:: Integration of Some Special Trigonometric Functions.

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[Formulae]

i) $\int \operatorname{cosec} x \, dx = \ln \tan \frac{x}{2} + c$

ii) $\int \sec x \, dx = \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c.$

iii) $\int \tan x \, dx = \ln(\sec x) + c$

iv) $\int \cot x \, dx = \ln(\sin x) + c.$

v) $\int \frac{dx}{a + b\cos x}$:

If $a < b$: $\frac{1}{\sqrt{b^2 - a^2}} \ln \left| \frac{\sqrt{a+b} + \sqrt{b-a} \tan x/2}{\sqrt{a+b} - \sqrt{b-a} \tan x/2} \right| + c$

If $a > b$: $\frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\frac{\sqrt{a-b}}{\sqrt{a+b}} \tan \frac{x}{2}\right] + c.$

:: Working Note

i) Convert D^r into homogenous of 2^{nd} degree of sin & cos

ii) Multiply N^r & D^r by $\sec^2 \frac{x}{2}$.

[Integral of hyperbolic function]

i) $\int \sinh x \, dx = \cosh x + c.$

ii) $\int \cosh x \, dx = \sinh x + c$

iii) $\int \operatorname{sech}^2 x \, dx = \tanh x + c$

iv) $\int \operatorname{cosech}^2 x \, dx = -\coth x + c$

v) $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c.$

vi) $\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + c.$

7) $\int \coth x \, dx = \ln \sinh x + c$

8) $\int \tanh x \, dx = \ln \cosh x + c$

9) $\int \operatorname{cosech} x \, dx = \ln\left(\tanh \frac{x}{2}\right) + c.$

10) $\int \operatorname{sech} x \, dx = 2\tan^{-1}\left(\tanh \frac{x}{2}\right) + c.$

∴ Exercise: 16.3

Evaluate.

1. $\int \frac{dx}{1+2\sin^2 x}$

$= \int \frac{dx}{3\sin^2 x + \cos^2 x}$

$= \int \frac{\sec^2 x \, dx}{3\tan^2 x + 1}$

Let $\tan x = t$

$dt = \sec^2 x \, dx$

$= \int \frac{dt}{3t^2+1}$

$= \frac{1}{3} \int \frac{dt}{t^2 + \left(\frac{1}{\sqrt{3}}\right)^2}$

$= \frac{1}{3 \cdot \frac{1}{\sqrt{3}}} \cdot \tan^{-1}\left(t \cdot \sqrt{3}\right) + c.$

$= \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}\tan x) + c.$

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2. $\int \frac{dx}{5+4\cos x}$

$= \int \frac{dx}{5\sin^2\frac{x}{2} + 5\cos^2\frac{x}{2} + 4\cos^2\frac{x}{2} - 4\sin^2\frac{x}{2}}$

$$= \int \frac{dx}{9\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}.$$

$$= \int \frac{\sec^2 x/2 \, dx}{9 + \tan^2\frac{x}{2}}$$

& Let,

$$\tan\frac{x}{2} = t$$

$$2dt = \sec^2\frac{x}{2}\, dx.$$

$$\Rightarrow \quad 2\int \frac{dt}{9+t^2}$$

$$= 2\int \frac{dt}{3^2+t^2}$$

$$= 2\cdot\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + c$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan x/2}{3}\right) + c$$

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3) $\int \frac{dx}{1-3\sin x}$

$$= \int \frac{dx}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} - 6\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int \frac{\sec^2 x/2 \, dx}{\tan^2\frac{x}{2} + 1 - 6\tan\frac{x}{2}}$$

let $\tan\frac{x}{2} = t \qquad 2dt = \sec^2\frac{x}{2}\, dx$

$$\int \frac{dx}{1-3\sin x}$$

$$= 2\int \frac{dt}{t^2-6t+1}$$

$$= 2\int \frac{dt}{t^2-6t+9-9+1}$$

$$= 2\int \frac{dt}{(t-3)^2-(\sqrt{8})^2}$$

$$= \cancel{2}\frac{1}{\cancel{2}\sqrt{8}} \ln\left(\frac{t-3-\sqrt{8}}{t-3+\sqrt{8}}\right)+c.$$

$$= \frac{1}{2\sqrt{2}} \ln\left(\frac{\tan x/2-3-2\sqrt{2}}{\tan x/2-3+2\sqrt{2}}\right)+c.$$

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4) $$\int \frac{dx}{a^2\sin^2 x-b^2\cos^2 x}$$

$$= \int \frac{dx.\sec^2 x}{a^2\tan^2 x-b^2}$$

let $\tan x = t$,

$dt = \sec^2 x\,dx$

$$\Rightarrow \int \frac{dt}{a^2t^2-b^2}$$

$$\frac{1}{a^2}\int \frac{dt}{t^2-\frac{b^2}{a^2}}$$

$$= \frac{1}{a^2}\frac{1}{2\cdot\frac{b}{a}} \cdot \ln\left(\frac{at-b}{at+b}\right)+c.$$

$$= \frac{1}{2ab} \ln \left(\frac{a\tan x - b}{a\tan x + b} \right) + c$$

5. $\int \frac{dx}{4\cos x - 1}$

$$= \int \frac{dx}{4\cos^2\frac{x}{2} - 4\sin^2\frac{x}{2} - \sin^2\frac{x}{2} - \cos^2\frac{x}{2}}$$

$$= \int \frac{dx}{3\cos^2\frac{x}{2} - 5\sin^2\frac{x}{2}}$$

$$= \int \frac{\sec^2 x/2\, dx}{3 - 5\tan^2 x/2}$$

$$= 1/5 \int \frac{\sec^2 x/2\, dx}{\frac{3}{5} - \tan^2 x/2}$$

let $\tan^2\frac{x}{2} = t$

$dt = 2\sec^2 x/2$

$$\Rightarrow 1/5 \int \frac{dt}{\left(\sqrt{\frac{3}{5}}\right)^2 - t^2}$$

$$= \frac{2}{2 \cdot \frac{\sqrt{3}}{\sqrt{5}} \cdot 5} \ln \left| \frac{\sqrt{3} + \sqrt{5}t}{\sqrt{3} - \sqrt{5}t} \right| + c$$

$$= \frac{1}{\sqrt{15}} \ln \left(\frac{\sqrt{3} + \sqrt{5}\tan x/2}{\sqrt{3} - \sqrt{5}\tan x/2} \right) + c$$

6. $\int \frac{dx}{2+3\cos x}$

$$= \int \frac{dx}{2\sin^2\frac{x}{2} + 2\cos^2\frac{x}{2} + 3\cos^2\frac{x}{2} - 3\sin^2\frac{x}{2}}$$

$$= \int \frac{dx}{5\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \int \frac{\sec^2 x/2 \, dx}{5 - \tan^2\frac{x}{2}}$$

let $t = \tan x/2$

$2dt = \sec^2\frac{x}{2} dx$

$$\Rightarrow 2\int \frac{dt}{(\sqrt{5})^2 - t^2}$$

$$= 2\int \frac{dt}{(\sqrt{5})^2 - (t)^2}$$

$$= \frac{\cancel{2}\cdot 1}{\cancel{2}\cdot\sqrt{5}} \ln\left|\frac{\sqrt{5}+t}{\sqrt{5}-t}\right| + c$$

$$= \frac{1}{\sqrt{5}} \ln\left|\frac{\sqrt{5}+\tan x/2}{\sqrt{5}-\tan x/2}\right| + c.$$

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7. $\int \frac{\sin x\cos x}{(\sin x+\cos x)^2} dx.$

$$= \frac{1}{2}\int \frac{2\sin x\cos x + \sin^2 x + \cos^2 x}{(\sin x+\cos x)^2} - \frac{1}{2}\int \frac{1}{(\sin x+\cos x)^2} dx$$

$$= \frac{x}{2} - \frac{1}{2}\int \frac{1}{\sin^2 x + \cos^2 x + 2\sin x \cos x}\, dx$$

$$= \frac{x}{2} - \frac{1}{2}\int \frac{\sec^2 x\, dx}{\tan^2 x + 1 + 2\tan x}$$

let $dt = \tan x$

$dt = \sec^2 x\, dx$

$$= \frac{x}{2} - \frac{1}{2}\int \frac{dt}{t^2 + 2t + 1}$$

$$= \frac{x}{2} - \frac{1}{2}\int \frac{dt}{(t+1)^2}$$

$$= \frac{x}{2} + \frac{(t+1)^{-1}}{-1\cdot 2}\, \cancel{dx}$$

$$= \frac{x}{2} + \frac{1}{2(t+1)} + c$$

$$= \frac{x}{2} + \frac{1}{2(\tan x + 1)} + c$$

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8. $\int \frac{dx}{\sin x + \cos x}$

$$= \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= \int \frac{\sec^2 x/2\, dx}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}$$

let $\tan x/2 = t$

$2dt = \sec^2 x/2\, dx$

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$$= 2\int \frac{dt}{2t+1-t^2}$$

$$= 2\int \frac{dt}{2-(t-1)^2}$$

$$= \cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{2}}\cdot \ln\left|\frac{2+t-1}{2-t+1}\right| + c$$

$$= \frac{1}{\sqrt{2}} \ln\left|\frac{1+\tan x/2}{3-\tan x/2}\right| + c.$$

9. $$\int \frac{dx}{1+\sin x+\cos x}$$

$$= \int \frac{dx}{\cancel{\sin^2\frac{x}{2}} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2} - \cancel{\sin^2\frac{x}{2}}}$$

$$= \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2} + \sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \frac{1}{2}\int \frac{\sec^2 x\, dx}{1+\tan x/2}.$$

$\frac{1}{2}$ let $\tan\frac{x}{2} = t$

$2dt = \sec^2 x/2$

$$= \int \frac{dt}{1+t}$$

$$= \ln\left(\tan\frac{x}{2}+1\right) + c$$

10. $\int \frac{dx}{3+2\sin x+\cos x}$

$= \int \frac{dx}{3\sin^2\frac{x}{2}+3\cos^2\frac{x}{2}+4\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}$

$= \int \frac{dx}{2\sin^2\frac{x}{2}+4\cos^2\frac{x}{2}+4\sin\frac{x}{2}\cos\frac{x}{2}}$

$= \int \frac{\sec^2 x\,dx}{2\tan^2\frac{x}{2}+4+4\tan\frac{x}{2}}$

let $\tan\frac{x}{2}=t$

$2dt=\sec^2\frac{x}{2}\,dx$

$\Rightarrow \frac{\cancel{2}}{2}\int \frac{dt}{\cancel{\tan^2} t^2+2+2t}$

$= \int \frac{dt}{(t+1)^2+(1)^2}$

$= \frac{1}{1}\tan^{-1}\left(\frac{t+1}{1}\right)+C$

$= \tan^{-1}\left(\tan\frac{x}{2}+1\right)+C$ //

12. $\int \frac{dx}{2+\cos x-\sin x}$

$= \int \frac{dx}{2\sin^2\frac{x}{2}+2\cos^2\frac{x}{2}-\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}$

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Multiplying D^r & N^r by $\sec^2\frac{x}{2}$,

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2} + 3 - 2\tan x/2}$$

let $\tan x/2 = t$

$\cdot 2dt = \sec^2\frac{x}{2}\,dx$

$$\Rightarrow 2\int \frac{dt}{t^2+3-2t}$$

$$= 2\int \frac{dt}{(t-1)^2+(\sqrt{2})^2}$$

$$= 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-1}{\sqrt{2}}\right)+c$$

$$= \sqrt{2}\tan^{-1}\left(\frac{\tan x/2 - 1}{\sqrt{2}}\right)+c_{11}$$

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13. $$\int \frac{dx}{\cos x - \sqrt{3}\sin x}$$

$$= \int \frac{dx}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2} - 2\sqrt{3}\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int \frac{\sec^2 x/2\,dx}{1-\tan^2\frac{x}{2} - 2\sqrt{3}\tan x/2}$$

let $\tan x/2 = t$

$\therefore 2dt = \sec^2\frac{x}{2}\,dx$

$$= 2\int \frac{dt}{1-t^2-2\sqrt{3}\,t}$$

$$= 2\int \frac{dt}{4-(t+\sqrt{3})^2}$$

$$= 2.\frac{1}{2.2} \ln \left|\frac{2+t+\sqrt{3}}{2-(t+\sqrt{3})}\right| + c.$$

$$= \frac{1}{2} \ln \left| \frac{\tan x_{/2}+\sqrt{3}+2}{2-\tan x_{/2}-\sqrt{3}} \right| + c$$

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15. $\int \frac{dx}{2+\sin x}$

$$= \int \frac{dx}{2\sin^2\frac{x}{2}+2\cos^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int \frac{\sec^2 x_{/2}\,dx}{2\tan^2\frac{x}{2}+2-2\tan x_{/2}}$$

let $\tan\frac{x}{2} = t$

$\therefore 2dt = \sec^2 x_{/2}\,dx$

$$= \frac{1}{2}\int \frac{2dt}{t^2+1-t}$$

$$= \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{\sqrt{4}}\right)^2}$$

$$= \frac{1}{\sqrt{3}}\,2 \tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c.$$

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$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right) + c$$ //

16. $\int \frac{dx}{4+3\sinh x}$

$$= \int \frac{dx}{4\cosh^2\frac{x}{2} - 4\sinh^2\frac{x}{2} + 6\sinh\frac{x}{2}\cdot\cosh\frac{x}{2}}$$

$$= \int \frac{\operatorname{sech}^2 x/2 \, dx}{4 - 4\tanh^2\frac{x}{2} + 6\tanh\frac{x}{2}}$$

let $\tanh\frac{x}{2} = t$

$2\,dt = \operatorname{sech}^2\frac{x}{2}\,dx$

$$= \frac{2}{2}\int \frac{dt}{2-2t^2+3t}$$

$$= \frac{1}{2}\int \frac{dt}{1-t^2+\frac{3}{2}t}$$

$$= \frac{1}{2}\int \frac{dt}{\left(\frac{5}{4}\right)^2 - \left(t-\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot\frac{5}{4}} \ln\left|\frac{\frac{5}{4}+t-\frac{3}{4}}{\frac{5}{4}-t+\frac{3}{4}}\right| + c$$

$$= \frac{1}{5} \ln \left| \frac{1 + 2\tanh x/2}{4 - 2\tanh x/2} \right| + c.$$

17. $\int \frac{dx}{4 + 3\cosh x}$

$$= \int \frac{dx}{4\cosh^2\frac{x}{2} - 4\sinh^2\frac{x}{2} + 3\cosh^2\frac{x}{2} + 3\sinh^2\frac{x}{2}}$$

$$= \int \frac{\operatorname{sech}^2 x/2 \, dx}{7 - \tanh^2\frac{x}{2}}.$$

So, let $\tanh\frac{x}{2} = t$

$$2dt = \operatorname{sech}^2\frac{x}{2}\, dx$$

$$= 2 \int \frac{dt}{(\sqrt{7})^2 - t^2}$$

$$= 2 \cdot \frac{1}{2\sqrt{7}} \ln \left| \frac{\sqrt{7} + t}{\sqrt{7} - t} \right| + c$$

$$= \frac{1}{\sqrt{7}} \ln \left| \frac{\sqrt{7} + \tanh x/2}{\sqrt{7} - \tanh x/2} \right| + c$$

18 $\int \frac{\tanh x}{36\operatorname{sech} x + \cosh x} dx$

$$= \int \frac{\sinh x}{36 + \cosh^2 x} dx$$

let $\cosh x = t$

$$dt = \sinh x\, dx$$

$$= \int \frac{dt}{6^2+t^2}$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{t}{6}\right) + c$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{\cosh x}{6}\right) + c.$$

19. $\displaystyle\int \frac{\tanh x\,dx}{\cosh x + 64\operatorname{sech} x}$

$$= \int \frac{\sinh x\,dx}{\cosh^2 x + 64}$$

let $\cosh x = t$

$dt = \sinh x\,dx$

$$\Rightarrow \int \frac{dt}{t^2+8^2}$$

$$= \frac{1}{8} \tan^{-1}\left(\frac{t}{8}\right) + c$$

$$= \frac{1}{8} \tan^{-1}\left(\frac{\cosh x}{8}\right) + c$$

20. $\displaystyle\int \frac{\sinh x\,dx}{4\tanh x - \operatorname{cosech} x \operatorname{sech} x}$

$$= \int \frac{\sinh x\,dx}{\left(4\frac{\sinh x}{\cosh x} - \frac{1}{\sinh x \cosh x}\right)}$$

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$$= \int \frac{\sinh^2 x \cdot \cosh x \, dx}{4\sinh^2 x - 1}$$

Put $\sinh x = t$

$\cosh x \, dx = dt$

So,

$$I = \int \frac{t^2}{4t^2 - 1} dt$$

$$= \frac{1}{4} \int \frac{4t^2 \, dt}{4t^2 - 1}$$

$$= \frac{1}{4} \int \left(1 + \frac{1}{4t^2 - 1} \right) dt$$

$$= \frac{1}{4} \left\{ t + \frac{1}{4} \int \frac{dt}{t^2 - \left(\frac{1}{2}\right)^2} \right\}$$

$$= \frac{1}{4} t + \frac{1}{16} \ln \left(\frac{2t - 1}{2t + 1} \right) + c$$

$$= \frac{1}{4} \sinh x + \frac{1}{16} \ln \left| \frac{2\sinh x - 1}{2\sinh x + 1} \right| + C$$