum.
   c. What is the maximum profit?
   d. Sketch the graph of π.

Solution:

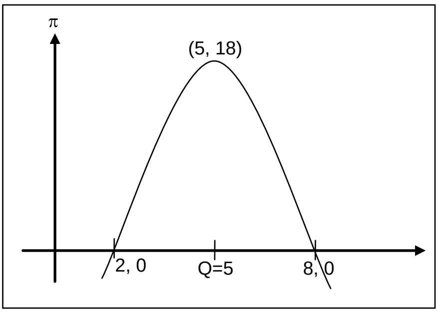
Fixed cost = 32 Variable cost = 5 per unti
$\therefore$ total cost for producing a units is given by, $TS = 5Q + 32$
Also, demand function $p = 25 - 2Q$
$\therefore$ total revenue function $TR = 25Q - 2Q^2$

a. For break-even TR = TC
$25Q - 2Q^2 = 5Q + 32$
$20Q - 2Q^2 - 32 = 0$
or, $Q^2 - 10Q + 16 = 0$
or, $Q^2 - 8Q - 2Q + 16 = 0$
$\therefore$ Q = 2 or 8

b. Profit function $\pi = TR - TC = 25Q - 2Q^2 - 5Q - 32$
$\pi = -2Q^2 + 20Q - 32$
Since coefficient of Q^2 is negative, parabola is concave downward,
Profit is maximum at $Q = -\frac{20}{-4} = 5$

c. Maximum profit $\pi_{max} = \frac{4ac - b^2}{4a} = \frac{4 \times (-2)\ (-32) - 400}{-8} = 18$

d.

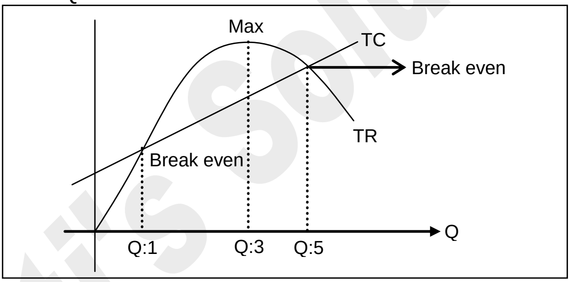


7. For a firm, the total revenue and the total cost are given by
$TR = -2Q^2 + 14Q$ $\qquad$ $TC = 2Q + 10$
   a. Find the value(s) of Q for which the firm (i) breaks even, (ii) maximizes profit.
   b. Sketch the graph of TR and TC on the same diagram and show the break-even point(s) and the maximum profit.

Solution:

a. $TR = -2Q^2 + 14Q$
$TC = 2Q + 10$
For break-even, TR = TC
$-2Q^2 + 14Q = 2Q + 10$
$2Q^2 - 12Q + 10 = 0$
$Q^2 - 6Q + 5 = 0$
$Q^2 - 5Q - Q + 5 = 0$
$Q = 1$ or 5
Now, profit function $\pi = TR - TC$
$\therefore \quad \pi = Q^2 - 6Q + 5$
It is quadratic function. Since a < 0 concave downward, gives max profit at
$Q = -\frac{b}{2a} = \frac{6}{2} = 3$

b. Find the value of Q for which π is maximum.



EXERCISE 22.2

1. Write the consumption matrix and the system of linear equations formed from the Leontief input-output model in the following cases

a.

Producers	Users		Final (External) demand
	C_1	C_2	
C_1	200	250	450
C_2	125	50	225

b.

Purchased from	Consumed by (in Crores Rs.)			External Demand (In Crores Rs.)
	Manufacturing	Agriculture	Services	
Manufacturing	250	140	30	80
Agriculture	100	105	15	130
Services	50	35	45	20

Solution:

a. Given, $x_{11} = 200$, $x_{12} = 250$, $d_1 = 450$
$\therefore \quad x_1 = x_{11} + x_{12} + d_1 = 900$
$x_{21} = 125$, $x_{22} = 50$, $d_2 = 225$

$\therefore\ x_2 = x_{21} + x_{22} + d_2 = 400$

$\therefore$ the consumption matrix or coefficient input matrix is given by $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ where

$a_{ij} = \dfrac{x_{ij}}{x_j}$ for all i, j

so, $a_{11} = \dfrac{x_{11}}{x_1} = \dfrac{200}{900} = \dfrac{2}{9}$

$a_{12} = \dfrac{x_{12}}{x_2} = \dfrac{125}{400} = \dfrac{5}{36}$ $\qquad a_{21} = \dfrac{x_{21}}{x_1} = \dfrac{125}{450} = \dfrac{5}{16}$

$a_{22} = \dfrac{x_{22}}{x_2} = \dfrac{50}{450} = \dfrac{1}{9}$

$\therefore$ Input coefficient matrix is $\begin{bmatrix} \dfrac{2}{9} & \dfrac{5}{16} \\ \dfrac{5}{36} & \dfrac{1}{9} \end{bmatrix}$

b. Given,

$x_{11} = 250,\ x_{12} = 140,\ x_{13} = 30,\ d_1 = 80$

Then total output $(x_1) = x_{11} + x_{12} + x_{13} + d_1 = 250 + 140 + 30 + 80 = 500$

$x_{21} = 100,\ x_{22} = 105,\ x_{23} = 15,\ d_2 = 130$

$\therefore$ Total output $(x_2) = x_{21} + x_{22} + x_{23} + d_2 = 350$

$x_{31} = 50,\ x_{32} = 35,\ x_{33} = 45,\ d_3 = 20$

$\therefore$ Total output $(x_3) = x_{31} + x_{32} + x_{33} + d_3 = 150$

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Matrix where $a_{ij} = \dfrac{x_{ij}}{x_j}$

$a_{11} = \dfrac{x_{11}}{x_1} = \dfrac{250}{500} = 0.5$ $\qquad a_{12} = \dfrac{x_{12}}{x_2} = \dfrac{140}{350} = 0.4$

$a_{13} = \dfrac{x_{13}}{x_3} = \dfrac{30}{150} = 0.2$ $\qquad a_{21} = \dfrac{x_{21}}{x_1} = \dfrac{100}{500} = 0.2$

$a_{22} = \dfrac{x_{22}}{x_2} = \dfrac{105}{350} = 0.3$ $\qquad a_{23} = \dfrac{x_{23}}{x_3} = \dfrac{15}{150} = 0.1$

$a_{31} = \dfrac{x_{31}}{x_1} = \dfrac{50}{500} = 0.1$ $\qquad a_{32} = \dfrac{x_{32}}{x_1} = \dfrac{35}{200} = 0.1$

$a_{33} = \dfrac{x_{33}}{x_3} = \dfrac{45}{150} = 0.3$

Therefore, $A = \begin{pmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{pmatrix}$

2. Given three sector economy with sectors sector 1, sector 2, sector 3, the consumption matrix is given as

$$A = \begin{bmatrix} 0.1 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.2 \\ 0.1 & 0.0 & 0.2 \end{bmatrix}$$

If the first sector decides to produce 200 units, what amounts will be consumed by consumed by it?

Solution:

Given, consumption input coefficient matrix is sector I, Sector II and Sector III

$$A = \begin{pmatrix} 0.1 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.2 \\ 0.1 & 0.0 & 0.2 \end{pmatrix} \begin{array}{l} \text{Sector I} \\ \text{Sector II} \\ \text{Sector III} \end{array}$$

If first sector decides to produce 200 units, then it consumes 0.1×200 units $= 20$ units of itself

and 0.4×200 units $= 80$ units of sectors 2

and 0.1×200 units $= 20$ units of sector 3

3. For a two-sector economy, the consumption matrix is $A = \begin{bmatrix} 0.1 & 0.6 \\ 0.5 & 0.2 \end{bmatrix}$, and the external demand is $D = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$. Find the production level to satisfy the external demand.

Solution:

Given,

Coefficient matrix $A = \begin{bmatrix} 0.1 & 0.6 \\ 0.5 & 0.2 \end{bmatrix}$ and external demand vector $D = \begin{bmatrix} 18 \\ 11 \end{bmatrix}$

Then technolny matrix $T = I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.6 \\ 0.5 & 0.2 \end{pmatrix}$

$$T = \begin{bmatrix} 0.9 & -0.6 \\ -0.5 & 0.8 \end{bmatrix}$$

$$|T| = \begin{vmatrix} 0.9 & -0.6 \\ -0.5 & 0.8 \end{vmatrix} = 0.72 - 0.30 = 0.42$$

$$\therefore T^{-1} = \frac{\text{Adj. (T)}}{|T|} = \frac{\begin{bmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{bmatrix}}{0.42}$$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the gross output to meet the final demand then,

$$X = T^{-1}D = \frac{1}{0.42}\begin{bmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{bmatrix}\begin{bmatrix} 18 \\ 11 \end{bmatrix} = \frac{1}{0.42}\begin{bmatrix} 21 \\ 18.9 \end{bmatrix} = \begin{bmatrix} 50 \\ 45 \end{bmatrix}$$

$\therefore$ The production level is 50 units and 45 units respectively.

4. If the input-output coefficient matrix $A = \begin{pmatrix} 0.2 & 0.4 \\ 0.5 & 0.3 \end{pmatrix}$. Find the Leontief's technology test and viability as per Hawkin's Simon's condition. Also find the demand vector D which is consistent with the output vector $\begin{pmatrix} 100 \\ 80 \end{pmatrix}$.

Solution:

Here, $A = \begin{pmatrix} 0.2 & 0.4 \\ 0.5 & 0.3 \end{pmatrix}$

The Leontif's technology matrix T is given by

$$T = I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.2 & 0.4 \\ 0.5 & 0.3 \end{pmatrix} = \begin{pmatrix} 1 - 0.2 & 0 - 0.4 \\ 0 - 0.5 & 1 - 0.3 \end{pmatrix}$$

$\therefore T = \begin{pmatrix} 0.8 & -0.4 \\ -0.5 & 0.7 \end{pmatrix}$ which is required Leontif's technology matrix.

For the viability test by Hawkin's Simon condition.

$T = I - A = \begin{pmatrix} 0.8 & -0.4 \\ -0.5 & 0.7 \end{pmatrix}$

Since each element in the leading diagonal of the matrix T is positive.

Also, $|T| = \begin{vmatrix} 0.8 & -0.4 \\ -0.5 & 0.7 \end{vmatrix} = 0.56 - 0.20 = 0.36 > 0$

Hence, the given input-output system is viable as per Hawkin's Simon condition.

By given $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 100 \\ 80 \end{pmatrix}$

Now, $D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ be the demand vector.

Then $(I - A)\, X = D$

$\Rightarrow \begin{pmatrix} 0.8 & -0.4 \\ -0.5 & 0.7 \end{pmatrix} \begin{pmatrix} 100 \\ 80 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 80 - 30 \\ -50 + 56 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 48 \\ 6 \end{pmatrix}$

$\therefore$ Demand vector (D) $= \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 48 \\ 6 \end{pmatrix}$

5. A and D, the input-output coefficient matrix and the demand vector respectively are given below:

   $A = \begin{pmatrix} 0.1 & 0.4 \\ 0.2 & 0.2 \end{pmatrix}$ and $D = \begin{pmatrix} 560 \\ 320 \end{pmatrix}$

   Find the Leontif's technology matrix and the total output.

Solution

Here, $A = \begin{pmatrix} 0.1 & 0.4 \\ 0.2 & 0.2 \end{pmatrix}$

$T = I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.4 \\ 0.2 & 0.2 \end{pmatrix}$

$= \begin{pmatrix} 1 - 0.1 & 0 - 0.4 \\ 0 - 0.2 & 1 - 0.2 \end{pmatrix}$

$T = = \begin{pmatrix} 0.9 & -0.4 \\ -0.2 & 0.8 \end{pmatrix}$, which is required Leontif's technology matrix.

$|T| = \begin{vmatrix} 0.9 & -0.4 \\ -0.2 & 0.8 \end{vmatrix} = 0.72 - 0.08 = 0.64 \neq 0$

$\therefore T^{-1}$ exists

T_{11} = Cofactor of 0.9 = 0.8, $T_{12} = -(-0.2) = 0.2$,

$T_{21} = -(-0.4) = 0.4$, $T_{22} = 0.9$

$\therefore$ Matrix of cofactors $= \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.9 \end{pmatrix}$

Adjoint of $T = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.9 \end{pmatrix}$

i.e. Adj. $T = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.9 \end{pmatrix}$

$$T^{-1} = \frac{\text{Adj. T}}{|T|} = \frac{\begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.9 \end{pmatrix}}{0.64}$$

$$= \frac{1}{0.64}\begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.9 \end{pmatrix}$$

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ where, x_1 and x_2 be the two outputs.

Then, $X = T^{-1}D$

$$= \frac{1}{0.64}\begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.9 \end{pmatrix}\begin{pmatrix} 560 \\ 320 \end{pmatrix}$$

$$= \frac{1}{0.64}\begin{pmatrix} 448 + 128 \\ 112 + 288 \end{pmatrix}$$

$$= \frac{1}{0.64}\begin{pmatrix} 576 \\ 400 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 900 \\ 625 \end{pmatrix}$$

$\therefore x_1 = 900$ and $x_2 = 625$

$\therefore$ The required total outputs to meet the future demands of the consumers are 900 units and 625 units.

6. A factory makes two goods, grommets and widgets. To make \$1 worth of grommets requires \$0.2 worth grommets and \$0.1 worth ofwidgets, and to make \$1 worth of widgets requires \$0.05 worth of grommets and \$0.1 worth of widgets. There is a market demand for \$750 worth of grommets and \$500 worth of widgets. What should the total production ofeach be to meet the market demand?

Solution:

Given, consumption input coefficient matrix be $\begin{vmatrix} 0.2 \\ 0.1 \end{vmatrix} \begin{vmatrix} 0.05 \\ 0.1 \end{vmatrix}$

i.e. $A = \begin{pmatrix} 0.2 & 0.05 \\ 0.1 & 0.1 \end{pmatrix}$

Also given market demand vector $D = \begin{bmatrix} 750 \\ 500 \end{bmatrix}$

Now, technolny matrix $T = I - A = \begin{bmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{bmatrix}$

$$T^{-1} = \frac{\text{Adj (T)}}{|T|}$$

$$|T| = \begin{vmatrix} 0.8 & 0.6 \\ 0.5 & 0.9 \end{vmatrix} = 0.72 - 0.005 = 0.715$$

$$T^{-1} = \frac{1}{|T|}\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.8 \end{bmatrix} = \frac{1}{0.715}\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.8 \end{bmatrix}$$

Using $x = T^{-1}D = \frac{1}{0.715}\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.8 \end{bmatrix}\begin{bmatrix} 750 \\ 500 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 979.02 \\ 664.34 \end{bmatrix}$$

$\therefore$ x_1 = Rs. 979.02, x_2 = Rs. 664.34

7. Suppose that for the production of Rs. 1 worth of C_1, we require Rs. 0.1 worth

of C_1, Rs. 0.2 worth of C_2, Rs. 0.3 worth of C_3. For the production of Rs. 1 worth of C_2, we require Rs. 0u.2 worth of C_1, Rs. 0.1 worth of C_2, Rs. 0.4 worth of C_3. For the production of Rs. 1 worth of C_3, we require Rs. 0.5 worth of C_1, Rs. 0.2 worth of C_2, Rs. 0.2 worth of C_3. Find the production level to meet the external demand worth Rs. 35 from C_1, Rs. 100 from C_3, and no demand from C_2.

Solution:

Given,

$$A = \begin{bmatrix} 0.1 & 0.2 & 0.5 \\ 0.2 & 0.1 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix} \qquad D = \begin{bmatrix} 35 \\ 0 \\ 100 \end{bmatrix}$$

Technolny matrix T = I – A

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.1 & 0.2 & 0.5 \\ 0.2 & 0.1 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.9 & -0.2 & -0.5 \\ -0.2 & 0.9 & -0.2 \\ -0.3 & -0.4 & 0.8 \end{bmatrix}$$

$$|T| = \begin{vmatrix} 0.9 & -0.2 & -0.5 \\ -0.2 & 0.9 & -0.2 \\ -0.3 & -0.4 & 0.8 \end{vmatrix} = 0.9\begin{vmatrix} 0.9 & -0.2 \\ -0.4 & 0.8 \end{vmatrix} + 0.2\begin{vmatrix} -0.2 & -0.2 \\ -0.3 & 0.8 \end{vmatrix} - 0.5\begin{vmatrix} -0.2 & 0.9 \\ -0.3 & -0.4 \end{vmatrix}$$

$$= 0.9\,(0.72 - 0.08) + 0.2\,(-0.16 - 0.06) - 0.5(0.08 + 0.27)$$

$$= 0.576 - 0.044 - 0.175 = 0.357$$

Let $\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$ be a cofactor matrix of T

Then

$$T_{11} = \text{cofactor of } 0.9 = \begin{vmatrix} 0.9 & -0.2 \\ -0.4 & 0.8 \end{vmatrix} = 0.64$$

$$T_{12} = \text{Cofactor of } -0.2 = -\begin{vmatrix} -0.2 & -0.2 \\ -0.3 & 0.8 \end{vmatrix} = 0.22$$

$$T_{13} = \text{Cofactor of } -0.5 = \begin{vmatrix} -0.2 & 0.9 \\ -0.3 & -0.{-}.4 \end{vmatrix} = 0.35$$

$$T_{21} = \text{Cofactor of } -0.2 = -\begin{vmatrix} -0.2 & -0.5 \\ -0.4 & 0.8 \end{vmatrix} = 0.36$$

$$T_{22} = \text{Cofactor } 0.9 = \begin{vmatrix} 0.9 & -0.5 \\ -0.3 & 0.8 \end{vmatrix} = 0.57$$

$$T_{23} = \text{Cofactor of } -0.2 = -\begin{vmatrix} 0.9 & -0.2 \\ -0.3 & -0.4 \end{vmatrix} = 0.42$$

$$T_{31} = \text{Cofactor of } -0.3 = \begin{vmatrix} 0.2 & -0.5 \\ 0.9 & -0.2 \end{vmatrix} = 0.49$$

$$T_{32} = \text{Cofactor of } -0.4 = -\begin{vmatrix} 0.9 & -0.5 \\ -0.2 & -0.2 \end{vmatrix} = 0.28$$

$$T_{33} = \text{Cofactor of } 0.8 = \begin{vmatrix} 0.9 & -0.2 \\ -0.2 & 0.9 \end{vmatrix} = 0.77$$

$\therefore$ Cofactor matrix is $\begin{bmatrix} 0.64 & 0.22 & 0.35 \\ 0.36 & 0.57 & 0.42 \\ 0.49 & 0.28 & 0.77 \end{bmatrix}$

$$\text{Adj (T)} = \begin{bmatrix} 0.64 & 0.36 & 0.49 \\ 0.22 & 0.57 & 0.28 \\ 0.35 & 0.42 & 0.77 \end{bmatrix}$$

$$T^{-1} = \frac{1}{0.357}\begin{bmatrix} 0.64 & 0.36 & 0.49 \\ 0.22 & 0.57 & 0.28 \\ 0.35 & 0.42 & 0.77 \end{bmatrix}$$

Using $x = T^{-1}D = \dfrac{1}{0.357}\begin{bmatrix} 35 \\ 0 \\ 100 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \\ 250 \end{bmatrix}$$

$\therefore$ $X_1 = 200$, $X_2 = 100$, $X_3 = 250$

EXERCISE 22.3

1. Find the consumer's surplus for the demand functions given by:

a. $P = 100 - Q^2$ at $Q = 8$

b. $P = \dfrac{80}{\sqrt[3]{Q}}$ at $Q = 64$

c. $Q = \dfrac{10 - P}{2P}$ at $P = 2$

d. $P = \dfrac{2Q}{Q^2 + 1}$ at $Q = 10$

Solution:

a. Given, Demand function $p = 100 - Q^2$

at $Q = 8$, $p = 100 - 64 = 36$

Consumer's surplus (C.S.) = $\int_0^8 (100 - Q^2)\, dQ - 36 \times 8$

$$= \left[100Q - \frac{Q^3}{3}\right]_0^8 - 288 = 341.33$$

b. Given demand function

$p = \dfrac{80}{\sqrt[3]{Q}}$

$p = 80Q^{-1/3}$

When $Q = 64$ then $p = 80(64)^{-1/3} = 20$

Consumers' surplus (c.s.) = $\int_0^Q p\,dQ - p \times Q$

C.S. = $\int_0^{64} 80\, Q^{-1/3}\, dQ - 20 \times 64$

$$= 80\,\frac{3}{2}\left[Q^{\frac{2}{3}}\right]_0^{64} - 1280 = 120 \times 16 - 1280 = 640$$

c. $Q = \dfrac{10 - p}{2p}$ at $p = 2$

$Q = \frac{5}{p} - \frac{1}{2} \Rightarrow p = \frac{10}{2Q+1}$

When p = 2 then Q = 2

Consumer's demand (C.S.) = $\int_0^Q pdQ - p\times Q = \int_0^2 \frac{10}{2Q+1} dQ - 2\times2$

$$= \frac{10}{2}[\ln(2Q+1)]_0^Q - 4 = 5\ln 5 - 4$$

d. Given, $p = \frac{2Q}{Q^2+1}$

When Q = 10, then $p = \frac{20}{101}$

Consumer's surplus (C.S.) = $\int_0^{10} p\,dQ - p \times Q$

$$= \int_0^{10} \frac{2Q}{Q^2+1} dQ - \frac{20}{101} \times 10[\ln(Q^2+1)]_0^{10} - \frac{200}{101}$$

$$= \ln 101 - \frac{200}{101} = 2.63$$

2. Find the producer's surplus for the supply functions given by:

a. P = 12 + 2Q at Q = 5 b. $P = 20\sqrt{Q} + 15$ at Q = 25

Solution:

a. Given, supply sunction p = 12 + 2Q

When Q = 5 then p = 22

Producer surplus (P.S.) = $P\times Q - \int_0^Q pdQ = 22\times5 - \int_0^5 (121 + 2Q)\,dQ$

$$= 110 - [12Q + Q^2]_0^5 = 110 - 85 = 25$$

b. Given,

$P = 20\sqrt{Q} + 15$ at Q = 25

When Q = 25 then p = 115

Then P.S. = $P \times Q - \int_0^Q pdQ = 115 \times 25 - \int_0^{25} (20\sqrt{Q} + 15)\,dQ$

$$= 2875 - \left[20\frac{Q^{3/2}}{3/2} + 15Q\right]_0^{25} = 2675 - \left(\frac{40}{3} \times 125 + 375\right)$$

$= 2875 - 2041.67 = 833.33$

3. Find the consumer's surplus and producer's surplus at the equilibrium

a. Supply equation: P = Q + 50, demand equation: $P = \frac{4000}{Q+20}$

b. $P = 74 - Q_D^2$, $P = Q_S^2 + 2$

c. Demand equation: $P = 100\,e^{-Q/5}$, Supply equation: $P = 20\,e^{2Q/5}$

Solution:

a. Given, Demand function $p = \frac{4000}{Q+20}$

Supply function p = Q + 50

For equilibrium

Supply = demand

i.e. (Q + 50) (Q + 20) = 4000

$Q^2 + 70Q - 3000 = 0 \Rightarrow Q = 30, -100$

Since output cannot be negative so Q = 30

When Q = 30 then p = 80

Now, consumer's surplus $= \int_0^Q$ demand function $- P \times Q$

$$= \int_0^{30} \frac{4000}{Q + 20}\, dQ - 80 \times 30$$

$$= 4000\,[\ln (Q + 20)]_0^{30} - 2400 = 400 \ln\left(\frac{50}{20}\right) - 2400 = 1265.16$$

Producer's surplus (P.S.) = $P \times Q - \int$ supply function

$$= 80 \times 30 - \int_0^{30} (Q + 50)\, dQ$$

$$= 2400 - \left[\frac{Q^2}{2} + 50Q\right]_0^{30}$$

$$= 2400 - (450 + 1500) = 450$$

b. Given, $p_d = 74 - Q^2_d$

$p_s = Q^2_s + 2$

For equilibrium, $p_d = p_s = p$

$Q_d = Q_s = Q$

$Q^2 + Q = 74 - Q^2$

$2Q^2 = 72$

$Q = 6$

When Q = 6 then p = 38

Consumer's surplus (C.S.) = $\int_0^6 p_d\, dQ - p \times Q$

$$\int_0^6 (74 - Q^2_d)\, dQ - 38 \times 6$$

$$\left[74Q - \frac{Q^3}{3}\right]_0^6 - 228 = 144$$

And, P.S. $= P \times Q - \int_0^Q p_s\, dQ$

$$= 228 - \int_0^6 (Q^2 + 2)\, dQ$$

$$= 228 - \left[\frac{Q^3}{3} + Q\right]_0^6 = 144$$

c. Given, demand function $p = 100\, e^{-Q/5}$

Supply function $p = 20\, e^{2Q/5}$

For market equilibrium,

Supply function = Demand function

i.e. $20\, e^{2Q/5} = 100\, e^{-Q/5}$

$e^{3Q/5} = 5$

$\frac{3Q}{5} = \ln 5$

$Q = 2.68$

When Q = 2.68 then $p = 20\, e^{1.073} = 58.48$

Now, consumer surplus (C.S.) $= \int_0^6 100e^{-Q/5}\,dQ - 58.48 \times 2.68$

$$= 100\left[\frac{e^{-Q/5}}{-1/5}\right]_0^{2.68} - 156.73$$

$$= -500\,(e^{-2.68/5} - e^0) - 156.73 = 50.73$$

Producer surplus (P.S.) $= P \times Q - \int_0^Q 20\,e^{2Q/5}\,dQ$

$$= 156.73 - \int_0^{2.68} 20\,e^{2Q/5}\,dQ$$

$$= 156.73 - 20 \times \frac{5}{2}\,[e^{2Q/5}]_0^{2.68}$$

$$= 156.73 - 50\,(e^{0.4\times 2.68} - e^0) = 60.7$$

4. For the supply function given by $P = 3 + 4Q$, the producer's surplus at $Q = \alpha$ is known to be 72. Find the value of α.

Solution:

Given, Supply function $p = 3 + 4Q$

Producer's surplus at $Q = \propto$ is 72

When $Q = \propto$ then $p = 4 \propto 3$

Now, using

$$\text{P.S.} = p \times Q - \int_0^Q (3 + 4Q)\,dQ$$

$72 = (4\alpha + 3)\,\alpha - [3Q + 2Q^2]_0^\alpha$

$72 = (4\alpha^2 + 3\alpha) - (3\alpha + 2\alpha^2)$

$72 = 4\alpha + 3\alpha - 2\alpha\ 2$

$2\alpha^2 = 72$

$\therefore\ \alpha = 6$

5. The demand function of a certain good is given by $P = 80 - 6\sqrt{Q}$. Find the change in consumer's surplus due to decrease in price from $P = 62$ to $P = 56$.

Solution:

Given, Demand $p = 80 - 6\sqrt{Q}$

when $p = 62$ then $62 = 80 - 6\sqrt{Q}$

$6\sqrt{Q} = 18$

$\therefore\ Q = 9$

$$\text{C.S.} = \int_0^9 (80 - 6\sqrt{Q})\,dQ - P\times Q$$

$$= \left[80Q - \frac{6Q^{3/2}}{3/2}\right]_0^9 - 62\times 9$$

$$= (720 - 4\times 27) - 558 = 54$$

Again,

when $p = 56$ then $56 = 80 - 6\sqrt{Q}$

$6r\sqrt{Q} = 24$

$\therefore\ Q = 16$

$$\text{C.S.} = \int_0^{16} (80 - 6\sqrt{Q})dQ - 56 \times 16$$

$= [80Q - 4Q^{3/2}]_0^{16} - 896$
$= (1280 - 256) - 896 = 128$
Change in C.S. is $128 - 54 = 74$

EXERCISE 22.4

1. The demand and supply functions of a new product in a competitive market are $Q_d = 120 - 2P$ and $Q_s = 3P - 40$ respectively. At the time of unequilibrium condition, the rate of price adjustment is $\frac{dp}{dt} = 0.25\ (Q_d - Q_s)$. Derive and solve the differential equation given that $P(0) = 20$
   a. Find the price when $t = 4$
   b. Is $P(4)$ close to the equilibrium condition?
   c. Examine the state of stability in the long run.

Solution:

$Q_d = 120 - 2P$ $\qquad$ $Q_s = 3P - 40$

$Q_d - Q_s = 120 - 2P - 3P + 40 = 160 - 5P$

and $\frac{dp}{dt} = 0.25\ (Q_d - Q_s) = 0.25\ (160 - 5P)$

$\Rightarrow \frac{dp}{dt} = 40 - 1.25P$

$\Rightarrow \frac{dp}{dt} + 1.25\ P = 40$

Here, $a = 1.25$, $b = 40$, $P = y$, $t = 5$

$\therefore$ The complete solution is $y = C.e^{-at} + \frac{b}{a}$

$\Rightarrow P = C.e^{-1.25t} + \frac{40}{1.25}$

$\Rightarrow P = C.e^{-1.25t} + 32$

When $t = 0$, $P = 20$

$20 = C.1 + 32$ $\qquad$ $\therefore C = -12$

$\therefore P = -12e^{-1.25t} + 32$

When $t = 4$

$P = -12e^{-1.25 \times 4} + 32 = 31.92$

When $t \to \infty$, $e^{-1.25t} \to 0$ and hence the first term tends to zero, so the price in the long run approaches to 32 and hence stable.

2. In a competitive, the demand and the supply functions are given by the equations $Q_d = 240 - 3P$ and $Q_s = 5P - 150$. Also the rate of change of price adjustment proportional to the process of demand is given by $\frac{dP}{dt} = 0.05\ (Q_d - Q_s)$. Solve the differential equation for the time path of $P(t)$, the initial price level P_0 being 50.
   a. Predict the price level for the time period 4.
   b. In how many time periods would its price level dropped by Rs. 6 than the initial price.

Solution:

$Q_d = 240 - 3P,$ $\qquad$ $Q_s = 6P - 150$

$Q_d - Q_s = 240 - 3P - 6P + 150 = 390 - 9P$

and $\frac{dp}{dt} = 0.05\ (Q_d - Q_s) = 0.05\ (390 - 9P)$

$\Rightarrow \frac{dp}{dt} = 19.5 - 0.45P$

$\Rightarrow \frac{dp}{dt} + 0.45\ P = 19.5$

Here, a = 0.45, b = 19.5, y = P, t = t

Now the complete solution is

$y = C.e^{-at} + \frac{b}{a}$

$\Rightarrow P = C.e^{-0.45t} + \frac{19.5}{0.45}$

$\Rightarrow P = C.e^{-0.45t} + \frac{130}{3}$

When t = 0, P =50

$50 = C.1 + \frac{130}{3}\ \therefore\ C = \frac{20}{3}$

$\therefore$ The solution is $P = \frac{20}{3}e^{-0.45t} + \frac{130}{3}$

When t = 4

$P = \frac{20}{3}e^{-0.45t} + \frac{130}{3}$

$= \frac{20}{3} \times e^{-1.8} + \frac{130}{3}$

$= \frac{20}{3} \times 0.1653 + \frac{130}{3} = 44.43$

When P = 50 – 6 = 44

$40 = \frac{20}{3}e^{-0.45t} + \frac{130}{3}$

$\Rightarrow \frac{2}{3} = \frac{20}{3} \times e^{-0.45t}$

$\Rightarrow 0.1 = e^{-0.45t}$

$\Rightarrow -0.45t = \log 0.1 = -2.3026$

$\therefore\ t = \frac{-2.304}{-0.45} = 5$ periods

3. If demand and supply function in a competitive market are $Q_d = 32 - 0.5P$ and $Q_s = -8 + 0.3P$ and the rate of adjustment of price when the market is out of equilibrium is $\frac{dP}{dt} = 0.2\ (Q_d - Q_s)$. Derive and solve the obtained differential equation to get a function for P in terms of t given that price is 12 in the time period 0. Comment on this market.

Solution:

Here,

$Q_d = 32 - 0.5P$

$Q_s = -8 + 0.3P$

Now, $\frac{dp}{dt} = 0.2\ (Q_d - Q_s)$

or, $\frac{dp}{dt} = 0.2\ (32 - 0.5P) - (-8 + 0.3P)$

or, $\frac{dp}{dt} = 0.2\ (32 – 0.5P + 8 – 0.3P)$

or, $\frac{dp}{dt} = 0.2\ (40 – 0.8P)$

or, $\frac{dp}{dt} = 8 – 0.16P$

or, $\frac{dp}{dt} = –0.16\ (P – 50)$

or, $\frac{dp}{P – 50} = – 0.16dt$

Integrating both side, we get

$\ln (P – 50) = –0.16t + \ln C$

or, $\ln (P – 50) – \ln C = –0.16t$

or, $\ln \left(\frac{P – 50}{C}\right) = – 0.16\ t$

or, $P – 50 = C.e^{–0.16t}$

or, $P = 50 + C.e^{–0.16t}$

By question, when t = 0, P = 12

The form (i), we get

$12 = 50 + C.e^{–0.16} \times 0$

or, $12 – 50 = C$

$\therefore\ C = –38$

Substituting the value of C in (i), we get

$P = 50 – 38.e^{–0.16t}$

which is required solution.

Here, m = –0.16 < 0. So, the market is stable.

4. The demand and supply functions in a competitive market are $Q_d = 500 – 5P$, $Q_s = –40 + 20P$ respectively. The initial part P_0 is Rs. 100. Derive a function for time path P and use it to predict price in time period 5 given that price adjust proportion to excess demand at the rate $\frac{dP}{dt} = 0.02\ (Q_d – Q_s)$. How many time periods would you like to wait for the price to drop by Rs. 40?

Solution:

We have,

$\frac{dp}{dt} = 0.02\ (Q_d – Q_s)$

or, $\frac{dp}{dt} = 0.02\ (500 – 5P – (–40 – 20P)$

or, $\frac{dp}{dt} = 0.02\ (500 – 5P + 40 – 20P)$

or, $\frac{dp}{dt} = 10.8 – 0.5P$

or, $\frac{dp}{dt} = –0.5\ (P – 21.6)$

or, $\frac{dp}{P – 21.6} = – 0.5dt$

Integrating both side, we get

$\ln(P - 21.6) = -0.5t + \ln C$

or, $\ln(P - 21.6) - \ln C = -0.5t$

or, $\ln\left(\frac{P - 21.6}{C}\right) = -0.5t$

or, $P - 21.6 = C.e^{-0.5t}$

or, $P = 21.6 + C.e^{-0.5t}$ ….. (i)

By question, when t = 0, P(0) = 100

The form (i), we get

$100 = 21.6 + C.e^{-0.5 \times 0}$

or, $100 - 21.6 = C$

$\therefore$ C = 78.4

From equation (i), we get

$P = 21.6 + 78.4e^{-0.5t}$ ….. (ii)

When P = 40, from equation (ii)

$40 - 21.6 + 78.4e^{-0.5t}$

or, $40 - 21.6 = 78.4e^{-0.5t}$

or, $\frac{17.4}{78.4} = e^{-0.5t}$

or, $e^{-0.5t} = \frac{23}{98}$

Taking ln on both sides, we get

$-0.5t = \ln 23 - \ln 98$

or, $-0.5t = -1.45$

or, $t = \frac{1.45}{0.5} = 2.9$

$\therefore$ t = 2.9 time period.

5. If the demand and supply functions in a competitive market are $Q_d = 35 - 0.5P$, $Q_s = -4 + 0.8P$ and the rate of adjustment of price when market is out of equilibrium is $\frac{dP}{dt} = 0.25(Q_d - Q_s)$. Derive and then solve the relevant differential equation to get a function for p in terms of t given that price is 37 in time period zero. Comment on the stability of this market.

Solution:

We have $Q_d = 35 - 0.5P$ ….. (i)

$Q_s = -4 + 0.8P$ ….. (ii)

and $\frac{dp}{dt} = 0.85(Q_d - Q_s)$ ….. (iii)

Substituting the value of Q_d and Q_s from (i) and (ii) in (iii), we get

$\frac{dP}{dt} = 0.25(35 - 0.5P + 4 - 0.8P)$

$\Rightarrow \frac{dP}{dt} = 0.25(39 - 1.3P)$

$\Rightarrow \frac{dP}{dt} = 0.25 \times 1.3(30 - P)$

$\Rightarrow \frac{dp}{30 - p} = 0.325\, dt$

$\therefore \dfrac{dp}{p-30} = -0.325\ dt$

Now integrating on both sides, we get

$\displaystyle\int \frac{dp}{p-30} = \int -0.325dt + C$

$\Rightarrow \ln(p-30) = -0.325t + C$

$\Rightarrow p - 30 = e^{-0.325t + C}$

$\Rightarrow p - 30 = Ae^{-0.325t}$; where $A = e^c$

$\therefore p = 30 + Ae^{-0.325t}$ ….. (iv) is a function of p in terms of t.

For the second part of the question

When t = 0, p(0) = 37 then from (iv)

$p(0) = 30 + A^\circ$

$\Rightarrow 37 = 30 + A \Rightarrow A = 7$

$\therefore$ Equation (iv) becomes

$p = 30 + 7e^{-0.325t}$ is the particular solution

For the third part of the question

As $t \to \infty$, $e^{-0.325} \to 0$

$\therefore P = 30 + 7 \times 0 = 30$ is a finite value

Therefore, the market is stable.

The demand and supply function of a product are given by

$Q_d = 122 - 4p + 6\dfrac{dp}{dt}$, $Q_s = -40 + 5p + 60\dfrac{dp}{dt}$

6. If initial price is Rs. 45 per unit, find the time path of price for dynamic equilibrium. What will be price after 5 months?

Solution:

We have, $Q_d = 122 - 4p + 6\dfrac{dp}{dt}$, $Q_s = -40 + 5p + 60\dfrac{dp}{dt}$

At the market equilibrium $Q_d = Q_s$

$\Rightarrow 122 - 4p + 6\dfrac{dp}{dt} = -40 + 5p + 60\dfrac{dp}{dt}$

$\Rightarrow -54\dfrac{dp}{dt} - 9P = -162$

$\Rightarrow \dfrac{dp}{dt} + \dfrac{1}{6}p = 3$ ….. (i)

$\Rightarrow \dfrac{dp}{dt} = 3 - \dfrac{P}{6} = \dfrac{18-p}{6}$

$\Rightarrow \dfrac{dp}{p-18} = -\dfrac{dt}{6}$

Now, integrating on the both sides, we get

$\displaystyle\int \frac{dp}{p-18} = -\frac{1}{6}\int dt$

$\Rightarrow \ln(p-18) = -\dfrac{1}{6}t + L_nC$

$\Rightarrow \ln\left(\dfrac{p-18}{c}\right) = -\dfrac{t}{6}$

$\Rightarrow \dfrac{p - 18}{c} = e^{-t/6}$

$\Rightarrow p = 18 + ce^{-1/6}$ ….. (i) is the general solution

For the second part of the question

When t = 0, p(0) = 45 then from (i)

$45 = 18 + ce^{\circ}$

$\Rightarrow c = 27$

$\therefore$ From (i), $p = 27e^{-1/6 + 18}$ is the particular solution

For the third part of the question

When t = 5 then $p = 27e^{-5/6} + 18 = 29.378$ which is finite. So, the market is stable.

EXERCISE 22.5

1. Form the difference equations from

   a. $y_t = 8(4^t) - 7$ b. $y_t = A(4^t) + B(7^t)$

Solution:

Form the difference equations from

a. $Y_t = 8(4^t) - 7 \Rightarrow Y^t + 7 = 8(4)^t$

Replacing T by T+1

$Y_{T+1} = 8(4^{t+1}) - 7 = 84^t.4 - 7$

$= 4(Y^t + 7) - 7 = 4Y_t + 28 - 7 = 4Y_t + 21$

Hence, the required difference equation is,

$Y_t + = 4Y_t + 21$

Equivalent $Y_t = 4Y_{t+1} + 21$

$\therefore \quad Y_T = 4Y_{t+1} + 21)$

b. $Y_t = A4^t + B.7^t$

We have,

$Y_t = A(4^t) + B(7^t)$

Replacing t by t+1

$Y_{t+1} = A(4^{t+1}) + B(7^{t+1})$

$= A4^t.(4) + B7^t.(7)$

From the given equation $(Y_t - A4^t) = B7^t$ so

$Y_{t+1} = 4A4^t + 7(Y_t - A4^t)$

$= 4A4^t + 7Y_t - 7A4^t$

$Y_{t+1} = 7Y_t - 3A4^t \Rightarrow 3A4^t = 7Y_t - y_{t+1}$

again,

Replacing t by t+1

$Y_{t+2} = 7Y_{t+1} - 3A(4^{t+1})$

From the given equation $(7Y_t - Y_{t+1}) = 4A4^{t+1})$

$= 7Y_{t+1} - 3(7Y_t - Y_{t+1})$

$= 7Y_{t+1} - 21Y_t + 3Y_{t+1}$

$= 10Y_{t+1} - 21Y_t$

so required difference equation is

$\therefore Y_{t+2} - 10Y_{t+1} - 21Y_t = 0$

2. Solve the following difference equations.

   a. $y_t = y_{t-1} + 2,\ y_0 = 0$ b. $y_{t+1} = -y_t + 6,\ y_0 = 4$

   c. $4y_t = y_{t-1} + 24$ d. $y_t = -0.5y_{t-1} + 1$

Solution:

a. $Y_T = Y_{T-1} + 2$, $Y_o = 2$

Comparing the equation with $Y_T = aY_{T-1} + b$, we have

$a = 1$

$b = 2$

Since $a \neq 1$

Another method,

The complementary function (c.f.) = $Aa^T = A(1)^T$

Let $Y_T = KT$ be a particulars solution

Then $Y_{T-1} = K.(T-1)$

Substituting the value Y_T and Y_{T-1}

$K_T = K(T - 1) + 2$

$K_T - K(T-1) = 2$

$K_T - K_T + K = 2$

$\therefore \quad K = 2$

So, PS = 2

The required general solution is

Y_T = CF + PS

$= A(1)^T + 2 \Rightarrow Y_T = A(1)^T + 2$

As given, $Y_o = 2$, then

$2 = A + 2$

$A = 0$

Now, $(Y_T = (1)^T + 2)$

b. $Y_{T+1} = -Y_T + 6$, $Y_o = 4$

$Y_T = -Y_{T+1} + 6$

Comparing the equation with $Y_T = aY_{T+1} + b$, we have

$a = -1$,

$b = 6$

Since $a \neq 1$, the required solution is

$Y_T = Aa^T + \dfrac{b}{1 - a}$, where A is constant.

Substituting the value of a, b

$Y_T = A(-1)^T + \dfrac{6}{1 + 1}$

$Y_T = A(-1)^T + 3$

Putting,

Given $Y_0 = 4$, $T = 0$

Then,

$4 = A(-1)^0 + 3$

$4 - 3 = A$

$A = 1$,

Now,

$Y_T = 1(-1)^T + 3$, is the required solution,

c. $4Y_T = Y_{T-1} + 24$

$Y_T = \frac{1}{4} Y_{T-1} + 6$

Comparing the equation with $Y_T = aY_{T+1} + b$, we have

$a = \frac{1}{4}$, $b = 6$

Since $a \neq 1$, the required solution is

$Y_T = A4^T + \frac{b}{1-a}$ where A is constant.

Substituting the values of a and b

$Y_T = A\left(\frac{1}{4}\right)^T + \frac{6}{1-\frac{1}{4}}$

$Y_T = A\left(\frac{1}{4}\right)^T + 8 = A(0.25)^t + 8$

d. $Y_T = -0.5Y_{T-1} + 1$

Comparing with the equation $Y_T = aY_{T+1} + b$, we have

$a = -0.5$

$b = 1$

Since $a \neq 1$ the required

Solution is

$Y_T = Aa^T + \frac{b}{1-a}$, where A is constant.

Substituting the value of a and b

$Y_T = A(-0.5)^T + \frac{1}{1+0.5}$

$Y_T = A(-0.5)^T + 0.66$

3. Consider the difference equation $y_t = 3y_{t-1} + 7$ with initial condition $y_0 = 2$.
   a. Find the value of y_1, y_2, y_3 without solving the difference equation.
   b. Solve the difference equation to find y_t as a function of t. Find y_1, y_2, y_3 using this solution.

Solution:

Given, $Y_T = 3Y_{T-1} + 7$, $Y_o = 2$

a. Find the value of Y_1, Y_2, Y_3 without solving d.e. when, Y_1 then, Y_2 then,

We have,

$Y_t = 3y_{t-1} + 7$

Put t = 1

$Y_1 = 3Y_0 + 7$

Given,

$Y_0 = 2$

$Y_1 = 3 \times 2 + 7 = 13$

Put t = 2

$Y_2 = 3Y_1 + 7$

$Y_2 = 3 \times 13 + 7 = 46$

Put t = 3

$Y_3 = 3Y_2 + 7$

$Y_3 = 3 \times 46 + 7 = 145$

b. Find Y_1, Y_2, Y_3 using this solution.

Comparing with the equation $Y_T = aY_{T-1} + b$, we have

$a = 3$

$b = 7$

Since, $a \neq 1$, the required solution is,

$$Y_T = Aa^T + \frac{b}{1-a}$$

Substituting the value and a, b

$$Y_T = A(3)^T + \frac{7}{1-3}$$

$Y_T = A(3)^T - 3.5$

When, $Y_0 = 2$, then

$2 = A(3)^0 - 3.5$

$2 + 3.5 = A$

$\therefore \quad A = 5.5$

Now, $Y_T = 5.5(3)^T - 3.5$

When, Y_1, Y_2, Y_3, 300

$Y_1 = 5.5(3)^1 - 3.5$	$Y_2 = 5.5(3)^2 - 3.5$	$Y_3 = 5.5(3)^3 - 3.5$
$Y_1 = 13$	$= 46$	$= 145$

$$\therefore \left(\begin{array}{c} Y_1 = 13 \\ Y_2 = 46 \\ Y_3 = 145 \end{array}\right)$$

4. A person borrows Rs. 1,50,000 from a bank at the interest rate of 9.6% per annum on the outstanding balance. The person wishes to pay Rs. 4,000 each month.
   a. How much does the person owe after 1 year?
   b. How long will it take to pay the loan?

Solution:

Given, $Y_T = 0.3Y_{T-1} + 0.4T + 5$

a. $y_t = 1.008\, y_{t-1} - 4{,}000$

$y_c = m = 1.008$

$y_c = A(1.008)^t$

Particular integral

(y_p) = let $y_t = k$ be

$y_{t-1} = k$

$k - 1.008k = -4{,}000$

$k = 5{,}00{,}000$

$\therefore \quad y_t = A(1.008)^t + 5{,}00{,}000$

$y_0 = 1{,}50{,}000$

$1{,}50{,}000 = A + 5{,}00{,}000$

$A = -3{,}50{,}000$

$y_t = -3{,}50{,}000\,(1.008)^t + 5{,}00{,}000$

$y_{12} = 114881.46$

b. We have, $y_t = A(1.008)^t + 5{,}00{,}000$

To pay the loan, $y_t = 0$

$3{,}50{,}000\ (1.008)^t = 5{,}00{,}000$

$(1.008)^t = 1.43$

$t = \dfrac{\ln(1.43)}{\ln(1.008)} = 45$ months .

EXERCISE 22.6

1. Consider the following cobweb model $Q_{st} = P_{t-1} - 8$, $Q_{Dt} = -2P_t + 22$
Find the expressions for P_t, Q_t when $P_0 = 11$. Also comment on the stability.

Solution:

Given,

$Q_{ST} = P_{t-1} - 8$

$Q_{dT} = -2P_t + 22$

For equilibrium

$Q_{ST} = Q_{dt}$

So, $P_t - 1 - 8 = -2P_t + 22$

$P_{t-1} - 8 + 2P_t - 22 = 0$

$2P_t = -P_{t-1} + 30$

$P_T = -\frac{1}{2} P_{t-1} + 15$

Comparing with $P_t = aP_{t-1} + b$

Now, $a = \frac{-1}{2}$, $b = 13$

The general solution Rs. $P_t = AaT + \dfrac{b}{1 - a}$

Where, A is constant.

Substituting the values,

$$P_T = A\left(\frac{-1}{2}\right)^t + \frac{15}{1 + \frac{1}{2}}$$

$$P_T = A\left(\frac{-1}{2}\right)^t + 10 \quad \ldots\ \ldots\ \ldots \text{(i)}$$

When, $P_o = 11$,

Now, $11 = A + 10$

$11 - 10 = A$

$\therefore \quad A = 1$

$$\text{Now, } P_t = 1\left(1 - \frac{1}{2}\right)^t + 10$$

Putting this expression in $Q_{dt} = -2_{pt} + 22$

$$= -2\left[1\left(\frac{-1}{2}\right)^t + 10\right] + 22 \quad = -2\left(1 - \frac{1}{2}\right)^t + 2$$

Since, $|a| = |-\frac{1}{2}| = \frac{1}{2} > 0$. So it is stable.

2. Given the demand and supply equations
$Q_{Dt} = -5P_t + 35$, $Q_{st} = 4P_{t-1} - 10$
find expressions for P_t and Q_t when $P_0 = 6$. Find the values of P and Q where the model converges.

Solution:

Given,

$Q_{2t} = -5P_t + 35$

$Q_{ST} = 4P_{t-1} - 10$

For equation

$Q_{dt} = Q_{St}$

$4P_{t-1} - 10 = -5P_t + 35$

$5P_t = -4P_{t-1} + 45$

$P_t = \frac{-4}{5} P_{t-1} + 9$

Comparing with $P_t = aP_{t-1} + b$,

so, $a = \frac{-4}{5}$, $b = 9$

The general solution is $P_t = Aa^t \neq \frac{Aa}{1-a}$ (A is constants)

Substitution the values

$$P_t = A\left(\frac{-4}{5}\right)^t + \frac{9}{1 + \frac{4}{5}} = A\left(\frac{-4}{5}\right)^t + 5$$

When, $P_o = 6$ then

$6 = A + 5$

$\therefore$ $A = 1$

$\therefore$ $P_t = 1\left(\frac{-4}{5}\right)^t + 5$

Putting this expression is $Q_{dt} = -5P_t + 35$

$$= -5\left[1\left(\frac{-4}{5}\right)^t + 3\right] + 33 = -5\left(\frac{-4}{5}\right)^t + 10$$

$\therefore$ $Q_{dt} = -5\left(\frac{-4}{5}\right)^t + 10$

$P_t = 1\left(\frac{-4}{5}\right)^t + 5 = (-0.8)^t + 5$

3. Consider the demand and supply equations
$Q_D = -4P + 10$
$Q_S = 6P - 10.$
Find the equilibrium price and quantity.

Solution:

Given,

$Q_d = -4p + 10$

$Q_s = 6p - 10$

a. For equilibrium, $Q_d \neq Q_s$

$-4p + 10 = 6p - 10$

$-10p = -20$

$p = 2$

Substituting the value of $Q_s = 6p - 10$

$6 \times 2 - 10$

$\therefore \quad Q = 2$

$\therefore \quad \begin{pmatrix} p = 2 \\ Q = 2 \end{pmatrix}$

4. Given the following model

$Y_t = C_t + I_t$

$C_t = 0.75Y_{t-1} + 400$

$I_t = 200$

$Y_0 = 400$

Find the value of C_2.

Solution:

Given,

$y_t \; = c_t + I_t = 0.75y_{t-1} + 400 + 200$

$\therefore \; y_t \; = 0.75y_{t-1} + 600$

If $t = 1$,

$y_1 \; = 0.75\, y_0 + 600 = 0.75 \times 400 + 600 = 900$

So, from $c_t = 0.75\, y_{t-1} + 400$

$c_2 \; = 0.75y_1 + 400 = 1075$

5. Consider the following model

$Y_t = C_t + I_t$

$C_t = 0.7Y_{t-1} + 400$

$I_t = 0.1Y_{t-1} + 100$

Given $Y_0 = 3000$, find an expression for Y_t and comment on the stability.

Solution:

We have,

$y_t = c_t + I_t$

$y_t = 0.7y_{t-1} + 400 + 0.1y_{t-1} + 100$

or, $y_t = 0.8y_{t-1} + 500$

$y_t - 0.8y_{t-1} = 500$ ... ... (i)

Solution of (i) is $y_t = y_c + y_p$ where

y_c = complementary function

y_p = particular integral

For complementary function (y_c) : Reduce (i) into homogeneous form as

$y_t - 0.8y_{t-1} = 0$ ... ... (ii)

Let $y_t = A(m)^t$ be a trial solution.

Then $y_{t-1} = Am^{t-1}$

from (ii)

$Am^t - 0.8\, Am^{t-1} = 0$

$Am^t\,(1 - 0.8m^{-1}) = 0$

$m = 0.8$ since $Am^t \neq 0$

$\therefore \quad y_c = A(0.8)^t$

For particular integral (y_p) :

Let $y_t = k$ be a trial solution of (i).

Then $y_{t-1} = k$

$\therefore$ (i) becomes

$k - 0.8k = 500$

$0.2k = 500$

$k = \dfrac{500}{0.2} = 2500$

$\therefore \quad y_p = 2500$

$\therefore \quad y_t = A(0.8)^t + 2500$ is general solution.

When t = 0 then $y_0 = A(0.8)^0 + 2500$

$300 = A + 2500$

$A = 500$

$\therefore \quad y_t = 500\,(0.8)^t + 2500$ is required particular solution for y_t.

6. A lagged-income model is given as

$Y_t = C_t + I_t$

$C_t = 0.8Y_{t-1} + 200$

$I_t = 1000$

where Y_t, C_t and I_t denote national income, consumption and investment in the period t. The initial value of income $Y_0 = 5000$.

a. Find out Y_2 and C_2 from the relations.

b. Construct the difference equation relating Y_t and Y_{t-1}

c. Solve difference equation so formed to find Y_t in terms of t.

Solution:

Given,

$y_t = c_t + I_t$

$y_t = (0.8y_{t-1} + 200) + 1000$

$y_t = 0.8y_{t-1} + 1200$

$y_t - 0.8y_{t-1} = 1200$ ... ... (i)

a. When t = 1 then

$y_1 - 0.8y_0 = 1200$

$y_1 - 0.8 \times 5000 = 1200$

$y_1 = 5200$

When t = 2

Then $y_2 = 0.8y_1 = 1200$

$y_2 \quad = 1200 + 0.8\ 5200 = 5360$

We have,

$c_t \quad = 0.8y_{t-1} + 200$

When t = 2

$c_2 \quad = 0.8 \times y_1 + 200 = 0.8 \times 5200 + 200 = 4360$

b. The difference equation relating $y_t + y_{t-1}$ is $y_t - 0.8\,y_{t-1} = 1200$ ... ... (i)

c. Its solution is $y_t = y_c + y_p$

For y_c:

$y_t - 0.8y_{t-1} = 0$

$Am^t\,(1 - 0.8\,m^{-1}) = 0$

$\therefore \quad m = 0.8$ since $Am^t \neq 0$

$y_c = A(0.8)^t$

For y_p:
Let $y_t = k$ be a solution
Then $y_{t-1} = k$
from (i)
$0.2k = 1200$
$k = 6000$
$\therefore \quad y_p = 6000$
Hence $y_t = A(0.8)^t + 6000$
When $t = 0$
$y_0 = A + 6,000$
$\therefore \quad A = -1,000$ since $y_0 = 5,000$
$\therefore \quad y_t = -1,000\ (0.8)^t + 6,000$
when $t = 2$
$y_2 = -1,000\ (0.8)^2 + 6,000$

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