20th Shrawan, 2079

[Anti-Derivatives]

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Standard Integrals

Formulae:

1. $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

2. $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

3. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{a-x}\right| + c$

4. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln|x+\sqrt{x^2+a^2}| + c$

5. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln|x+\sqrt{x^2-a^2}| + c$

6. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + c$

[For proof use trigonometric substitution)

∴ [Basic ∫ formulae)

$\int \tan x\,dx = -\ln\cos x + c \;/\; \ln\sec x + c$

$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$

$\int \frac{1}{ax+b}\,dx = \frac{\ln|ax+b|}{a} + c$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad [n \neq -1)$

$\int a^x\,dx = \frac{a^x}{\ln a} + c.$

$\int \cot x\,dx = \ln\sin x \;/\; -\ln\operatorname{cosec} x + c.$

$\int \sec x\,dx = \ln|\sec x + \tan x| + c.$

$\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + c.$

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$$\int \sin^2 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$$

$$\int \cos^2 dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c.$$

$$\int \tan^2 x\,dx = \tan x - x + c$$

$$\int \cot^2 x\,dx = -\operatorname{cosec} x - x + c.$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$$

$$\int f'(x)\sqrt{f(x)}\,dx = \frac{2}{3}\left(\sqrt{f(x)}\right)^{3/2} + c$$

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Exercise : 16.1

1. Evaluate.

a) $\int \frac{dx}{4x^2+9}$

$= \frac{1}{4} \int \frac{dx}{x^2+\left(\frac{3}{2}\right)^2}$

$= \frac{1}{4\cdot\frac{3}{2}} \tan^{-1}\left(\frac{x}{3/2}\right) + c$

$= \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right) + c.$

b) $\int \frac{x\,dx}{x^4+3}$

Soln

let

$x^2 = t,$

$dt = 2x\,dx$

Now.

$\int \frac{x\,dx}{x^4+3}$

$= \frac{1}{2} \int \frac{dt}{t^2+3}$

$= \frac{1}{2} \int \frac{dt}{t^2+(\sqrt{3})^2}$

$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c$

$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x^2}{\sqrt{3}}\right) + c.$

c) $\int \frac{(2x+3)}{4x^2+1} dx$

Sol.n

Here

~~let $2x = t$~~

$\frac{1}{4}\int \frac{8x}{4x^2+1} dx + \frac{3}{4}\int \frac{1}{x^2+\frac{1}{4}} dx$

$= \frac{1}{4} \ln|4x^2+1| + \frac{3}{4 \cdot \frac{1}{2}} \tan^{-1}(2x) + c$

$= \frac{1}{4} \ln|4x^2+1| + \frac{3}{2} \tan^{-1} 2x + c.$

d) $\int \frac{x^2 dx}{x^6 - 9}$

Soln

let, $x^3 = t$,

$dt = 3x^2 dx.$

$\Rightarrow \frac{1}{3}\int \frac{3x^2 dx}{x^6 - 9}$

$= \frac{1}{3}\int \frac{dt}{t^2 - 3^2}$

$= \frac{1}{3 \cdot 2 \cdot 3} \ln\left|\frac{t-3}{t+3}\right| + c$

$= \frac{1}{18} \ln\left|\frac{x^3-3}{x^3+3}\right| + c.$

e) $\int \frac{dx}{x^2+6x+8} dx$

$= \int \frac{dx}{(x+3)^2 - (1)^2}$

$= \frac{1}{2 \cdot 1} \ln\left|\frac{x+3-1}{x+3+1}\right| + c$

$\frac{1}{2} \ln \left| \frac{x+2}{x+4} \right| + c$

f. $\int \frac{\cos x \, dx}{\sin^2 x + 4\sin x + 5}$

Sol^n

let $\sin x = t$,

$dt = \cos x \, dx$

$\Rightarrow \int \frac{dt}{t^2 + 4t + 5}$

$= \int \frac{dt}{(t+2)^2 + 1}$

$= \frac{1}{1} \tan^{-1}(t+2) + c$

$= \tan^{-1}(\cancel{t} \sin x + 2) + c$

g) $\int \frac{x \, dx}{x^4 - x^2 - 1}$

Sol^n

let $x^2 = t$,

$dt = 2x \, dx$

$\Rightarrow \frac{1}{2} \int \frac{dt}{t^2 - t - 1}$

$= \frac{1}{2} \int \frac{dt}{(t - \frac{1}{2})^2 - \left(\frac{\sqrt{3}}{\sqrt{2}}\right)^2}$

$= \frac{1}{2 \cdot 2 \cdot \frac{\sqrt{3}}{\sqrt{2}}} \cdot \tan^{-1} \left(\frac{t - \frac{1}{2} - \frac{\sqrt{3}}{\sqrt{2}}}{t - \frac{1}{2} + \frac{\sqrt{3}}{\sqrt{2}}} \right) + c$

$$= \frac{1}{2}\int \frac{dt}{\left(t-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{\cancel{2}\cdot\frac{\sqrt{5}}{\cancel{2}}}\cdot \ln\left|\frac{t-\frac{1}{2}-\frac{\sqrt{5}}{2}}{t-\frac{1}{2}+\frac{\sqrt{5}}{2}}\right| + c$$

$$= \frac{1}{2\sqrt{5}} \ln\left|\frac{2x^2-1-\sqrt{5}}{2x^2-1+\sqrt{5}}\right| + c.$$

h) $\int \frac{e^x\,dx}{e^{2x}+2e^x+5}$

Sol-n

let,

$$e^x = t$$

$$dt = e^x dx$$

$$\Rightarrow \int \frac{dt}{t^2+2t+5}$$

$$= \int \frac{dt}{(t+1)^2+(2)^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right)+c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{e^x+1}{2}\right)+c$$

i) $\int \frac{dx}{9x^2+12x+13}$

Sol-n

Here,

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let $3x = t$
$dt = 3dx$

$$\Rightarrow \frac{1}{3}\int \frac{dt}{t^2+4t+4+9}$$

$$= \frac{1}{3}\int \frac{dt}{(t+2)^2+(3)^2}$$

$$= \frac{1}{3\cdot 3}\tan^{-1}\left(\frac{3x+2}{3}\right)+c.$$

$$= \frac{1}{9}\tan^{-1}\left(\frac{3x+2}{3}\right)+c$$

j) $\int \frac{dx}{1-6x-9x^2}$

Sol^n

let $3x = t$
$dt = 3dx$

$$\Rightarrow \frac{1}{3}\int \frac{dt}{1-2t-t^2}$$

$$= \frac{1}{3}\int \frac{dt}{2-(t+1)^2}$$

$$= \frac{1}{3\cdot 2\sqrt{2}}\ln\left|\frac{\sqrt{2}+t+1}{\sqrt{2}-t-1}\right|+c$$

$$= \frac{1}{6\sqrt{2}}\ln\left|\frac{\sqrt{2}+3x+1}{\sqrt{2}-3x-1}\right|+c$$

k) $\int \frac{3x+5}{x^2+4x+20}\,dx$

$$= \frac{3}{2}\int \frac{2x+4-1}{x^2+4x+20}\,dx$$

$$= \frac{3}{2}\ln(x^2+4x+20) - \frac{3}{2}\int \frac{1}{(x+2)^2+(4)^2}\,dx$$

$= \frac{3}{2} \ln(x^2+4x+20) - \frac{3}{2 \cdot 8} \tan^{-1}\left(\frac{x+2}{4}\right) + c.$

i) $\int \frac{2x+2}{3+2x-x^2} dx$

Sol.n

$\int \frac{2x+2}{3+2x-x^2} dx$

$-\int \frac{2-2x-4}{3+2x-x^2} dx$

$= -\int \frac{2-2x}{3+2x-x^2} dx + 4 \int \frac{dx}{3+2x-x^2}$

$= -\ln(3+2x-x^2) + 4 \int \frac{dx}{(2)^2-(x-1)^2}$

$= -\ln(3+2x-x^2) + \frac{4}{2 \cdot 2} \ln\left|\frac{2+x-1}{2-x+1}\right| + c.$

$= \ln\left|\frac{x+1}{3-x}\right| - \ln(3+2x-x^2) + c.$

2 Evaluate.

a) $\int \frac{dx}{\sqrt{x^2-4}}$

Sol.n

$\int \frac{dx}{\sqrt{x^2-2^2}}$

$= \ln|x+\sqrt{x^2-4}| + c.$

b) $\int \frac{dx}{\sqrt{x^2+x-2}}$

Sol:n

$$\int \frac{dx}{\left(x+\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}$$

$$= \ln\left|x+\frac{1}{2}+\sqrt{x^2+x-2}\right|+c.$$

$$= \ln\left(\frac{2x+1}{2}+\sqrt{x^2+x+2}\right|+c$$

c) $\int \frac{dx}{\sqrt{2x^2+3x+4}}$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+2x\cdot\frac{3}{4}+\left(\frac{3}{4}\right)^2-\frac{9}{16}+2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\left(x+\frac{3}{4}\right)^2-\left(\frac{\sqrt{23}}{4}\right)^2}$$

$$= \frac{1}{\sqrt{2}}\ln\left(\frac{3x+4}{3}+\sqrt{x^2+\frac{3}{2}x+2}\right)+c.$$

d) $\int \frac{dx}{\sqrt{2ax+x^2}}$

$$= \int \frac{dx}{\sqrt{x^2+2ax+a^2-a^2}}$$

$$= \int \frac{dx}{\sqrt{(x+a)^2-a^2}}$$

$$= \ln\left|x+a+\sqrt{2ax+x^2}\right|+c.$$

e) $\int \frac{dx}{\sqrt{5-x+x^2}}$

$$= \int \frac{dx}{x^2-2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2-\frac{1}{4}+5}$$

$$= \int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{19}}{2}\right)^2}$$

$$= \ln\left(x-\frac{1}{2}+\sqrt{5-x+x^2}\right)+c.$$

f. $\int \frac{dx}{\sqrt{6+x-x^2}}$

$$= \int \frac{dx}{-\left(x^2-2.x.\frac{1}{2}+\frac{1}{4}\right)+\frac{1}{4}+6}$$

$$S, = \int \frac{dx}{\left(\frac{5}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

$$= \sin^{-1}\frac{2x-1}{5}+c$$

g) $\int \frac{dx}{\sqrt{2-2x-x^2}}$

$$= \int \frac{dx}{\sqrt{(\sqrt{3})^2-(x+2)^2}}$$

$$= \sin^{-1}\frac{x+1}{\sqrt{3}}+c.$$

h) $\int \frac{dx}{\sqrt{9x^2+6x+10}}$

$$= \frac{1}{3}\int \frac{dx}{x^2+2x.\frac{1}{3}+\frac{1}{9}-\frac{1}{9}+\frac{10}{9}}$$

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$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{891}}{3}\right)^2}$$

$$= \frac{1}{3}\ln\left(\frac{3x+1}{3} + \sqrt{x^2+\frac{2}{3}x+\frac{10}{9}}\right) + c$$

i) $\int \frac{x\,dx}{\sqrt{a^4+x^4}}$

Soln

let $x^2 = t$

$dt = 2x\,dx$

$$\Rightarrow \frac{1}{2}\int \frac{2x\,dx}{\sqrt{a^4+x^4}}$$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{(a^2)^2+(t)^2}}$$

$$= \frac{1}{2}.\ln|t+\sqrt{a^4+t^2}| + c.$$

$$= \frac{1}{2}\ln(x^2+\sqrt{a^4+x^4}) + c.$$

j) $\int \frac{x\,dx}{\sqrt{x^4+2x^2+10}}$

Soln

let $x^2 = t$

$dt = 2x\,dx$

$$\Rightarrow \frac{1}{2}\int \frac{dt}{\sqrt{t^2+2t+1+9}}$$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{(t+1)^2+3^2}}$$

$$= \frac{1}{2}\ln\left(t+1+\sqrt{(t+1)^2+9}\right) + c$$

$$= \frac{1}{2}\ln\left(x^2+1+\sqrt{x^4+2x^2+10}\right) + c$$

~~2)~~

$$\int \frac{x-2}{\sqrt{2x^2-8x+5}}\,dx.$$

~~Sol^n~~

~~let $2x^2-8x=t$~~

~~$dt=$~~

$$\int \frac{x-2}{\sqrt{2x^2-8x+5}}\,dx$$

$$= \int \frac{x}{\sqrt{2x^2-8x+5}}\,dx - 2\int \frac{1}{\sqrt{2x^2-8x+5}}\,dx$$

l)

$$\int \frac{x-2}{\sqrt{2x^2-8x+5}}\,dx$$

$$= \frac{1}{4}\int \frac{4x-8}{\sqrt{2x^2-8x+5}}\,dx$$

$$= \frac{1}{4}\cdot 2\sqrt{2x^2-8x+5} + c.$$

$$= \frac{1}{2}\sqrt{2x^2-8x+5} + c.$$

m.

$$\int \frac{x}{\sqrt{7+6x-x^2}}\,dx$$

$$= \frac{-1}{2}\int \frac{-2x+6-6}{\sqrt{7+6x-x^2}}$$

$$= -\frac{1}{2}\int \frac{-2x+6}{\sqrt{7+6x-x^2}} + 3\int \frac{dx}{\sqrt{16-x^2+6x-9}}$$

$$= -\frac{1}{2}\cdot 2\sqrt{7+6x-x^2} + 3.\sin^{-1}\frac{(x-3)}{4} + c.$$

$$= 3\sin^{-1}\left(\frac{x-3}{4}\right) - \sqrt{7+6x-x^2} + c.$$

n. $\int \frac{dx}{\sqrt{(x+a)(x+b)}}$

Sol^n

Put $x+a = y^2$

$\therefore dx = 2y\,dy.$

$\Rightarrow \int \frac{2y\,dy}{\sqrt{y^2(y^2-a+b)}}$

$= 2\int \frac{\cancel{y}\,dy}{\cancel{y}\sqrt{y^2+(\sqrt{b-a})^2}}$

$= 2.\ln\left(y + \sqrt{y^2+(\sqrt{b-a})^2}\right) + c.$

$= 2\ln\left(\sqrt{x+a} + \sqrt{x+a+b-a}\right) + c$

$= 2\ln\left(\sqrt{x+a} + \sqrt{x+b}\right) + c.$

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o) $\int \frac{dx}{(11+x)\sqrt{2+x}}$

Sol^n

Here,

let $2+x = y^2$

$\cancel{dy =}$ $dx = 2y\,dy.$

$\Rightarrow \int \frac{2y\,dy}{(9+y^2).y^2}$

$= 2\int \frac{dy}{y^2+3^2}$

$\cancel{= 2\ln(y +}$

$= \frac{2}{3}\tan^{-1}\left(\frac{y}{3}\right) + c$

$= \frac{2}{3}\tan^{-1}\left(\frac{\sqrt{2+x}}{3}\right) + c.$

9) $\int \sqrt{\frac{1+x}{1-x}}\,dx$

$= \int \sqrt{\frac{(1+x)^2}{1-x^2}}\,dx$

$= \int \frac{1+x}{\sqrt{1-x^2}}\,dx$

$= \int \frac{1}{\sqrt{1-x^2}}\,dx - \frac{1}{2}\int \frac{-x \cdot 2}{\sqrt{1-x^2}}\,dx$

$= \sin^{-1}x - \frac{1}{2} \cdot 2\sqrt{1-x^2} + c.$

$= \sin^{-1}x - \sqrt{1-x^2} + c$

∴ <u>Formulae</u>

1) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln|x+\sqrt{x^2+a^2}| + c$

2) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + c.$

3) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

4) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + c.$

5) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\overset{\cos}{\cancel{\sin}}\,bx] + c$