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∴ Integration of Rational Function by Partial Fractions

Rational Fraction	Partial Fraction
i) $\frac{mx+n}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
ii) $\frac{mx+n}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
iii) $\frac{mx^2+nx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
iv) $\frac{mx^2+nx+r}{(x-a)^2+(x-b)}$	$\frac{A}{(x-b)} + \frac{B}{(x-a)} + \frac{C}{(x-a)^2}$
v) $\frac{mx^2+nx+r}{(x-1)(ax^2+bx+c)}$	$\frac{A}{(x-1)} + \frac{Bx+C}{ax^2+bx+c}$
vi) $\frac{x^3}{(x-a)(x-b)(x-c)}$	$1 + \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$

:: Exercise : 16.4

Evaluate

1. $\int \frac{2x}{(2x+3)(3x+5)} dx$

$\frac{2x}{(2x+3)(3x+5)} = \frac{A}{(2x+3)} + \frac{B}{(3x+5)}$ [let]

where A & B are the values to be determined

i.e. $2x = A(3x+5) + B(2x+3)$. —①

Put $x = -\frac{5}{3}$ in eq①, then,

$-\frac{10}{3} = 0 + B\left(\frac{-10}{3} + 3\right)$

or, $-\frac{10}{3} = B \cdot -\frac{1}{3}$

$\therefore B = 10.$

Again,

Put $x = -\frac{3}{2}$ in ①,

$2.\left(-\frac{3}{2}\right) = A\left(3.\left(-\frac{3}{2}\right) + 5\right) + B.0$

or, $-3 = -\frac{9+10}{2} A$

$\therefore A = -6.$

So,

$\int \frac{2x}{(2x+3)(3x+5)} dx = \int \left(\frac{-6}{(2x+3)} + \frac{10}{(3x+5)}\right) dx$

$\int \frac{2x}{(2x+3)(3x+5)} dx = \frac{-6}{2} \ln|2x+3| + \frac{10}{3} \ln|3x+5| + C.$

2) $\int \frac{3x}{(x-a)(x-b)} dx$

Sol^n

let $\frac{3x}{(x-a)(x-b)} \cancel{dx} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$ let,

where A & B are values to be determined.

i.e $3x = A(x-b) + B(x-a)$. —①

Put $x = b$ in eq ①,

Then, $3b = B(b-a)$

$\therefore B = \frac{3b}{b-a}$.

Put $x = a$ in eq ①,

$3a = A(a-b)$.

$\therefore A = \frac{3a}{(a-b)}$.

$$\therefore \int \frac{3x}{(x-a)(x-b)} = \int \left(\frac{3a}{(a-b)(x-a)} - \frac{3b}{(a-b)(x-b)} \right) dx$$

$$= \frac{3a}{(a-b)} \int \frac{1}{(x-a)} dx - \frac{3b}{(a-b)} \int \frac{1}{(x-b)} dx$$

$$= \frac{3a \ln|x-a| - 3b \ln|x-b|}{(a-b)} + c \quad \text{Ans}_{\text{"}}$$

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3. $\int \frac{1}{(x+2)(x+3)^2} dx$

let $\frac{1}{(x+2)(x+3)^2} = \frac{A}{(x+2)} + \left(\frac{B}{x+3}\right) + \frac{C}{(x+3)^2}$

+C,

$\therefore 1 = A(x+3)^2 + B(x+2)(x+3) + C(x+1)$ —①

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Put $x=3$. in ①, then
$$1 = 0+0+C(-3+1)$$
$$\therefore C=-1$$

Put $x=2$,
$$\therefore 1 = A(-2+3)^2+0+0$$
$$\therefore A=1.$$

Put $x=0$, then
$$1 = A(3)^2+6B+2C$$
or $1 = 9+6B+2(-1)$

or $1-7=6B$
$$\therefore B=-1$$

$$\therefore \int\frac{1}{(x+2)(x+3)^2}dx = \int\frac{1}{(x+2)}dx+\int\frac{-1}{(x+3)}+\int\frac{-1}{(x+3)^2}dx$$

$$= \ln|x+2| - \ln|x+3| + \frac{1}{(x+3)} + C$$

$$= \ln\left|\frac{x+2}{x+3}\right| + \frac{1}{(x+3)} + C$$

4. $\int\frac{x^2dx}{(x-a)(x-b)(x-c)}$

Soln

let $\frac{x^2}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$

$\therefore x^2 = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)$ - ①

Put $x=b$, then from ①,
$$b^2 = A\times 0 + B(b-a)(b-c)+0.$$
$$\therefore B = \frac{b^2}{(b-a)(b-c)}$$

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Put $x=a$, then,

$$a^2 = A(a-b)(a-c)$$

$$\therefore A = \frac{a^2}{(a-b)(a-c)}$$

Put $x=c$ in (1), then,

$$c^2 = C((c-a)(c-b))$$

$$\therefore C = \frac{c^2}{(c-a)(c-b)}.$$

$$\therefore \int \frac{x^2\,dx}{(x-a)(x-b)(x-c)}$$

$$= \frac{a^2}{(a-b)(a-c)} \ln|x-a| + \frac{b^2}{(b-a)(b-c)} \ln|x-b|$$

$$+ \frac{c^2}{(c-a)(c-b)} \ln|x-c| + c.$$

5. $\int \frac{x^2+1}{x-1}\,dx$

$$= \int (x+1)\,dx + 2\int \frac{1}{(x-1)}\,dx$$

$$= \frac{(x+1)^2}{2} + 2\ln|x-1| + c_{//}$$

6. $\int \frac{dx}{1+x+x^2+x^3}$

$$= \int \frac{dx}{(1+x)+(1+x^2)} = \int \left(\frac{A}{(1+x)} + \frac{Bx+C}{(1+x^2)}\right)dx$$

(let) where A & B are constants to be found

$\therefore 1 = A(1+x^2) + Bx(1+x) + C(1+x) \quad -(1)$

Put $x = -1$,

$1 = A(1+1)$

$\therefore A = \frac{1}{2}$.

Put $x = 0$,

$1 = A + C$

or, $1 = \frac{1}{2} + C$

$\therefore C = \frac{1}{2}$.

Put $x = 1$.

$1 = \frac{1}{\cancel{2}} \cdot \cancel{2} + B.2 + \frac{1}{\cancel{2}} . \cancel{2}$

$\therefore B = -\frac{1}{2}$.

Now,

$\frac{1}{2}\left(\int \frac{1}{(1+x)} dx + \int \frac{(1-x)}{(1+x^2)} dx \right)$

$= \frac{1}{2}\left(\ln|1+x| \cancel{0 - \frac{1}{2 \cdot 2}\int} + \int \frac{1}{1+x^2} dx - \int \frac{x}{(1+x^2)} dx \right)$

$= \frac{1}{2}\left(\ln|1+x| + \tan^{-1}x - \frac{1}{2}\ln(1+x^2) \right) + C$.

8. $\int \frac{x^2-1}{x^4+x^2+1} dx$

Soln

Here,

$I = \int \frac{x^2-1}{x^4+x^2+1} dx$

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$$= \int \frac{x^2(1-1/x^2)}{x^2(x^2+1+1/x^2)}\,dx$$

$$= \int \frac{1-1/x^2}{x^2+2.x\cdot\frac{1}{x}+\left(\frac{1}{x}\right)^2-1}\,dx$$

$$= \int \frac{(1-1/x^2)}{\left(x+\frac{1}{x}\right)^2-1}\,dx$$

Put $x+\frac{1}{x}=t$

$\therefore dt = \left(1-\frac{1}{x^2}\right)dx$

$$\therefore I = \int \frac{dt}{t^2-1}$$

$$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+c$$

$$= \frac{1}{2}\left|\frac{x^2+1-x}{x^2+1+x}\right|+c.$$

9. $\int \frac{1}{x^4-1}\,dx$

$$= \int \frac{1}{(x-1)(x+1)(x^2+1)}$$

let,

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{(x-1)}+\frac{B}{(x+1)}+\frac{(Cx+D)}{x^2+1}$$

$\therefore 1 = A(x-1)(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)(x-1)$

Now,

when $x=1$,

$1 = 0 + B(1+1)(1^2+1) + 0$

$\therefore B = \frac{1}{4}$

when $x=-1$,

$1 = A(-1-1)((-1)^2+1) + 0 + 0$

$\therefore A = \frac{1}{4}$.

When $x=0$ and putting value of A & B.

$1 = -\frac{1}{4}(-1) + \frac{1}{4}\cdot 1 + D(-1)$

$D = \frac{1}{2} - 1$

$D \therefore -\frac{1}{2}$.

Putting $x=2$ and value of A, B & D.

or, $1 = -\frac{1}{4}(2-1)(2^2+1) + \frac{1}{4}(2+1)(2^2+1) + \left(C\cdot 2\left(-\frac{1}{2}\right)\right)(2+1)(2-1)$

or $1 = -\frac{1}{4}\times 5 + \frac{1}{4}\times 15 + C\cdot 2\left(-\frac{1}{2}\right)\cdot 3$

or, $1 - \frac{10}{4} = (4C-1)\frac{3}{2}$

$\therefore C = 0.$

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$$= \frac{-1}{4(x+1)} + \frac{1}{4(x+1)} + \frac{-1}{2(x^2+1)}$$

$$I = -\frac{1}{4}\int \frac{1}{x+1}\,dx + \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{1}{4}\ln\left(\frac{x-1}{x+1}\right) - \frac{1}{2}\tan^{-1}x + C.$$

10. $\displaystyle\int \frac{x^3}{(x^2+a^2)(x^2+b^2)}\,dx$

Sol.n

let $\displaystyle\int \frac{x^3}{(x^2+a^2)(x^2+b^2)}\,dx = I.$

$$\frac{x^3}{(x^2+a^2)(x^2+b^2)} = \frac{A}{x^2+a^2} + \frac{B}{x^2+b^2} \quad -(1)$$

$\therefore x^3 = A(x^2+b^2) + B(x^2+a^2)$

When $x=0$,

$0 = Ab^2 + Ba^2$

$\therefore A = -\dfrac{Ba^2}{b^2}$

Again, when $x=1$,

$1 = A(1+b^2) + B(1+a^2)$

$1 = -B\dfrac{a^2}{b^2}(1+b^2) + B(1+a^2)$

or, $b^2 = B(-a^2 - a^2b^2 + b^2 + a^2b^2)$

$\therefore B = \dfrac{b^2}{b^2-a^2} \qquad \therefore A = \dfrac{a^2}{a^2-b^2}$

Now,

$$I = \int \frac{a^2}{(a^2-b^2)(x^2+a^2)}\,dx + \int \frac{b^2}{(b^2-a^2)(x^2+b^2)}\,dx$$

$$= \frac{a^2}{(a^2-b^2)} \int \frac{1}{x^2+a^2}\,dx + \frac{b^2}{b^2-a^2}\int \frac{1}{x^2+b^2}\,dx$$

$$= \frac{a^2 \tan^{-1}\frac{x}{a}}{a^2-b^2} + \frac{b}{b^2-a^2}\tan^{-1}\frac{x}{b} + c.$$

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11. $\int \frac{x^2+4}{x^4+16}\,dx$

$$= \int \frac{1+4/x^2}{x^2+\frac{16}{x^2}}\,dx$$

$$= \int \frac{(1+4/x^2)}{\left(x-\frac{4}{x}\right)^2+2\cdot x\cdot\frac{4}{x}}\,dx$$

$$= \int \frac{(1+4/x^2)}{(x-4/x)^2+8}\,dx$$

let $x - \frac{4}{x} = y$

$$\frac{dy}{dx} = \left(1+\frac{4}{x^2}\right)$$

$$\therefore dy = \left(1+\frac{4}{x^2}\right)dx$$

$$I = \int \frac{1\,dy}{y^2+(\sqrt{8})^2}$$

$= \frac{1}{2\sqrt{2}} \tan^{-1} \frac{y}{\sqrt{8}} + c.$

$= \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{x - 4/x}{2\sqrt{2}} \right) + c$

$= \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 4}{2x\sqrt{2}} \right) + c.$

12. $\int \frac{x^3}{2x^4 - x^2 - 10} \, dx$

$I = \int \frac{x^2 \cdot x}{2(x^2)^2 - x^2 - 10} \, dx$

Put $x^2 = t$

$\therefore x\,dx = dt/2$

$\therefore I = \int \frac{t \cdot 1/2 \, dt}{2t^2 - t - 10}$

$= \frac{1}{2} \int \frac{t\,dt}{(t+2)(2t-5)}$

let $\frac{t}{(t-2)(2t+5)} = \frac{A}{(t+2)} + \frac{B}{(2t-5)}$

$\therefore t = A(2t-5) + B(t+2)$

let $t = -2$,

$-2 = A(-2 \cdot 2 - 5) + 0$

$\therefore A = \frac{2}{9}.$

let $t = 0$,

$0 = A(-5) + 2B$

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$\therefore B = \frac{5}{9}.$

Now

$$I = \frac{1}{2}\int \frac{2/9\, dt}{t+2} + \frac{1}{2}\int \frac{5/9}{2t+5} dt$$

$$= \frac{2}{18} \ln(x^2+2) + \frac{5}{36} \ln(2x^2-5) + c_{11}$$

14. $\int \frac{dx}{(x-1)^2(x-3)^2}$

Soln

Put $(x-1) = t(x-3)$

or, $(x-1) = tx - 3t.$

$\therefore x = \frac{1-3t}{1-t}$ — ①

Note (Integral of type) $\frac{1}{(x-a)^m(x-b)^m} : m \neq n$

Put $x-a = t(x-b)$

Diff. ① w.r.t. t,

$$\frac{dx}{dt} = \frac{1}{(1-t)^2}\left[(1-t)\frac{d(1-3t)}{dt} - (1-3t)\frac{d}{dt}(-t+1)\right]$$

$$= \frac{1}{(1-t)^2}(-3+3t+1-3t)$$

$$\therefore dx = \frac{-2dt}{(1-t)^2}$$

Also,

$$(x-3) = \frac{1-3t}{1-t} - 3$$

$$= \frac{1-3t-3+3t}{1-t} = \frac{-2}{1-t} \text{ — ②}$$

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So,

$$I = -2\int \frac{dt}{t^2(1-t)^2 \cdot \left(\frac{-2}{1-t}\right)^4}$$

$$= -\frac{2}{16}\int \frac{dt}{t^2(1-t)^2(1-t)^{-4}}$$

$$= -\frac{1}{8}\int \frac{dt}{t^2(t-2)^{-2}}$$

$$= \cancel{-\frac{1}{8}\frac{(1-t)^2}{t^2}\,dt}$$

$$= -\frac{1}{8}\int \frac{(1-t)^2\,dt}{t^2}$$

$$= -\frac{1}{8}\left[\int \frac{1}{t^2}\,dt - \int \frac{2}{t}\,dt + \int 1\,dt\right]$$

$$= -\frac{1}{8}\left[-\frac{1}{t} - 2\ln|t| + t\right] + C$$

$$= \frac{1}{8}\left(\frac{x-3}{x-1}\right) + 2\ln\left(\frac{x-1}{x-3}\right) - \frac{x-1}{x-3} + C_4$$

15. $\int \frac{dx}{(x-1)^2(x-4)^3}$

Soln

let, $(x-1) = t(x-4)$

$x-1 = tx - 4t$

$\therefore x = \frac{1-4t}{1-t}$ — ①.

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Now,

$$\frac{dx}{dt} = \frac{1}{(1-t^2)}\left[(1-t)\frac{d}{dt}(1-4t) - (1-4t)\frac{d}{dt}(1-t)\right]$$

$$= \frac{1}{(1-t)^2}\left[(1-t)\times(-4) - (1-4t)\times 1\times(-1)\right]$$

$$= \frac{1}{(1-t)^2}(-4+4t+1-4t)$$

$$\therefore dx = \frac{-3dt}{(1-t)^2}$$

Now,

$$x-4 = \frac{1-4t}{1-t} - 4 = \frac{-3}{1-t} \quad \text{—(1)}$$

$$\text{So, } I = \int \frac{1}{t^2 (x-4)^2 (x-4)^3} \times \left(\frac{-3}{(1-t)^2}\right) dt$$

$$= -3\int \frac{dt}{(1-t)^2 \cdot t^2 \left(\frac{-3}{1-t}\right)^5}$$

$$= -3\int \frac{dt\,(1-t)^3}{t^2\cdot(-3)^5}$$

$$= \frac{1}{81}\int \frac{(1-t)^3}{t^2}\,dt$$

$$= \frac{1}{81}\int\left(\frac{1 - 3t + 3t^2 - t^3}{t^2}\right) dt$$

$$= \frac{1}{81}\left[\frac{-1}{t} + 3t - 3\ln|t| - \frac{t^2}{2}\right] + c$$

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$$= \frac{1}{81}\left\{ \frac{x-4}{x-1} - 3\ln\left(\frac{x-1}{x-4}\right) + 3\left(\frac{x-1}{x-4}\right) - 2\left(\frac{x-1}{x-4}\right) \right\} + c.$$

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