

Set-I

Group A

- 1) Which of the following is true?
a) All of the above
- 2) Sum of all binomial coefficients is
a) 2^n
- 3) Product of fourth roots of unity is ...?
b) -1
- 4) If the roots of the equation $(b-c)x^2 + (c-a)x + (a-b) = 0$ are equal, then, a, b, c are in
a) A.P.
- 5) Which of the statements is incorrect?
d) $\sec \theta = 1/2$
- 6) The sum of the roots of a quadratic equation is 1 and sum of their square is 13, then the equation is ...
a) $x^2 - x - 6 = 0$
- 7) $\int \frac{e^x}{1+e^x} dx$ is equal to
b) $\ln(1+e^x) + C$

8) The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ represents a horizontal ellipse if

a) $a^2 > b^2$

9) What is the value of $\vec{i} \cdot (\vec{j} \times \vec{k}) + \vec{j} \cdot (\vec{k} \times \vec{i}) + \vec{k} \cdot (\vec{i} \times \vec{j})$ is equal to

d) 3

10) The demand function for a certain commodity is $P = 60 - 2x - x^2$, where P is price and x the quantity. Then the consumer surplus when demand is 6 is

11) The derivative of $2 \tan^{-1} \tan \frac{x}{2}$ with respect to x is

b) $\sec x$

Group B

12-a) A committee of 5 members is to be formed from 6 boys and 4 girls. In how many ways can it be done so as to include

- i) exactly 1 girl ii) at least two girls

i) No. of ways = $C(4, 1) \times C(6, 4)$
 $= \frac{4!}{(4-1)! 1!} \times \frac{6!}{(6-4)! 4!}$

$$= \frac{4! \times 4}{3!} \times \frac{4! \times 5 \times 6}{2! \times 4!}$$

$$= 4 \times 5 \times 3$$

$$= 60 \text{ ways}$$

Hence, Committee can be formed in 60 ways if only exactly 1 girl is included.

ii)	Girls	Boys	No. of ways (s)
	2	3	$C(4,2) \times C(6,3)$
	3	2	$C(4,3) \times C(6,2)$
	4	1	$C(4,4) \times C(6,1)$

$$\text{No. of ways} = C(6,2) \times C(4,3) + C(6,3) \times C(4,2) + C(6,4) \times C(4,1)$$

$$= \frac{4!}{(4-2)!2!} \times \frac{6!}{(6-3)!3!} + \frac{4!}{(4-3)!3!} \times \frac{6!}{(6-2)!2!} + \frac{4!}{(4-4)!4!} \times \frac{6!}{(6-1)!1!}$$

$$= 60 + 120 + 60 + 6$$

$$= 246 \text{ ways}$$

Hence, Committee can be formed in 246 ways if at least two girls are included.

b) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ show that $x+y+z = xyz$.

Solⁿ

We have,

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$$

$$\text{or, } \tan^{-1} \left(\frac{x+y}{1-xy} \right) + \tan^{-1} 2 = \pi$$

$$\text{or, } \tan^{-1} \left(\frac{\frac{x+y}{1-xy} + 2}{1 - \frac{x+y}{1-xy} \cdot 2} \right) = \pi$$

$$\text{or, } \frac{\frac{x+y+2-2xy}{1-xy}}{1 - \frac{x+y+2-2xy}{1-xy}} = \tan \pi = 0$$

$$\text{or, } x+y+2-2xy=0$$

$$\text{or, } x+y+2=2xy \quad \# \text{ proved}$$

b) A dice is rolled 3 times. Getting 5 or 6 is numbered as success. Find the probability of getting

i) 2 successes

ii) 3 successes

Solⁿ

$$p = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} = \frac{1}{3}, n = 3$$

$$q = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(r) =$$

$$P(r) = {}^n C_r p^r q^{n-r}$$

$$\begin{aligned} \text{i) } P(r=2) &= {}^3 C_2 p^2 q^{3-2} \\ &= 3 \times \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^1 \\ &= 3 \times \frac{1}{9} \times \frac{2}{3} \\ &= \frac{2}{9} \end{aligned}$$

$$\begin{aligned} \text{ii) } P(r=3) &= {}^3 C_3 p^3 q^{3-3} \\ &= 1 \times \left(\frac{1}{3}\right)^3 \cdot q^0 \\ &= \frac{1}{27} \end{aligned}$$

Hence, the probability of getting 2 success is $\frac{2}{9}$ and 3 successes is $\frac{1}{27}$.

14.9) Derive the condition that the quadratic equation $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$ may have one root common.

Solⁿ

Given equations are:

$$ax^2+bx+c=0 \text{ and}$$

$$a'x^2+b'x+c'=0$$

Let α be the common root. Then,

$$a\alpha^2+b\alpha+c=0 \text{ --- (1)}$$

$$a'\alpha^2+b'\alpha+c'=0 \text{ --- (2)}$$

Solving (1) and (2) by using cross-multiplication,

$$\frac{\alpha^2}{bc'-b'c} = \frac{\alpha}{a'c-ac'} = \frac{1}{ab'-a'b}$$

Taking 1st two ratios,

$$\frac{\alpha^2}{bc'-b'c} = \frac{\alpha}{a'c-ac'}$$

$$\Rightarrow \alpha = \frac{bc'-b'c}{a'c-ac'} \text{ --- (*)}$$

Taking last two ratios,

$$\frac{\alpha}{a'c-ac'} = \frac{1}{ab'-a'b}$$

$$\text{or, } \alpha = \frac{a'c-ac'}{ab'-a'b} \text{ --- (**)}$$

From (*) and (**)

$$\frac{bc'-b'c}{a'c-ac'} = \frac{a'c-ac'}{ab'-a'b}$$

$$\Rightarrow (bc'-b'c)(ab'-a'b) = (ca'-c'a)^2$$

which is the required condition for the equations

to have one root common.

- b) If α , β and γ are the angles which a line makes with the co-ordinate axes. Prove that $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$

15) A mixture is to be made of three foods A, B and C which contain nutrients P, Q and R as shown in the table below. How will you form a mixture, which will have 45 units of P, 54 units of Q and 45 units of R. If the cost per kg of the foods A, B, C are Rs. 40, Rs. 60 and Rs. 80 respectively. Form system of equation and find by matrix method the total cost of the mixture.

Foods		Units of nutrients per kg of food		
		P	Q	R
A	A	2	2	4
B	B	3	5	0
C	C	4	3	5

Solⁿ

Let x, y and z be the nutrients contained in the foods A, B and C respectively.

The system of equations are:

$$2x + 3y + 4z = 45 \quad \text{--- (1)}$$

$$2x + 5y + 3z = 54 \quad \text{--- (2)}$$

$$4x + 0y + 5z = 45 \quad \text{--- (3)}$$

This can be written as

$$\begin{bmatrix} 2 & 3 & 4 \\ 2 & 5 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 45 \\ 54 \\ 45 \end{bmatrix}$$

$$\text{i.e. } Ax = B$$

where

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 5 & 3 \\ 4 & 0 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}, \quad B = \begin{bmatrix} 45 \\ 54 \\ 45 \end{bmatrix}$$

Now,

$$|A| = 4 \begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} - 0 + 5 \begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix}$$

$$= 4(9 - 20) + 5(10 - 6)$$

$$= 4 \times (-11) + 5 \times 4$$

$$= -44 + 20$$

$$= -24$$

$$\therefore |A| = -24 \neq 0$$

So, A^{-1} exists.

Now,

we find cofactors,

$$\text{Cofactor of } (a_{11} = 2) = C_{11} = \begin{vmatrix} 5 & 3 \\ 0 & 5 \end{vmatrix} = 25$$

$$\text{Cofactor of } (a_{12} = 3) = C_{12} = - \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -(10 - 12) = 2$$

$$\text{Cofactor of } (a_{13} = 4) = C_{13} = \begin{vmatrix} 2 & 5 \\ 4 & 0 \end{vmatrix} = -5 \times 4 = -20$$

$$\text{Cofactor of } (a_{21} = 2) = C_{21} = - \begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} = -3 \times 5 = -15$$

$$\text{Cofactor of } (a_{22} = 5) = C_{22} = \begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} = 10 - 16 = -6$$

$$\text{Cofactor of } (a_{23} = 3) = C_{23} = - \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = -(-12) = 12$$

$$\text{Cofactor of } (a_{31} = 4) = C_{31} = \begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} = 9 - 20 = -11$$

$$\text{Cofactor of } (a_{32} = 0) = - \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} = -(6 - 8) = 2$$

$$\text{Cofactor of } (a_{33} = 5) = \begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} = 10 - 6 = 4$$

$$\begin{aligned} \text{Matrix of cofactors} &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \\ &= \begin{bmatrix} 25 & 2 & -20 \\ -15 & -6 & 12 \\ -11 & 2 & 4 \end{bmatrix} \end{aligned}$$

$$\text{Adjoint of } A = \begin{bmatrix} 25 & 2 & -20 \\ -15 & -6 & 12 \\ -11 & 2 & 4 \end{bmatrix}^T$$

$$\therefore \text{Adj}(A) = \begin{bmatrix} 25 & -15 & -11 \\ 2 & -6 & 2 \\ -20 & 12 & 4 \end{bmatrix}$$

We know that,

$$A^{-1} = \frac{1}{|A|} = \text{Adj}(A)$$

$$= \frac{1}{-24} \begin{bmatrix} 25 & -15 & -11 \\ 2 & -6 & 2 \\ -20 & 12 & 4 \end{bmatrix}$$

Now,

$$x = A^{-1} B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-24} \begin{bmatrix} 25 & -15 & -11 \\ 2 & -6 & 2 \\ -20 & 12 & 4 \end{bmatrix} \begin{bmatrix} 45 \\ 54 \\ 45 \end{bmatrix}$$

$$= -\frac{1}{24} \begin{bmatrix} 25 \times 45 - 15 \times 54 - 11 \times 45 \\ 2 \times 45 - 6 \times 54 + 2 \times 45 \\ -20 \times 45 + 12 \times 54 + 4 \times 45 \end{bmatrix}$$

$$= -\frac{1}{24} \begin{bmatrix} -180 \\ -144 \\ -72 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7.5 \\ 6 \\ 3 \end{bmatrix}$$

$$\therefore x = 7.5, y = 6, z = 3$$

$$\text{Now, Total cost} = \begin{bmatrix} 40 & 60 & 80 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 7.5 & 6 & 3 \end{bmatrix} \begin{bmatrix} 40 \\ 60 \\ 80 \end{bmatrix}$$

$$= 7.5 \times 40 + 6 \times 60 + 3 \times 80 \\ = 900$$

Hence, the total cost is Rs. 900

14) Define correlation. From the following table, calculate the coefficient of correlation by Karl Pearson's method:

x	6	2	2	10	4	8
y	9	11	11	-	8	17

With arithmetic mean of x and y series are 6 and 8 respectively.
solⁿ

Calculation of Correlation coefficient

x	y	xy	x ²	y ²
6	9	54	36	81
2	11	22	4	121
2	11			
10	a=5	50	100	25
4	8	32	16	64
8	7	56	64	49
$\Sigma x = 30$	$\Sigma y = 35 + a$	$\Sigma xy = 214$	$\Sigma x^2 = 220$	$\Sigma y^2 = 340$

Let a be the unknown value in series y.

Given $\bar{x} = 6$, $\bar{y} = 8$, $\Sigma x = 30$, $\Sigma y = 35 + a$, $N = 5$

$$\bar{x} = \frac{\Sigma x}{N}$$

$$\bar{y} = \frac{\Sigma y}{N}$$

$$\Rightarrow 6 = \frac{30}{N}$$

$$\therefore N = 5$$

$$\Rightarrow 8 = \frac{35 + a}{5}$$

$$\Rightarrow 40 - 35 = a$$

$$\therefore a = 5$$

$$\therefore \Sigma y = 40$$

Here, $N=5$, $\Sigma xy = 214$, $\Sigma x = 30$, $\Sigma y = 40$,
 $\Sigma x^2 = 220$, $\Sigma y^2 = 340$

$$\begin{aligned}\text{Correlation coefficient (r)} &= \frac{N \Sigma xy - \Sigma x \Sigma y}{\sqrt{N \Sigma x^2 - (\Sigma x)^2} \times \sqrt{N \Sigma y^2 - (\Sigma y)^2}} \\ &= \frac{5 \times 214 - 30 \times 40}{\sqrt{5 \times 220 - 30^2} \sqrt{5 \times 340 - 40^2}} \\ &= -0.92\end{aligned}$$

* Defn: Correlation is the degree of relationship between dependent variable (say y) and independent variable (say x).

17-g) Find the equation of the ellipse in the standard form with a vertex at $(0, 8)$ and passing through $(3, 32/5)$.

Solⁿ

Given: vertex = $(0, b) = (0, 8) \Rightarrow b = 8$
and through $(3, \frac{32}{5})$

Let equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{8^2} = 1 \quad \text{--- (1)}$$

~~Since it passes~~

Since (1) passes through $(3, \frac{32}{5})$

$$\Rightarrow \frac{32}{a^2} + \frac{\left(\frac{32}{5}\right)^2}{8^2} = 1$$

$$\Rightarrow \frac{9}{a^2} + \frac{1024}{25 \times 64} = 1$$

$$\Rightarrow \frac{9}{a^2} + \frac{\cancel{1024} 16}{25} = 1$$

$$\Rightarrow \frac{9}{a^2} = 1 - \frac{16}{25}$$

$$\Rightarrow \frac{9}{a^2} = \frac{9}{25}$$

$$\therefore a^2 = 25$$

Putting the value of a^2 in eqn (1),

$$\frac{x^2}{25} + \frac{y^2}{64} = 1$$

b) Evaluate using L' Hospital rule: $\lim_{x \rightarrow 0} \frac{x - \tan x}{x^2}$

Solⁿ

$$\lim_{x \rightarrow 0} \frac{x - \tan x}{x^2}$$

Using L' Hospital rule,

$$= \lim_{x \rightarrow 0} \frac{1 - \sec^2 x}{2x}$$

Again, using L' Hospital rule,

$$= \lim_{x \rightarrow 0} \frac{-2\sec^2 x \cdot \tan x}{2}$$

$$= -1 \times 0$$

$$= 0$$

$$\therefore \lim_{x \rightarrow 0} \frac{x - \tan x}{x^2} = 0$$

18) Define vector product. Prove by vector method that $\sin(A-B) = \sin A \cos B - \cos A \sin B$

→ The vector product of vectors $\vec{a}$ and $\vec{b}$ is denoted and defined as

$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is the unit normal vector perpendicular to both vectors $\vec{a}$ and $\vec{b}$.

Proof: Let O be the origin and Ox and Oy be two perpendicular lines.

Here,

$$OP = |\vec{OP}| = r_1, \quad \angle POx = B, \quad \angle POy = A$$

$$OQ = |\vec{OQ}| = r_2, \quad \angle QOx = A - B$$

Then, the co-ordinates of OP & OQ are:

$$OP = (r_1 \cos B, r_1 \sin B, 0)$$

$$OQ = (r_2 \cos(A-B), r_2 \sin(A-B), 0)$$

$$OP \times OQ = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_1 \cos B & r_1 \sin B & 0 \\ r_2 \cos(A-B) & r_2 \sin(A-B) & 0 \end{vmatrix} = (0-0)\hat{k} - \hat{j}(0-0) + \hat{i}(r_1 r_2 \cos B \sin(A-B) - r_1 r_2 \sin B \cos(A-B))$$

$$\Rightarrow |OP \times OQ| = \sqrt{0^2 + 0^2 + (r_1 r_2 \cos B \sin(A-B) - r_1 r_2 \sin B \cos(A-B))^2}$$

$$= r_1 r_2 (\sin A \cos B - \sin B \cos A)$$

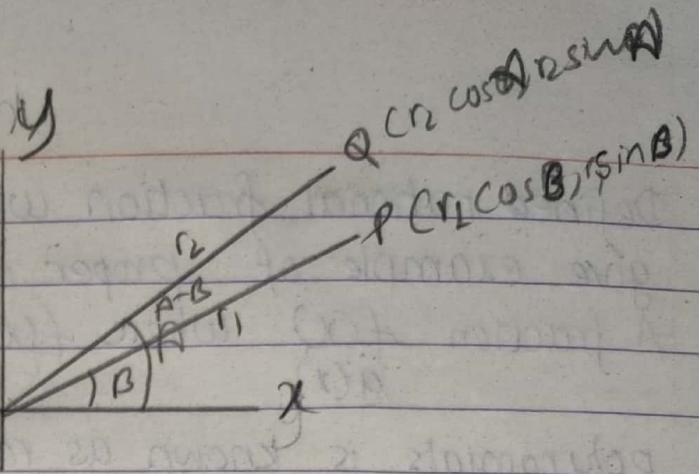
Since $A-B$ is the angle between OP & OQ ,

$$\sin(A-B) = \frac{|OP \times OQ|}{|\vec{OP}| |\vec{OQ}|}$$

$$= \frac{r_1 r_2 (\sin A \cos B - \sin B \cos A)}{r_1 r_2}$$

$$= \sin A \cos B - \sin B \cos A$$

$$\therefore \sin(A-B) = \sin A \cos B - \cos A \sin B$$



19. a) Define rational fraction with example. Also give example of proper and improper fraction.
 → A fraction $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are

polynomials is known as rational fraction.

Example of proper fraction: $\frac{3x^2 + 4x}{4x^3 + 3x}$

Example of improper fraction: $\frac{4x^3 + 3x}{3x^2 + 4x}$

b) Integrate $\int \frac{dx}{a + b \cos x}$ $a > b$.

Solⁿ

$$I = \int \frac{dx}{a + b \cos x}, \quad a > b$$

$$I = \int \frac{dx}{a \left(\frac{\sin^2 x}{2} + \frac{\cos^2 x}{2} \right) + b \left(\frac{\cos^2 x}{2} - \frac{\sin^2 x}{2} \right)}$$

$$= \int \frac{dx}{\frac{a \sin^2 x}{2} + \frac{a \cos^2 x}{2} + \frac{b \cos^2 x}{2} - \frac{b \sin^2 x}{2}}$$

$$= \int \frac{dx}{(a+b) \frac{\cos^2 x}{2} + (a-b) \frac{\sin^2 x}{2}}$$

Multiplying Num^r and Den^r by $\sec^2 x$, we

get,

$$I = \int \frac{\sec^2 x}{(a+b) + (a-b) \tan^2 \frac{x}{2}} dx$$

$$\text{Let } \tan^2 \frac{x}{2} = t$$

$$\Rightarrow \frac{1}{2} \sec^2 x = \frac{dt}{dx}$$

$$\therefore \frac{dt}{2} \sec^2 x dx = 2 dt$$

$$\text{So, } I = 2 \int \frac{dt}{(a+b) + (a-b)t}$$

$$= \frac{2}{a-b} \int \frac{dt}{\left(\frac{a+b}{a-b}\right)^2 + (t)^2}$$

$$= \frac{2}{a-b} \cdot \frac{1}{\sqrt{\frac{a+b}{a-b}}} \tan^{-1} \left(\frac{\sqrt{t}}{\sqrt{\frac{a+b}{a-b}}} \right) + C$$

$$= \frac{2}{\sqrt{(a+b)(a-b)(a+b)}} \tan^{-1} \sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} + C$$

$$= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right) + C$$

$$\therefore \int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right) + C$$

Group C

20a) Show that the function $f(x) = |x-1|$ is continuous but not differential at $x=1$.

Solⁿ

$$f(x) = |x-1|$$

we know, $|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \\ 1-x & \text{if } x-1 < 0 \end{cases}$

for left hand limit,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} |x-1| = \lim_{x \rightarrow 1^-} (1-x) = 1-1 = 0$$

for right hand limit,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} |x-1| = \lim_{x \rightarrow 1^+} (x-1) = 1-1 = 0$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = 0$$

$$\lim_{x \rightarrow 1} f(x)$$

for functional value,

$$f(1) = 1-1 = 0$$

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1) = 0$$

Hence, $f(x)$ is continuous at $x=1$.

For left hand derivative,

$$L(f'(x)) = \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{h}{-h}$$

$$= -1$$

For right hand derivative,

$$R(f'(x)) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h}{h}$$

$$= 1$$

$$\therefore L(f'(x)) \neq R(f'(x))$$

Here, $f(x)$ is not differentiable at $x=1$

Therefore, $f(x) = |x-1|$ is continuous but not differentiable at $x=1$.

b) If $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$, then prove that $C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \dots + C_nC_0 = \frac{2n!}{n!n!}$

Solⁿ

We know,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n \quad \text{--- (1)}$$

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n \quad \text{--- (2)}$$

Multiplying (1) & (2), we get,

$$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n) \times (C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n) \quad \text{--- (3)}$$

By actual multiplication, the term containing x^n in the RHS of (3) is

$$\cancel{C_0 + C_1x + \dots}$$

$$(C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \dots + C_nC_0)x^n$$

Also, the term containing x^n in the expansion of $(1+x)^{2n}$ is,

$${}^{2n}C_n x^n$$

Since eqn (3) is an identity, equating the coefficients of like terms, we get,

$$C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \dots + C_nC_0 = {}^{2n}C_n$$

$$= \frac{(2n)!}{n!n!}$$

$$= \frac{2n!}{n!n!}$$

$$= \frac{2n!}{n!n!}$$

$$= \frac{2n!}{n!n!}$$

$\therefore \text{LHS} = \text{RHS} \# \text{ proved.}$

21.b) State De-Moivre's theorem. Is this theorem true for any real numbers? Using De-Moivre's theorem, find the square roots of: $4+4\sqrt{3}i$.

Solⁿ

De-Moivre theorem states that
 $[r(\cos\theta + i\sin\theta)]^n = r^n (\cos n\theta + i\sin n\theta)$

Yes, this theorem is true for any real numbers.

Comparing $4+4\sqrt{3}i$ with $x+iy$, we get,

$$x=4, y=4\sqrt{3}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{4^2 + (4\sqrt{3})^2} = \sqrt{16 + 16 \times 3} = 8$$

$$\tan\theta = \frac{y}{x} = \frac{4\sqrt{3}}{4} = \sqrt{3}$$

$$\therefore \theta = 60^\circ \text{ [}(x,y) \text{ lies in 1st quadrant]}$$

$$\therefore 4+\sqrt{3}i = 8(\cos 60^\circ + i\sin 60^\circ)$$

$$= 8(\cos(360n + 60^\circ) + i\sin(360n + 60^\circ))$$

$$\sqrt[4]{4+\sqrt{3}}$$

$$\begin{aligned}\sqrt{4+4\sqrt{3}i} &= [8(\cos(360n + 60^\circ) + i\sin(360n + 60^\circ))]^{1/2} \\ &= 2\sqrt{2}(\cos(180n + 30^\circ) + i\sin(180n + 30^\circ))\end{aligned}$$

When $n=0$,

$$\sqrt{4+4\sqrt{3}i} = 2\sqrt{2} \left(\frac{\sqrt{3}}{2} + i\frac{1}{2} \right)$$

$$= \sqrt{6} + \sqrt{2}i$$

When $n=1$,

$$\sqrt{4+4\sqrt{3}i} = 2\sqrt{2} \left(-\frac{\sqrt{3}}{2} - i\frac{1}{2} \right) = -\sqrt{6} - \sqrt{2}i$$

Hence, the square roots are $\pm\sqrt{2}$ and $\pm\sqrt{2}i$

22-a) Using simplex method, find the maximum value of $P = x + 3y$ subject to $x + y \leq 4$ and $-x + y \geq -1$, $x, y \geq 0$.

Solⁿ

We have to maximize,

$$P = x + 3y$$

Subject to

$$x + y \leq 4$$

$$-x + y \leq 1$$

Introducing slack variables s and t , we get,

$$x + y + s = 4$$

$$x - y + t = 1$$

$$-x - 3y + P = 0$$

Canonical form of the given equation,

$$x + y + s + 0t + 0p = 4$$

$$x - y + 0s + t + 0p = 1$$

$$-x - 3y + 0s + 0t + p = 0$$

The canonical form yields the following simplex tableau:

BV	x	y	s	t	p	RHS	Ratio
s	1	1	1	0	0	4	4
t	1	-1	0	1	0	1	-1
	-1	-3	0	0	1	0	

Here, the most negative entry in the bottom row is -3 . And, the least positive ratio is 4 . Hence, 1 is the pivot element.

Performing $R_2 \rightarrow R_2 + R_1$ and $R_3 \rightarrow R_3 + 3R_1$

BV	x	y	s	t	p	RHS
y	1	1	1	0	0	4
t	2	0	1	1	0	5
	0	0	3	0	1	12

Here, all entries in the bottom row are positive. So, the solution given by the tableau is maximum. Hence, maximum value of $P = 12$ at $(0, 4)$.

o)

Set - II

Group A

- 1) How many numbers are there between 100 and 1000 such that each digit is either 2 or 9?
a) 8
- 2) If a student has to pass each of the five subjects to get through. In how many ways can the candidate fail?
a) 31
- 3) If the coefficient of x^{-1} in the expansion of $\left(x + \frac{k}{x^2}\right)^5$ is 90, then the value of k is
c) both a and b
- 4) The set of all integers is not group under multiplication because
- 5) Conjugate of a complex number is $\frac{1}{(i-1)}$. Then the complex number is
d) $\frac{-1}{i+1}$

4) If $1-p$ is a root of $x^2 + px + 1-p = 0$ then its roots are
b) $0, -1$

7) The sum of the focal distances from any point on the ellipse.

a) 8

8) $\int \tan^3 x \sec^2 x \, dx$ equals.

d) $\frac{1}{4} \tan^4 x + C$

9) Which of the following is not true?

b) Cross product of any vector $\vec{a}$ with itself is $\perp$.

10) The foot of perpendicular drawn from point b (a, b, c) upon the yz -plane is ...

11) The derivative of $\tanh^{-1} x$ with respect to x is

c) $\frac{1}{1-x^2}$

Group B

120. A committee of 5 members is to be formed from 6 boys and 4 girls. In how many ways can it be done so as to include.

- i) exactly 1 boy ii) at least two boys

Solⁿ

i) No. of ways = $C(6, 1) \times C(4, 4)$
 $= 6$

ii)	Boys (6)	Girls (4)	No. of ways
	2	3	$C(6, 2) \times C(4, 3)$
	3	2	$C(6, 3) \times C(4, 2)$
	4	1	$C(6, 4) \times C(4, 1)$
	5	0	$C(6, 5) \times C(4, 0)$

$$\begin{aligned}\therefore \text{No. of ways} &= C(6, 2) \times C(4, 3) + C(6, 3) \times C(4, 2) \\ &\quad + C(6, 4) \times C(4, 1) + C(6, 5) \times C(4, 0) \\ &= \frac{6!}{(6-2)!2!} \times \frac{4!}{(4-3)!3!} + \frac{6!}{(6-3)!3!} \times \frac{4!}{(4-2)!2!} \\ &\quad + \frac{6!}{(6-4)!4!} \times \frac{4!}{(4-1)!1!} + \frac{6!}{(6-5)!5!} \times \frac{4!}{(4-0)!4!} \\ &= 246 \text{ ways.}\end{aligned}$$

b) Probability of hitting a ~~the~~ target is $\frac{1}{4}$. If 5 hitting are made, find the probability that:

i) None will strike the target

ii) Exactly one will strike the target

So r^n

$$p = \frac{1}{4}$$

$$q = 1 - \frac{1}{4} = \frac{3}{4}$$

$$n = 5$$

$$P(r) = {}^nC_r p^r q^{n-r}$$

$$i) P(r=0) = {}^5C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{5-0}$$

$$= 1 \times \frac{243}{1024} = \frac{243}{1024}$$

$$ii) P(r=1) = {}^5C_1 \left(\frac{1}{4}\right)^1 \times \left(\frac{3}{4}\right)^{5-1}$$

$$= 5 \times \frac{1}{4} \times \frac{3^4}{4^4}$$

$$= \frac{405}{1024}$$

- 14) A transport company has three types of truck A, B and C which are designed to carry three different sizes of boxes X, Y and Z per load as shown below:

Trucks	Boxes		
	X	Y	Z
A	2	5	0
B	3	2	5
C	1	5	4

Find by matrix method, the number of trucks each type that should be used to carry 17 boxes of size X, 33 boxes of size Y and 23 boxes of size Z.

If the charges of carrying the boxes by trucks A, B, C are Rs. 1200, Rs. 1500 and Rs. 1350 respectively, find the total income made by the transport company.

Solⁿ

Let the required number of trucks of size X, Y and Z be x , y and z respectively.

Then the system of equations:

$$2x + 3y + z = 17$$

$$5x + 2y + 6z = 33$$

$$0x + 5y + 4z = 23$$

This can be written as

$$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 6 \\ 0 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17 \\ 33 \\ 23 \end{bmatrix}$$

i.e. $AX = B$

where,

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 6 \\ 0 & 5 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 17 \\ 33 \\ 23 \end{bmatrix}$$

Now,

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 5 & 2 & 6 \\ 0 & 5 & 4 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 2 & 6 \\ 5 & 4 \end{vmatrix} - 3 \begin{vmatrix} 5 & 4 \\ 0 & 4 \end{vmatrix} + 0$$

$$= 2(8 - 30) - 3(20 - 0)$$

$$= 2(-22) - 60$$

$$= -44 - 60 = -104 \neq 0. \text{ So, } A^{-1} \text{ exists.}$$

~~But~~ Now, we find cofactors,

$$\text{Cofactor of } (a_{11} = 2) = \begin{vmatrix} 2 & 6 \\ 5 & 4 \end{vmatrix} = 8 - 30 = -22$$

$$\text{Cofactor of } (a_{12} = 3) = - \begin{vmatrix} 5 & 6 \\ 0 & 4 \end{vmatrix} = -20 + 0 = -20$$

$$\text{Cofactor of } (a_{13} = 1) = \begin{vmatrix} 5 & 2 \\ 0 & 5 \end{vmatrix} = 25$$

$$\text{Cofactor of } (a_{21} = 25) = - \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} = -(12 - 5) = -7$$

$$\text{Cofactor of } (a_{22} = 2) = \begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} = 8 - 0 = 8$$

$$\text{Cofactor of } (a_{23} = 6) = - \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -10 + 0 = -10$$

$$\text{Cofactor of } (a_{31} = 0) = \begin{vmatrix} 3 & 1 \\ 2 & 6 \end{vmatrix} = 18 - 2 = 16$$

$$\text{Cofactor of } (a_{32} = 5) = - \begin{vmatrix} 2 & 1 \\ 25 & 6 \end{vmatrix} = -(12 - 25) = 13$$

$$\text{Cofactor of } (a_{33} = 4) = \begin{vmatrix} 2 & 3 \\ 25 & 2 \end{vmatrix} = 10 - 75 = -65$$

$$\text{Matrix of cofactors} = \begin{bmatrix} -22 & -20 & 25 \\ -7 & 8 & -10 \\ 16 & 13 & -65 \end{bmatrix}$$

$$\text{Adjoint of } A = \begin{bmatrix} -22 & -20 & 25 \\ -7 & 8 & -10 \\ 16 & 13 & -65 \end{bmatrix}^T$$

$$\therefore \text{Adj}(A) = \begin{bmatrix} -22 & -7 & 16 \\ -20 & 8 & 13 \\ 25 & -10 & -65 \end{bmatrix}$$

We know that,

$$A^{-1} = \frac{1}{|A|} \text{Adj}(A)$$

$$= \frac{1}{-124} \begin{bmatrix} -22 & -7 & 16 \\ -8 & 20 & -28 \\ 10 & -25 & 4 \end{bmatrix}$$

Now,

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \frac{1}{-124} \begin{bmatrix} -22 & -7 & 16 \\ -8 & 20 & -28 \\ 10 & -25 & 4 \end{bmatrix} \begin{bmatrix} 17 \\ 33 \\ 23 \end{bmatrix}$$

$$= \frac{1}{-124} \begin{bmatrix} -22 \times 17 - 7 \times 33 + 16 \times 23 \\ -8 \times 17 + 20 \times 33 - 28 \times 23 \\ 10 \times 17 - 25 \times 33 + 4 \times 23 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \frac{1}{-79} \begin{bmatrix} -22 & -7 & 16 \\ -20 & 8 & -7 \\ 25 & -10 & -11 \end{bmatrix} \begin{bmatrix} 17 \\ 33 \\ 23 \end{bmatrix}$$

$$= -\frac{1}{79} \begin{bmatrix} -22 \times 17 - 7 \times 33 + 16 \times 23 \\ -20 \times 17 + 8 \times 33 - 7 \times 23 \\ 25 \times 17 - 10 \times 33 + 11 \times 23 \end{bmatrix}$$

$$= -\frac{1}{79} \begin{bmatrix} -237 \\ -237 \\ -158 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$$

$$\therefore x=3, y=3, z=2$$

$$\text{Total income} = \begin{bmatrix} 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1200 \\ 1500 \\ 1350 \end{bmatrix}$$

Q. a) Find the n term and sum to the n terms of the series :

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots$$

Solⁿ

$$\begin{aligned} n^{\text{th}} \text{ term of } 1, 2, 3, \dots &= 1 + (n-1)1 \\ &= 1 + n - 1 \\ &= n \end{aligned}$$

$$\begin{aligned} n^{\text{th}} \text{ term of } 2, 3, 4, \dots &= 2 + (n-1)1 \\ &= 2 + n - 1 \\ &= n + 1 \end{aligned}$$

$$\begin{aligned} n^{\text{th}} \text{ term of the series} &= n(n+1) \\ &= n^2 + n \end{aligned}$$

$$\begin{aligned} \text{Sum to } n \text{ terms} &= \sum (n^2 + n) \\ &= \sum n^2 + \sum n \end{aligned}$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(2n+1) + 3n(n+1)}{6}$$

$$= \frac{n(n+1)(2n+1+3)}{6}$$

$$= \frac{n(n+1)(2n+4)}{6}$$

$$\therefore n \text{ term} = n^2 + n$$

$$\therefore \text{Sum of } n \text{ term} = \frac{n(n+1)(2n+4)}{6}$$

b) Using principle of mathematical induction prove that $3^{2n}-1$ is divisible by 8.

Solⁿ

Let $P(n): 3^{2n}-1$ is divisible by 8.

When $n=1$,

$$3^{2 \times 1} - 1 = 9 - 1 = 8 \text{ which is divisible by 8.}$$

$\therefore P(1)$ is true.

Assume that $P(n)$ is true for $n=k$.

i.e. $P(k) = 3^{2k}-1$ which is divisible by 8.

We will show that $P(k+1)$ is true whenever $P(k)$ is true.

$$\begin{aligned} \text{i.e. } P(k+1) &= 3^{2(k+1)} - 1 \\ &= 3^{2k} \cdot 9 - 1 \\ &= 3^{2k} \cdot 9 - 9 + 9 - 1 \\ &= 9(3^{2k} - 1) + 8 \end{aligned}$$

which is divisible by 8 as $9(3^{2k}-1)$ and 8 are divisible by 8.

b) Find the foci and eccentricity of the hyperbola
 $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Solⁿ

Given equation of hyperbola:

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \quad \text{--- (1)}$$

Comparing (1) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

we get, $a = 3$, $b = 4$

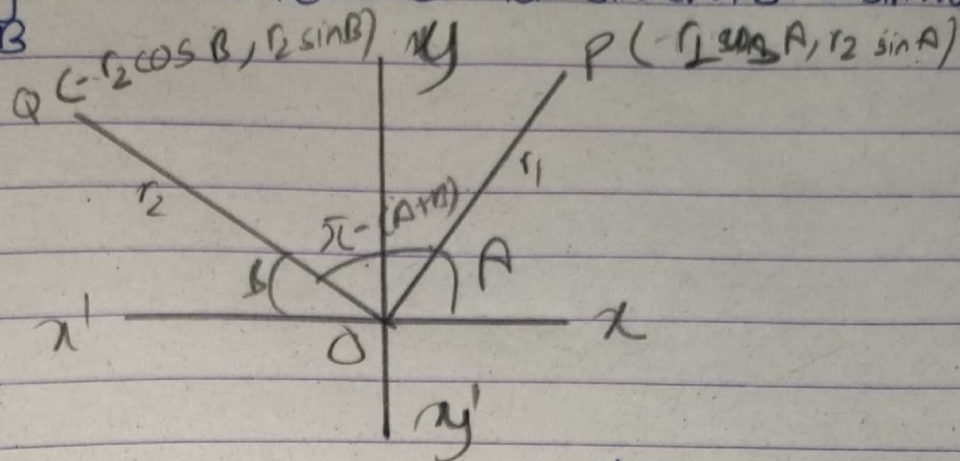
Since coeff. of x^2 in equation (1) is +ve, transverse axis is along x-axis.

$$\begin{aligned} \text{So, eccentricity} = e &= \sqrt{1 + \frac{b^2}{a^2}} \\ &= \sqrt{1 + \frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3} \end{aligned}$$

$$\text{Coordinate of foci} = (\pm ae, 0) = (\pm 3 \cdot \frac{5}{3}, 0) = (\pm 5, 0)$$

$\therefore$ coordinate of foci is $(\pm 5, 0)$ and eccentricity is $\frac{5}{3}$

(17) Prove by vector method that $\sin(A+B) = \sin A \cos B + \cos A \sin B$



proof: let O be the origin and xOx' and yOy' be two mutually perpendicular lines.

Here, $OP = |\vec{OP}| = r_1$

$OQ = |\vec{OQ}| = r_2$

$\angle POx = A$, $\angle QOx' = B$, $\angle POQ = \pi - (A+B)$

Hence, the co-ordinates of $\vec{OP}$ and $\vec{OQ}$ are:

$$\vec{OP} = (r_1 \cos A, r_1 \sin A, 0)$$

$$\vec{OQ} = (-r_2 \cos B, r_2 \sin B, 0)$$

$$\vec{OP} \times \vec{OQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_1 \cos A & r_1 \sin A & 0 \\ -r_2 \cos B & r_2 \sin B & 0 \end{vmatrix}$$

$$= (0-0)\vec{i} - \vec{j}(0-0) + \vec{k}(r_1 r_2 \cos A \sin B + r_1 r_2 \sin A \cos B)$$

$$|\vec{OP} \times \vec{OQ}| = \sqrt{0^2 + 0^2 + (r_1 r_2 \cos A \sin B + r_1 r_2 \sin A \cos B)^2}$$

$$= r_1 r_2 (\sin A \cos B + \cos A \sin B)$$

Since $\pi - (A+B)$ is the angle between OP and OQ ,

$$\sin(\pi - (A+B)) = \frac{|\vec{OP} \times \vec{OQ}|}{|\vec{OP}| |\vec{OQ}|}$$

$$\Rightarrow \sin(A+B) = \frac{r_1 r_2 (\sin A \cos B + \cos A \sin B)}{r_1 r_2}$$

$$= \sin A \cos B + \cos A \sin B$$

(19) Using simplex method, maximize, $U = 25x + 45y$ subject to
 $x + 3y \leq 21$, $2x + 3y \leq 24$, $x, y \geq 0$
 solⁿ

We have to maximize,

$$U = 25x + 45y$$

Subject to

$$x + 3y \leq 21$$

$$2x + 3y \leq 24$$

Introducing slack variables s and t , we get,

$$x + 3y + s = 21$$

$$2x + 3y + t = 24$$

$$-25x - 45y + U = 0$$

Canonical form of the above system:

$$x + 3y + s + 0t + 0u = 21$$

$$2x + 3y + 0s + t + 0u = 24$$

$$-25x - 45y + 0s + 0t + u = 0$$

The above canonical form yields the following simplex tableau:

Ratio	BV	x	y	s	t	u	RHS
7	s	1	3	1	0	0	21
8	t	2	3	0	1	0	24
		-25	-45	0	0	1	0

-45 is the most negative entry and 7 is the least positive ratio. Hence, 3 is the pivot element.

Performing $R_1 \rightarrow R_1/3$,

BV	x	y	s	t	u	RHS
y	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	7
t	2	3	0	1	0	24
	-25	-45	0	0	1	0

Performing $R_2 \rightarrow R_2 - 3R_1$ & $R_3 \rightarrow R_3 + 45R_1$,

Ratio	BV	x	y	s	t	u	RHS
21	y	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	7
3	t	1	0	-1	1	0	3
		-10	0	15	0	1	345

-10 is the most negative entry and 3 is the least positive ratio. So, 1 is the pivot element

performing $R_1 \rightarrow R_1 - R_2/3$ & $R_3 \rightarrow R_3 + 10R_2$

BV	x	y	s	t	u	RHS
y	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	6
x	1	0	-1	1	0	3
	0	0	5	10	1	345

So maximum value of u is 345 at (3,6)

Group - C

20a) State binomial theorem, then write down the formula for finding general term of binomial theorem. Also show that sum of all binomial coefficients is 2^n .

Solⁿ

Binomial theorem states that for any positive integer n ,

$$(a+x)^n = {}^nC_0 a^n x^0 + {}^nC_1 a^{n-1} x^1 + {}^nC_2 a^{n-2} x^2 + {}^nC_3 a^{n-3} x^3 + \dots + {}^nC_r a^{n-r} x^r + \dots + {}^nC_n a^{n-n} x^n$$

$$\text{i.e. } (a+x)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} x + {}^nC_2 a^{n-2} x^2 + \dots + {}^nC_n x^n$$

$$\text{where } {}^nC_r = C(n,r) = \frac{n!}{(n-r)!r!}$$

The general term in the expansion of $(a+x)^n$ is $(r+1)^{\text{th}}$ term

$$\text{i.e. } t_{r+1} = {}^nC_r a^{n-r} x^r$$

We know,

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$$

when $x=1$,

$$(1+1)^n = C_0 + C_1(1) + C_2(1)^2 + C_3(1)^3 + \dots + C_n(1)^n$$

$$\Rightarrow 2^n = C_0 + C_1 + C_2 + C_3 + \dots + C_n$$

Hence, the sum of all binomial coefficients is 2^n .

b) Show that the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$ is $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{n!} (-2)^n$.

Solⁿ

We have,

$$\left(x - \frac{1}{x}\right)^{2n}$$

Here, $m=2n$ which is even. So, there is only one middle term. The middle term is

$$\binom{m+1}{2} = \binom{2n+1}{2} = (n+1)^{\text{th}} \text{ term}$$

$$\therefore T_{n+1} = {}^{2n}C_n (x)^{2n-n} \left(-\frac{1}{x}\right)^n$$

$$= \frac{(2n)!}{(2n-n)! \cdot n!} x^n (-1)^n$$

$$= \frac{(2n)!}{n! \cdot n!} (-1)^n$$

$$= \frac{2n(2n-1)(2n-2)(2n-3) \dots 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot (-1)^n}{n! \cdot n!}$$

$$= \frac{[2n(2n-2)(2n-4) \dots 6 \cdot 4 \cdot 2] [(2n-1)(2n-3) \dots 5 \cdot 3 \cdot 1] (-1)^n}{n! \cdot n!}$$

$$= \frac{(2 \times 2 \times 2 \dots \times 2 \times 2 \times 2) [n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1] [1 \cdot 3 \cdot 5 \dots (2n-1)]}{(-1)^n}$$

$$= \frac{2^n \times n! \cdot [1 \cdot 3 \cdot 5 \dots (2n-1)] (-1)^n}{n! \cdot n!}$$

$$= \frac{1 \cdot 3 \cdot 5 \dots (2n-1) (-2)^n}{n!}$$

Hence the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$ is $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)(-2)^n}{n!}$

21) a) If α and β are complex cube roots of unity, show that

$$\alpha^4 + \beta^4 + \alpha^2 \beta^2 = 0$$

Solⁿ

Let $\alpha = \omega$ & $\beta = \omega^2$

$$\begin{aligned} \text{LHS} &= \alpha^4 + \beta^4 + \alpha^2 \beta^2 \\ &= \omega^4 + (\omega^2)^4 + \omega^2 (\omega^2)^2 \\ &= \omega^3 \cdot \omega + \omega^8 + \omega^2 \omega^4 \\ &= 1 \cdot \omega + \omega^6 \cdot \omega^2 + \omega^6 \\ &= \omega + 1 \cdot \omega^2 + 1 \\ &= \omega + \omega^2 + 1 = 0 = \text{RHS} \end{aligned}$$

b) Find the regression coefficients b_{yx} and b_{xy} for the data given below:

$$\Sigma x = 24, \Sigma y = 12, \Sigma x^2 = 374, \Sigma y^2 = 97, \Sigma xy = 157, n = 7$$

Solⁿ

We know,

$$\begin{aligned} b_{yx} &= \frac{n \Sigma xy - \Sigma x \Sigma y}{n \Sigma x^2 - (\Sigma x)^2} \\ &= \frac{7 \times 157 - 24 \times 12}{7 \times 374 - 24^2} \\ &= 0.3971 \end{aligned}$$

Also,

$$\begin{aligned}b_{yx} &= \frac{n \sum xy - \sum x \sum y}{n \sum y^2 - (\sum y)^2} \\&= \frac{7 \times 157 - 24 \times 12}{7 \times 97 - 12^2} \\&= 1.5158\end{aligned}$$

c) Find the regression of x on y from the following data. Estimate the value of x when $y = 18$.
Solⁿ

22.a) Find, by definition, the derivative of $e^{\cos x}$ at $x=0$.

Solⁿ

$$\text{let } f(x) = e^{\cos x}$$

$$\text{then } f(x+h) = e^{\cos(x+h)}$$

where h is small increment in x .

By definition,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^{\cos(0+h)} - e^{\cos 0}}{h} \quad \text{--- (*)}$$

$$\text{let } e^{\cos 0} = u$$

$$e^{\cos(0+h)} = u+k$$

$$\Rightarrow k = e^{\cos(0+h)} - e^{\cos 0}$$

When $h \rightarrow 0$, $k \rightarrow 0$.

from (*),

$$f'(0) = \lim_{h \rightarrow 0} \frac{e^{u+k} - e^u}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^u \cdot e^k - e^u}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^u (e^k - 1)}{h} \cdot \frac{k}{k}$$

$$= \lim_{h \rightarrow 0} e^{\cos 0} \cdot \frac{\cos(0+h) - \cos 0}{h}$$

$$= e^1 \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{\sin(0+h+0)}{2} \cdot \frac{\sin(0+h-0)}{2}$$

$$= e \lim_{h \rightarrow 0} \frac{\sin h}{\frac{h}{2}} \cdot \frac{\sin h}{2}$$

$$= e \times 1 \times \frac{\sin 0}{2}$$

$$= e \times 0$$

$$= 0$$

$$\therefore \frac{d}{dx} f'(0) = 0$$

b) Find the derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to x .

Solⁿ

Put $x = \tan \theta$

$$\therefore y = \tan^{-1}\left(\frac{2 \tan \theta}{1 - \tan^2 \theta}\right)$$

$$= \tan^{-1}(\tan 2\theta)$$

$$= 2\theta$$

$$\therefore y = 2 \tan^{-1} x \quad \text{--- (1)}$$

Differentiating (1) w.r.t. x , we get,

$$\frac{dy}{dx} = 2 \frac{d \tan^{-1} x}{dx} = \frac{2}{1+x^2}$$

$$\therefore \frac{d}{dx} \left[\tan^{-1} \left(\frac{2x}{1-x^2} \right) \right] = \frac{2}{1+x^2}$$

c) Evaluate: $\int \frac{5x^2}{3x^4 + x^2 - 2} dx$

Solⁿ

$$\text{let } I = \int \frac{5x^2}{3x^4 + x^2 - 2} dx$$

$$\text{put } x^2 = t$$

$$\Rightarrow 2x = \frac{dt}{dx}$$

$$\Rightarrow$$

Set-III

Group A

- 1) Third term in the expansion of $\left(x^2 - \frac{1}{x^3}\right)^n$ is ~~dit~~ independent of x , then $n = ?$
 - a) 3
- 2) In how many ways letters of the word "ARRANGE" be arranged so that two R's come together
 - c) 360
- 3) Amplitude of $2 - \sqrt{7}$ is ... 2
- 4) If the roots of the equation $lx^2 + nx + n = 0$ be ~~the~~ in the ratio of $p : q$, then $\sqrt{p/q} + \sqrt{q/p} + \sqrt{n/l} =$
 - b) 0
- 5) $2 \tan^{-1} x = \dots 2$
- 6) The parametric equation $x = a \cos \theta$, and $y = b \sin \theta$ represents
 - b) ellipse

7) $\int \frac{|x|}{x} dx$ is equal to

8) The probability of having exactly two tails in 6 tosses of coin is.

9) $15/64$

9) If $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a}$ is non-zero vector then

a) $\vec{b} = \vec{c}$

10) When two regression lines are perpendicular to each other, then there is

b) no correlation

11) The equation of normal of a curve $y = 2x^3 - 5x^2$ at $(2, 4)$ is

Group B

12) a) In how many ways can the letters of the word 'SUNDAY' be arranged? How many of these arrangements do not begin with S? How many begin with S and do not end with A?

In the word 'SUNDAY' there are 6 letters.

* 6 letters can be arranged in $6!$ ways = 720 ways.
Hence, the letters of the word SUNDAY can be arranged in 720 ways.

* For those which do not begin with S,
no. of arrangements = $5 \times P(5,5) = 5 \times 5! = 600$ ways

* For those beginning with S and not ending with Y,
no. of arrangements = $1 \times P(4,4) \times 4 = 4! \times 4 = 96$ ways

Hence 600 arrangements do not begin with S
and 96 begin with S and do not end with Y.

b) let $G = \{ \dots, -3, -2, 0, 1, 2, 3, \dots \}$ and the operation defined on G be of addition find the identity elements of 1 and 2.

- b) Find the equation of the ellipse whose distance between two foci is 8 and the semi latus rectum is 6.

Solⁿ

Here,

distance between foci = 8

$$\Rightarrow 2ae = 8$$

$$\Rightarrow ae = 4$$

$$\text{Also, semi-latus rectum} = \frac{1}{2} \left(\frac{2b^2}{a} \right)$$

$$\Rightarrow 6 = \frac{b^2}{a}$$

$$\Rightarrow b^2 = 6a \quad \text{--- (1)}$$

We know,

$$b^2 = a(1 - e^2)$$

$$\Rightarrow b^2 = a^2 - a^2e^2$$

$$\Rightarrow 6a = a^2 - 4^2$$

$$\Rightarrow a^2 - 6a - 16 = 0$$

$$\Rightarrow (a - 8)(a + 2) = 0$$

$$\Rightarrow a = -2, 8$$

Since a cannot be negative, $a = 8$, $a^2 = 64$.

$$\text{From (1), } b^2 = \sqrt{6 \times 8} = \sqrt{48} = 4\sqrt{3} = 6 \times 8 = 48$$

The required equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{64} + \frac{y^2}{48} = 1$$

$$\Rightarrow \frac{3x^2 + 4y^2}{192} = 1$$

$\Rightarrow 3x^2 + 4y^2 = 192$ which is the required equation of ellipse.

14. a) Given quadratic equation $ax^2 + bx + c = 0$ then write down the condition of

- i) roots equal in magnitude but opposite sign.
- ii) Reciprocal roots
- iii) One root zero

Solⁿ

Given equation is $ax^2 + bx + c = 0$ — (1)

~~Let the roots of the equation be α and β .~~

i) if the roots are equal in magnitude but opposite in sign, let the roots be α and $-\alpha$.

$$\Rightarrow \alpha + (-\alpha) = -\frac{b}{a}$$

$$\Rightarrow 0 = -\frac{b}{a}$$

$$\Rightarrow a = -b$$

$\Rightarrow a + b = 0$ which is the condition for roots to be equal in magnitude but opposite sign.

ii) if the roots are reciprocal, let them be α and $\frac{1}{\alpha}$.

$$\Rightarrow \alpha \cdot \frac{1}{\alpha} = \frac{c}{a}$$

$\Rightarrow a = c$ which is the condition for reciprocal roots.

iii) let the roots be α and 0.

$$\Rightarrow \alpha \cdot 0 = \frac{c}{a}$$

$$\Rightarrow 0 = \frac{c}{a}$$

$\Rightarrow c = 0$ which is the condition for one root zero.

b) Solve by matrix inverse method $x+y=2$ and $x-y=0$.
Solⁿ.

The matrix equation of given system is $AX=B$.
where, $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \neq 0$$

So, A^{-1} exists.

Now,

$$\text{Adj}(A) = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} -1 \times 2 & -1 \times 0 \\ -1 \times 2 & 1 \times 0 \end{pmatrix}$$

$$= \frac{1}{-2} \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\therefore x=y=1$$

16.a) Find the n th term and then sum to n terms of the series: $1 + 3 + 6 + 10 + \dots$

Solⁿ

Given series is:

$$1 + 3 + 6 + 10 + \dots$$

Let t_n be the n th term and S_n be the sum of n terms. Then,

$$S_n = 1 + 3 + 6 + 10 + \dots + t_n$$

$$- S_n = -1 - 3 - 6 - 10 - \dots - t_{n-1} - t_n$$

$$\text{or, } 0 = 1 + 2 + 3 + 4 + \dots + (t_n - t_{n-1}) - t_n$$

$$\text{or, } t_n = 1 + 2 + 3 + 4 + \dots + (t_n - t_{n-1})$$
$$= 1 + 2 + 3 + 4 + \dots + n^{\text{th}} \text{ term}$$

$$\therefore t_n = \frac{n(n+1)}{2}$$

$$= \frac{1}{2}n^2 + \frac{1}{2}n$$

Now,

$$S_n = \sum t_n$$

$$= \sum \left(\frac{1}{2}n^2 + \frac{1}{2}n \right)$$

$$= \frac{1}{2} \sum n^2 + \frac{1}{2} \sum n$$

$$= \frac{1}{2} \frac{n(n+1)(2n+1)}{6} + \frac{1}{2} \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{4} \left[\frac{2n+1}{3} + 1 \right]$$

$$= \frac{n(n+1)}{4} \left(\frac{2n+4}{3} \right)$$

$$= \frac{n(n+1)(n+2)}{6}$$

$$= \frac{n(n+1)(n+2)}{6}$$

$$\therefore n^{\text{th}} \text{ term} = \frac{1}{2}n^2 + \frac{1}{2}n$$

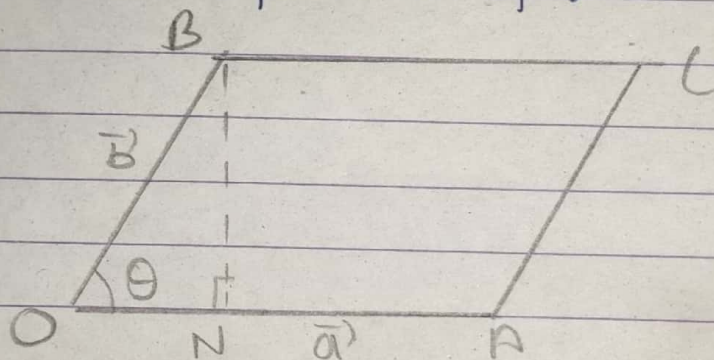
$$\text{and sum to } n \text{ terms} = \frac{n(n+1)(n+2)}{6}$$

b)

(18) Define vector product. Interpret it geometrically and hence find the area of the parallelogram whose adjacent sides are the vectors $\vec{u} = 2\vec{i} + 3\vec{j}$ and $\vec{v} = \vec{i} + 4\vec{j}$.

→ Vector product of vectors $\vec{a}$ and $\vec{b}$ is denoted and defined as $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is the unit normal vector perpendicular to both vectors $\vec{a}$ and $\vec{b}$.

Geometrical interpretation of vector product.



Let OACB be a parallelogram having adjacent sides $\vec{OA}$ and $\vec{OB}$ where $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$

$$|\vec{OA}| = OA = a \quad \text{and} \quad |\vec{OB}| = OB = b$$

Let θ be the angle between $\vec{OA}$ and $\vec{OB}$. From B, drawn BN perpendicular to OA. From figure, (In $\triangle ONB$)

$$\sin \theta = \frac{BN}{OB}$$

$$\Rightarrow BN = (OB) \sin \theta \quad \text{--- (*)}$$

We know that,

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$$
$$= (OA)(OB) \sin \theta$$

$$= (OA) \times (BN) \quad \text{Using (*)}$$

$$= (\text{base}) \times (\text{height})$$

$$= \text{Area of parallelogram OACB}$$

Thus, geometrically, magnitude of $\vec{a} \times \vec{b}$ is the area of the parallelogram having adjacent sides $\vec{a}$ and $\vec{b}$.

Given: the adjacent sides of parallelogram are:

$$\vec{u} = 2\vec{i} + 3\vec{j}$$

$$\vec{v} = \vec{i} + 4\vec{j}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ 1 & 4 & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(8-3)$$

$$= 5\vec{k}$$

$$|\vec{u} \times \vec{v}| = \sqrt{5^2}$$

$$= 5 \text{ sq. units.}$$

19. a) Discuss the applicability of the Rolle's theorem for $f(x) = \frac{1}{x^2 - 1}$ in $[-2, 2]$.

Solⁿ

Solⁿ

$$\text{Here, } f(x) = \frac{1}{x^2 - 1}$$

where $x = 1 \in [-2, 2]$, $f(x)$ is undefined.

So, $f(x)$ is discontinuous at $x = 1 \in [-2, 2]$.

So, mean value theorem is not applicable.

b) Evaluate $\int \frac{\sinh x \, dx}{4 \tanh x - \operatorname{cosech} x \operatorname{sech} x}$

Solⁿ

$$\text{Let } I = \int \frac{\sinh x \, dx}{4 \tanh x - \operatorname{cosech} x \operatorname{sech} x}$$

$$= \int \frac{\sinh x \, dx}{4 \frac{\sinh x}{\cosh x} - \frac{1}{\sinh x} \cdot \frac{1}{\cosh x}}$$

$$= \int \frac{\sinh x \, dx}{4 \sinh^2 x - 1}$$

$$= \int \frac{\sinh^2 x \cosh x \, dx}{4 \sinh^2 x - 1}$$

$$\text{Put } 2 \sinh x = t$$

$$\Rightarrow 2 \cosh x = \frac{dt}{dx}$$

$$\Rightarrow \cosh x \, dx = \frac{1}{2} dt$$

$$\text{So, } I = \int \frac{\frac{t^2}{4} \cdot \frac{dt}{2}}{t^2 - 1} \quad \left[\begin{array}{l} \because 2 \sinh x = t \\ \Rightarrow \sinh x = t/2 \\ \Rightarrow \sinh^2 x = t^2/4 \end{array} \right]$$

$$= \frac{1}{8} \int \frac{t^2 dt}{t^2 - 1}$$

$$= \frac{1}{8} \int \frac{(t^2 - 1) + 1}{t^2 - 1} dt$$

$$= \frac{1}{8} \int \frac{(t^2 - 1)}{(t^2 - 1)} dt + \frac{1}{8} \int \frac{1}{t^2 - 1} dt$$

$$= \frac{1}{8} \int 1 dt + \frac{1}{8 \cdot 2} \ln \left(\frac{t-1}{t+1} \right) + C$$

$$= \frac{1}{8} \cdot t + \frac{1}{16} \ln \left(\frac{t-1}{t+1} \right) + C$$

$$= \frac{1}{8 \cdot 4} \cdot 2 \sinh x + \frac{1}{16} \ln \left(\frac{2 \sinh x - 1}{2 \sinh x + 1} \right) + C$$

$$= \frac{1}{4} \sinh x + \frac{1}{16} \ln \left(\frac{2 \sinh x - 1}{2 \sinh x + 1} \right) + C$$

Group-C

20.a) If the successive coefficients in the expansion of $(1+x)^n$ are 28, 56 and 70, find n .

solⁿ

Let C_{r-1} , C_r and C_{r+1} be three consecutive coefficients in the expansion of $(1+x)^n$.

Then, by given condition,

$$C_{r-1} = 28 \quad \text{--- (1)}$$

$$C_r = 56 \quad \text{--- (2)}$$

$$C_{r+1} = 70 \quad \text{--- (3)}$$

from (1) & (2),

$$\frac{C_{r-1}}{C_r} = \frac{28}{56}$$

$$\Rightarrow \frac{\frac{n!}{(n-(r-1))! (r-1)!}}{n!} = \frac{28}{56}$$

$$\Rightarrow \frac{(n-r)! r!}{(n-r+1)! (r-1)!} = \frac{28}{56}$$

$$\Rightarrow \frac{r}{n-r+1} = \frac{28}{56}$$

$$\Rightarrow 2r = n-r+1$$

$$\Rightarrow 3r = n+1 \quad \text{--- (4)}$$

from (2) and (3),

$$\frac{C_r}{C_{r+1}} = \frac{56}{70}$$

$$\Rightarrow \frac{\frac{n!}{(n-r)! r!}}{(n-r+1)! (r+1)!} = \frac{4}{5}$$

$$\Rightarrow \frac{(n-r-1)!(r+1)!}{(n-r)! \cdot r!} = \frac{4}{5}$$

$$\Rightarrow \frac{(n-r-1)(r+1)}{(n-r)r} = \frac{4}{5}$$

$$\Rightarrow 5r+5 = 4n-4r$$

$$\Rightarrow 4n = 9r+5$$

$$\Rightarrow n = \frac{1}{4}(9r+5) \quad \text{--- (5)}$$

Solving (4) and (5),

$$3r = \frac{1}{4}(9r+5) + 1$$

$$\Rightarrow 12r = 9r+5+4$$

$$\Rightarrow 12r = 9r+9$$

$$\Rightarrow 3r = 9$$

$$\therefore r = 3$$

From (4),

$$\Rightarrow 3 \times 3 = n+1$$

$$\Rightarrow 9-1 = n$$

$$\therefore n = 8$$

b) Prove that $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots = 2e$

Solⁿ

$$\ln \text{ of } 1^2, 2^2, 3^2, \dots = n^2$$

$$\ln \text{ of } 1!, 2!, 3!, 4!, \dots = n!$$

$$\ln \text{ of LHS} = \frac{n^2}{n!}$$

$$= \frac{n^2}{n(n-1)!}$$

$$= \frac{n+1}{(n-1)!} + \frac{1}{(n-1)!}$$

$$= \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$$

$$\therefore S_n = \sum \ln$$

$$= \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots \right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots \right)$$

$$= e + e = 2e = \text{RHS}$$

21) a) If α and β are complex cube roots of unity, show that:

$$\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = 0$$

Solⁿ

$$\text{let } \alpha = \omega \text{ and } \beta = \omega^2$$

$$\text{LHS} = \alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1}$$

$$= \omega^4 + (\omega^2)^4 + \frac{1}{\omega} \cdot \frac{1}{\omega^2}$$

$$= \omega + \omega^2 + 1$$

$$= 0 = \text{RHS} \# \text{ proved.}$$

b) State De-Moivre's theorem. Using De-Moivre's theorem, solve $z^4 + 1 = 0$

Solⁿ

For positive integer n ,

$$[r(\cos \theta + i \sin \theta)]^n = r^n [\cos n\theta + i \sin n\theta]$$

Given: $z^4 + 1 = 0$

$$\Rightarrow z^4 = -1$$

Let $x + iy = -1 + 0i = r(\cos \theta + i \sin \theta)$

then, $x = -1$, $y = 0$

$$r = \sqrt{x^2 + y^2} = 1$$

$$\tan \theta = \frac{y}{x} = \frac{0}{-1} = 0 = \tan 180^\circ$$

$$\therefore \theta = 180^\circ$$

Thus,

$$-1 = (\cos 180^\circ + i \sin 180^\circ)$$

$$\Rightarrow z^4 = \cos (360n + 180) + i \sin (360n + 180)$$

$$\Rightarrow z = \cos (90n + 45) + i \sin (90n + 45)$$

where $n = 0, 1, 2, 3$

When $n = 0$

$$z = \cos 45^\circ + i \sin 45^\circ$$

$$= \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}$$

When $n = 1$,

$$z = \cos 135^\circ + i \sin 135^\circ$$

$$= -\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}$$

When $n = 2$,

$$z = \cos 225^\circ + i \sin 225^\circ$$

$$= -\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}$$

When $n = 3$,

$$z = \cos 315^\circ + i \sin 315^\circ$$

$$= \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}$$

$$\therefore Z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} \text{ and } \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

22.9) Using simplex method, maximize $z = 3x + 4y$ subject to $x + y \leq 10, x \leq 4, y \leq 8, x, y \geq 0$.

Solⁿ

We have to maximize,

$$Z = 3x + 4y$$

subject to $x + y \leq 10$

$$x \leq 4$$

$$y \leq 8$$

Introducing non-negative slack variables s, t, u , we get system of equations:

$$x + y + s = 10$$

$$x + t = 4$$

$$y + u = 8$$

$$-3x - 4y + z = 0$$

Canonical form of above system of equations is:

$$x + y + s + 0t + 0u + 0z = 10$$

$$x + 0y + 0s + t + 0u + 0z = 4$$

$$0x + y + 0s + 0t + u + 0z = 8$$

$$-3x - 4y + 0s + 0t + 0u + z = 0$$

Above canonical form yields following simplex tableau:

BV	x	y	s	t	u	z	RHS
s	1	1	1	0	0	0	10
t	1	0	0	1	0	0	4
u	0	1	0	0	1	0	8
	-3	-4	0	0	0	1	0

Here, 1 is the pivot element. So, performing $R_1 \rightarrow R_1 - R_3$ and $R_4 \rightarrow R_4 + 4R_3$

BV	x	y	s	t	u	z	RHS
s	1	0	1	0	-1	0	2
t	1	0	0	1	0	0	4
y	0	1	0	0	1	0	8
	-3	0	0	0	4	1	32

Now, performing $R_2 \rightarrow R_2 - R_1$ and $R_4 \rightarrow R_4 + 3R_1$

BV	x	y	s	t	u	z	RHS
x	1	0	1	0	1	0	2
t	0	0	-1	1	1	0	2
y	0	1	0	0	1	0	8
	0	0	0	0	0	1	38

Here, all the entries in the bottom row are non-negative. So, solution given by the above tableau is optimum.

Therefore, maximum value of $z = 38$ at $(2, 8)$.

- b) State Lagrange's mean value theorem. Verify that $f(x) = x - x^3$ on $[-2, 1]$ satisfies conditions of the mean value theorem and find c prescribed in the theorem.

Solⁿ

Lagrange's Mean Value Theorem

If $y = f(x)$ be a function

- 1) continuous on closed interval $[a, b]$
- 2) differentiable on open interval (a, b) . Then there exists at least one point $c \in (a, b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Here, $f(x) = x - x^3$
 $= x(1 - x^2) \quad \text{--- (I)}$

- 1) Here, $f(x)$ being a cubic function, is continuous on $[-2, 1]$
- 2) $f'(x) = 1 - 3x^2 \quad \text{--- (*)}$ which is defined in $[-2, 1]$. So, $f(x)$ is differentiable on $[-2, 1]$.

As $f(x)$ satisfies both conditions of mean value theorem, we can find $c \in (-2, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(1) - f(-2)}{-2 - 1} \quad (**)$$

Here, when $x = 1$, from (1), $f(1) = 0$

when $x = -2$, from (1), $-2 - (-2)^3 = -2 + 8 = 6$

When $x = c$, from (**), $f'(c) = 1 - 3c^2$.

Now, from (**),

$$1 - 3c^2 = \frac{0 - 6}{-2 - 1}$$

$$\Rightarrow 1 - 3c^2 = \frac{-6}{-3} = 2$$

$$\Rightarrow 3c^2 = 1$$

$$\therefore c^2 = \frac{1}{3} = 1$$

$$\therefore c = \pm 1 \notin (-2, 1)$$

~~but~~ $-1 \notin (-2, 1)$ but $-1 \in (2, 1)$.

$$\therefore c = 1$$

Set-IV

Group A

- 1) What is the amplitude of a complex number i ?
b) 90°
- 2) How many three-digit numbers can be formed by using the integers 4, 5, 6, 8 with repetition?
a) 64
- 3) Which one of the following is equal to $\cot^{-1} x^2$?
- 4) Which one of the following expressions is a trigonometric equation?
- 5) What is the area of a triangle with vertices $(1, -1, 0)$, $(0, -1, 0)$ and $(0, 0, 1)$?
c) $\frac{1}{\sqrt{2}}$
- 6) In a school ground, the area is the set of all points in a plane. The sum of whose distances from two fixed points is constant. The enclosed area represented gives a
c) ellipse

7) What is the probability of getting 53 Fridays or Saturdays or both in a leap year?

c) $\frac{3}{7}$

8) If the degree of the differential equation $\frac{d^4 y}{dx^4} +$

$$2\left(\frac{d^3 y}{dx^3}\right)^2 + 3\left(\frac{d^2 y}{dx^2}\right)^3 + 4\left(\frac{dy}{dx}\right)^4 + 5 = 0$$
 is 1, what is

its order?

9) What is the point on the curve $y = 2x^2 - 4x - 3$ at which the tangent of the curve is parallel to the line $4x - y + 2 = 0$?

a) $(2, -3)$

10) While solving a system of three linear equations in the three variables x, y, z by using Gauss elimination method, what happens to the system if the third variable z is found to be a free variable?

11) If x be the output vector, D be the ^{demand} output vector, A be the input coefficient matrix and I be the unit matrix then the Leontief's technology matrix is given by

Group B

12) The numbers $1, 2, 3, 4, \dots, \infty$ are said to be natural numbers. These numbers are used to form various series.

a) Write an expression for the sum of first $(n+2)$ natural numbers.

$$\Sigma(n+2).$$

b) Write the formula for the ^{sum of} first $(n+1)$ even natural numbers.

$$\text{Sum of first } n \text{ even numbers} = n(n+1)$$

$$\begin{aligned}\text{sum of first } (n+1) \text{ even numbers} &= (n+1)(n+2) \\ &= n^2 + 2n + n + 2 \\ &= n^2 + 3n + 2 \\ &= n(n+3) + 2\end{aligned}$$

c) Write the formula for the general term of the sequence formed by the first 'n' odd natural numbers.

$$\begin{aligned}n^{\text{th}} \text{ term} &= 1 + (n-1)2 \\ &= 1 + 2n - 2 \\ &= 2n - 1\end{aligned}$$

d) Write the formula for the sum of squares of first $(n-1)$ natural numbers.

The formula for the sum of squares of first $(n-1)$ natural numbers is $\frac{(n-1)(n-2)(2n-3)}{6}$

e) Write the formula for the sum of cubes of first 'n' natural numbers.

$$\left(\frac{n(n+1)}{2}\right)^2$$

13) If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, prove that

$$C_0C_3 + C_1C_4 + C_2C_5 + \dots + C_{n-3}C_n = \frac{(2n)!}{(n-n')!(n+3)!}$$

Solⁿ

We have,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_{n-1}x^{n-1} + C_nx^n \quad (1)$$

$$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_{n-1}x + C_n \quad (2)$$

Multiplying (1) and (2),

$$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_{n-1}x^{n-1} + C_nx^n) \times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_{n-1}x + C_n) \quad (3)$$

By actual multiplication, the term containing x^{n-3} in the RHS of (3) is

$$(C_0C_3 + C_1C_4 + \dots + C_{n-3}C_n)x^{n-3}$$

Also, the term containing x^{n-2} in the expansion of LHS of (1) is

$${}^{2n}C_{n-3} x^{n-3}$$

Since eqn (3) is an identity, equating corresponding coefficients, we get,

$$C_0C_3 + C_1C_4 + \dots + C_{n-3}C_n = {}^{2n}C_{n-3}$$

$$= \frac{(2n)!}{(2n-n+3)!(n-3)!} = \frac{(2n)!}{(n+3)(n-3)!} = \text{RHS}$$

14. a) Find the value of $\sin \left(\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} \right)$

$$\text{Let } \cos^{-1} \frac{4}{5} = A$$

$$\text{and } \cos^{-1} \frac{12}{13} = B$$

$$\Rightarrow \cos A = \frac{4}{5}$$

$$\Rightarrow \cos B = \frac{12}{13}$$

$$\Rightarrow \sin A = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$$

$$\Rightarrow \sin B = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \frac{5}{13}$$

Now,

$$\sin \left(\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} \right)$$

$$= \sin A \cos B + \cos A \sin B$$

$$= \frac{3}{5} \cdot \frac{12}{13} + \frac{4}{5} \cdot \frac{5}{13}$$

$$= \frac{36}{65} + \frac{20}{65}$$

$$= \frac{56}{65}$$

$$\therefore \sin \left(\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} \right) = \frac{56}{65}$$

b) Find the unit vector perpendicular to each of the vectors $\vec{a} (1, 0, 2)$ and $\vec{B} (2, 0, 1)$

So $\hat{r}$

The vector perpendicular to both the vectors $\vec{a}$ & $\vec{B}$ is $\vec{a} \times \vec{B}$

$$\vec{a}' \times \vec{b}' = \begin{vmatrix} \vec{i}' & \vec{j}' & \vec{k}' \\ 1 & 0 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}'(0-0) - \vec{j}'(1-4) + \vec{k}'(0-0)$$

$$= 3\vec{j}'$$

So, the vector perpendicular to $\vec{a}'(1,0,2)$ and $\vec{b}'(2,0,1)$ is $\{0, 3, 0\}$.

- 15) The following table gives the information about the ages (in years) and corresponding weights (in kg) of six students of grades 8, 10 and 12 selected for a survey of dietary purpose.

Ages (yrs)	13	15	14	16	18	21
Weights (in kg)	45	41	32	45	50	55

Estimate the weight of a student of age 17 years.
What type of correlation is between the ages and the corresponding weights?

Solⁿ

Ages (x)	Weights (y)	xy	x ²	y ²
13	45	585	169	2025
15	41	615	225	1681
14	32	448	196	1024
16	45	720	256	2025
18	50	900	324	2500
21	55	1150	441	3025
$\Sigma x = 97$	$\Sigma y = 268$	$\Sigma xy = 4418$	$\Sigma x^2 = 1611$	$\Sigma y^2 = 12280$

Now, we know, regression eqn of y on x ,

$$y - \bar{y} = b_{yx}(x - \bar{x}) \quad (1)$$

Here, to find x, y , & b_{yx} ,
we have,

$$\Sigma x = 97, \quad \Sigma y = 268, \quad \Sigma xy = 4418, \quad \Sigma x^2 = 1611, \quad N = 6$$

$$\text{Now, } \bar{x} = \frac{\Sigma x}{N} = \frac{97}{6}$$

$$\bar{y} = \frac{\Sigma y}{N} = \frac{268}{6} = \frac{134}{3}$$

$$\begin{aligned} b_{yx} &= \frac{N \Sigma xy - \Sigma x \Sigma y}{N \Sigma x^2 - (\Sigma x)^2} \\ &= \frac{6 \times 4418 - 97 \times 268}{6 \times 1611 - 97^2} \\ &= \frac{512}{257} \end{aligned}$$

Now, from (1),

$$\begin{aligned} y - \bar{y} &= b_{yx}(x - \bar{x}) \\ y - \frac{134}{3} &= \frac{512}{257} \left(x - \frac{97}{6} \right) \end{aligned}$$

$$\therefore y = \frac{512x}{257} + \frac{3202}{257}$$

When $x = 17$,

$$y = \frac{512 \times 17}{257} + \frac{3202}{257} = 46.3$$

Hence, ~~the~~ the weight of a student of age 17 years is estimated to be 46.3 kgs.

We know,

$$\begin{aligned} r &= \frac{N \sum xy - \sum x \sum y}{\sqrt{N \sum x^2 - (\sum x)^2} \sqrt{N \sum y^2 - (\sum y)^2}} \\ &= \frac{6 \times 4418 - 97 \times 268}{\sqrt{6 \times 1611 - 97^2} \sqrt{6 \times 12200 - 268^2}} \\ &= 0.860 \end{aligned}$$

Hence, there is high degree of positive correlation between the ages and corresponding weights.

16) A residential student of a school is putting the following questions to his roommates. Answer these questions.

a) What is the relation between continuity and differentiability of a function at a point.

→ All differentiable functions are continuous but the converse may not be true.

b) What is the first principle of differentiation?

→ ~~The~~ By first principle of differentiation, the derivative of a function $f(x)$ is given by

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

where h is the change in x .

7.9) Find $\frac{dy}{dx}$ of $x^{\sin y} = y^{\cos x}$

Solⁿ:

Here,

$$x^{\sin y} = y^{\cos x}$$

• Taking $\ln$ on both sides,

$$\ln x^{\sin y} = \ln y^{\cos x}$$

$$\Rightarrow \sin y \ln x = \cos x \ln y$$

Differentiating both sides with respect to x , we get,

$$\frac{d \sin y \ln x}{dx} = \frac{d \cos x \ln y}{dx}$$

$$\Rightarrow \sin y \frac{d \ln x}{dx} + \ln x \frac{d \sin y}{dx} = \cos x \frac{d \ln y}{dx} + \ln y \frac{d \cos x}{dx}$$

$$\Rightarrow \sin y \cdot \frac{1}{x} + \ln x \cdot \frac{d \sin y}{dy} \cdot \frac{dy}{dx} = \cos x \frac{d \ln y}{dy} \cdot \frac{dy}{dx} + \ln y \sin x$$

$$\Rightarrow \frac{\sin y}{x} + \ln x \cdot \cos y \frac{dy}{dx} = \cos x \cdot \frac{1}{y} \frac{dy}{dx} + \ln y \sin x$$

$$\Rightarrow \ln x \cos y \frac{dy}{dx} - \frac{\cos x}{y} \frac{dy}{dx} = -\frac{\sin y}{x} - \ln y \sin x$$

$$\Rightarrow \frac{dy}{dx} \left(\ln x \cos y - \frac{\cos x}{y} \right) = -\frac{\sin y}{x} - \ln y \sin x$$

$$\Rightarrow \frac{dy}{dx} \left(\frac{y \ln x \cos y - \cos x}{y} \right) = \left(\frac{-\sin y - x \ln y \sin x}{x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{-\sin y - x \ln y \sin x}{y \ln x \cos y - \cos x} \right)$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x} \left(\frac{x \ln y \sin x + \sin y}{y \ln x \cos y - \cos x} \right)$$

b) Integrate: $\int \frac{dx}{x(1-x)}$

Here, $I = \int \frac{dx}{x(1-x)}$

$$= \int \frac{dx}{(x - x^2)}$$

$$= - \int \frac{dx}{(x^2 - x)}$$

$$= - \int \frac{dx}{(x^2 - 2 \cdot \frac{1}{2} \cdot x + (\frac{1}{2})^2) - (\frac{1}{2})^2}$$

$$= - \int \frac{dx}{(x - \frac{1}{2})^2 - (\frac{1}{2})^2}$$

$$= \int \frac{dx}{(\frac{1}{2})^2 - (x - \frac{1}{2})^2}$$

$$= \frac{1}{2 \cdot \frac{1}{2}} \ln \left(\frac{\frac{1}{2} + x - \frac{1}{2}}{\frac{1}{2} - x + \frac{1}{2}} \right) + C$$

$$= \ln \left(\frac{x}{1-x} \right) + C$$

18) Using Simplex method, maximize $F = 18x + 8y$
 subject to $x + 2y \leq 25$, $3x + 2y \leq 28$, $x, y \geq 0$.
 Solⁿ

Introducing slack variable s, t , we get,

$$x + 2y + s = 25$$

$$3x + 2y + t = 28$$

$$-18x - 8y + F = 0$$

The canonical form of above equations:

$$x + 2y + s + 0t + 0F = 25$$

$$3x + 2y + 0s + t + 0F = 28$$

$$-18x - 8y + 0s + 0s + F = 0$$

The above canonical form gives the following initial tableau:

BV	x	y	s	t	F	RHS	Ratio
s	1	2	1	0	0	25	$25/1 = 25$
t	3	2	0	1	0	28	$28/3$
	-18	-8	0	0	1	0	

~~Perform~~ Here, -18 is the most negative entry in the bottom row and $28/3$ is the least positive ratio. So, 3 is the pivot element

performing $R_2 \rightarrow R_2/3$

BV	x	y	s	t	F	RHS
s	1	2	1	0	0	25
x	1	$2/3$	0	$1/3$	0	$28/3$
	-18	-8	0	0	1	0

Performing $R_1 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_3 + 18R_2$

BV	x	y	s	t	F	RHS
RS	0	$\frac{1}{2}$	$1 - \frac{1}{3}$	0		$\frac{47}{3}$
x	1	$\frac{5}{2}$	0	$\frac{1}{3}$	0	$\frac{28}{3}$
	0	19	0	6	1	168

Hence Here, all entries in the bottom row are non-negative. So, solution given by above tableau is optimum. Hence, maximum value of $F = 168$ at $(\frac{28}{3}, 0)$.

17)

Group C

- 20) A factory produces three articles A, B and C each of which is processed by three machines P, Q and R. The time (in hours) required for per unit of A, B and C each of which is processed by three machines P, Q and R. The in the machines P, Q and R are given below.

Articles	Machines		
	P	Q	R
A	3	4	2
B	4	2	3
C	2	3	4

If each of the machine is operated for 18 hours, find the number of articles A, B and C produced daily by performing the following tasks:

- Express the above information in form of equations and solve them by Cramer's rule.
- If the cost of production of each unit of the articles A, B and C are Rs. 200, Rs. 150 and Rs. 120 respectively, find the total cost by using the matrix multiplication.

Solution:

Let x , y and z be the number of articles of A, B and C respectively.

i) The required system of equations is:

$$3x + 4y + 2z = 18$$

$$4x + 2y + 3z = 18$$

$$2x + 3y + 4z = 18$$

Coefficient of x	Coefficient of y	Coefficient of z
3	4	2
4	2	3
2	3	4

Constants

18

18

18

$$D = \begin{vmatrix} 3 & 4 & 2 \\ 4 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} - 4 \begin{vmatrix} 4 & 3 \\ 2 & 4 \end{vmatrix} + 2 \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 3(8 - 9) - 4(16 - 6) + 2(12 - 4)$$

$$= -3 - 4 \times 10 + 2 \times 8$$

$$= -3 - 40 + 16$$

$$= -27$$

$$D_x = \begin{vmatrix} 18 & 4 & 2 \\ 18 & 2 & 3 \\ 18 & 3 & 4 \end{vmatrix}$$

$$= 18 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} - 4 \begin{vmatrix} 18 & 3 \\ 18 & 4 \end{vmatrix} + 2 \begin{vmatrix} 18 & 2 \\ 18 & 3 \end{vmatrix}$$

$$= 18(8-9) - 4(72-54) + 2(54-36)$$

$$= -18 - 72 + 2 \times 18$$

$$= -54$$

$$D_y = \begin{vmatrix} 3 & 18 & 2 \\ 4 & 18 & 3 \\ 2 & 18 & 4 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 18 & 3 \\ 18 & 4 \end{vmatrix} - 18 \begin{vmatrix} 4 & 3 \\ 2 & 4 \end{vmatrix} + 2 \begin{vmatrix} 4 & 18 \\ 2 & 18 \end{vmatrix}$$

$$= 3(18) - 18(16-6) + 2(36)$$

$$= -54$$

$$D_z = \begin{vmatrix} 3 & 4 & 18 \\ 4 & 2 & 18 \\ 2 & 3 & 18 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 2 & 18 \\ 3 & 18 \end{vmatrix} - 4 \begin{vmatrix} 4 & 18 \\ 2 & 18 \end{vmatrix} + 18 \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 3(36-54) - 4(72-36) + 18(12-4)$$

$$= -54$$

Using Cramer's rule,

$$x = \frac{D_x}{D} = \frac{-54}{-27} = 2, \quad y = \frac{D_y}{D} = \frac{-54}{-27} = 2$$

$$z = \frac{-54}{-27} = 2$$

$$\therefore x = y = z = 2$$

$$ii) \begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 200 \\ 160 \\ 120 \end{bmatrix}$$

$$= [2 \times 200 + 2 \times 160 + 2 \times 120]$$

$$= 960$$

Hence, the total cost is Rs 960

$$21. a) \text{ If } \left(\frac{x_1 + x_2 + x_3}{3} \right),$$