[::Notes in note copy]

2079-04-15

::Rolle's Theorem & Mean Value Theorem

Exercise : 15.5

1. Verify Rolle's theorem for each of the following function.

a) $f(x) = x^2 + 2$ in $[-2, 2]$

Sol^n

Here,

$f(x) = x^2 + 2$ in $[-2, 2]$ —— ①

i) Here, $f(x)$ being polynomial function is continuous in $[-2, 2]$.

ii) Diff ① w.r.t x,

$f'(x) = 2x$ which is defined on $(-2, 2)$. So, $f(x)$ is differentiable on $(-2, 2)$.

iii) When $x = -2$,

$f(-2) = (-2)^2 + 2 = 6$

$f(2) = 2^2 + 2 = 6$

i.e. $f(-2) = f(2)$.

Since, $f(x)$ holds all three conditions of Rolle's theorem. We can find at least one point $c \in (-2, 2)$ such that $f'(c) = 0$.

or $f'(c) = 2c$

or $2c = 0$

$\therefore c = 0$

Here, $c = 0 \in [-2, 2]$

$\therefore$ Theorem is verified.

Date |
Page |

b) $f(x) = x^3 - 4x$ in $[0,2]$

Soln

Given,

$$f(x) = x^3 - 4x \text{ in } [0,2] \text{ --- (i)}$$

1) Here $f(x)$ being polynomial is continuous on $[0,2]$.

2) Diff (i) w.r.t x,

$f'(x) = 3x^2 - 4$ --- (ii) which is defined on $(0,2)$. So, $f(x)$ is differentiable on $(0,2)$.

3) When $x = 0$, $y = f(0) = 0$.

When $x = 2$, $f(x) = 2^3 - 4\cdot 2$

$= 0$.

Here

$f(0) = f(2)$

Since, $f(x)$ holds all three conditions of Rolle's theorem, we can find at least one point $c \in (0,2)$ such that,

$f'(c) = 0$.

or, $3c^2 - 4 = 0$ [from (ii)]

or, $c^2 = \frac{4}{3}$

$\therefore c = \pm\frac{2}{\sqrt{3}}$.

Here $c = \frac{2}{\sqrt{3}} \in [0,2]$

$c = \frac{-2}{\sqrt{3}} \notin [0,2]$.

Date |
Page |

Hence,
$c = \frac{2}{\sqrt{3}}$. So, theorem is verified.

---

c) $f(x) = \sin 2x$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Sol:n
Here,
$f(x) = \sin 2x$ —(i)

1) $f(x)$ being sine function is continuous in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

2) Diff $f(x)$ w.r.t x,
$f'(x) = 2\cos 2x$ —(ii) which is defined on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So, $f(x)$ is differentiable on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

3) When $x = -\frac{\pi}{2}$, $f\left(-\frac{\pi}{2}\right) = 0$.
When $x = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = 0$.
$\therefore f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$.

Since, $f(x)$ holds all three conditions of Rolle's theorem, there exists at least one point $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that,

$f'(c) = 0$
or $2\cos 2c = 0$
or $\cos 2c = \cos 90^\circ, \cos 270^\circ$

$\therefore 2C = \frac{\pi}{2}, -\frac{\pi}{2}$

$\therefore C = \frac{\pi}{4}, -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Date |

Page |

~~$\therefore C = -\frac{\pi}{2}, \frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.~~

Hence,
Theorem is verified.

e. $f(x) = \sqrt{25-x^2}$ in $[-5, 5]$

Sol^n

$$f(x) = \sqrt{25-x^2} \quad \text{—(1)}$$

1) Here, $f(x)$ is being polynomial continuous on $[-5, 5]$.

2) Diff (1) w.r.t x,

$$f'(x) = \frac{-2x}{2\sqrt{25-x^2}} \quad \text{—(2)}$$

which is defined on $(-5, 5)$.
So, $f(x)$ is differentiable on $(-5, 5)$.

3) When, $x = 5$, $f(5) = 0$.
When $x = -5$, $f(-5) = 0$.
Here,

$$f(-5) = f(5)$$

Since $f(x)$ holds all three conditions of Rolle's theorem, we can find at least one point $C \in [5, 5]$ such that,

$$f'(C) = 0$$

or, $\frac{-2C}{2\sqrt{25-C^2}} = 0$

or, $-2C = 0$

$\therefore C = 0 \in [-5, 5]$. verified

Date |
Page |

f) $f(x) = \sin x + \cos x$ in $[0, 2\pi]$

Soln

$f(x) = \sin x + \cos x$ — (1)

1) Here, $f(x)$ being trigonometric function is continuous on $[0, 2\pi]$

2) Diff (1) w.r.t. x,

$f'(x) = \cos x - \sin x$ — (2).

which is defined on $(0, 2\pi)$. So, $f(x)$ is differentiable on $(0, 2)$.

3) When $(x) = 0$ $f(0) = 1$

When $x = 2\pi$ = $f(2\pi) = 1$

$\therefore f(0) = f(2\pi)$

Since $f(x)$ holds all three conditions of Rolle's theorem, we can find atleast one point $c \in (0, 2\pi)$ such that,

$f'(c) = 0$

or, $\cos c - \sin c = 0$

or, $\cos c = \sin c$.

or, $\cos\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ $\sin\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ [$\cos c = \sin c$ at $c = \frac{\pi}{4}, \frac{5\pi}{4}$.

$\therefore c = \frac{\pi}{4}, \frac{5\pi}{4} \in [0, 2\pi]$.

$\therefore$ Theorem is verified.

2. By using Rolle's theorem, find a point on each of the curves given by the following where tangent is parallel to x-axis.

a) $f(x) = 6x - x^2$ in $[0, 6]$

Sol:n

$f(x) = 6x - x^2$ — ①

1) Here $f(x)$ being polynomial is continuous on $[0, 6]$.

2) Diff ① w.r.t x,

$f'(x) = 6 - 2x$ — ② which is defined on $(0, 6)$. So, $f(x)$ is differentiable on $(0, 6)$.

3) When $x = 0$, $f(0) = 0$

When $x = 6$, $f(6) = 0$

$\therefore f(0) = f(6)$.

Since $f(x)$ holds all three conditions of Rolle's theorem. We can find at least one point on $c \in (0, 6)$ such that,

$f'(c) = 0$.

or, $6 - 2c = 0$

or $6 = 2c$

$\therefore c = 3$.

When $c = 3$,

$f(c) = 6c - c^2$

$= 6 \times 3 - 3^2$

$= 9$

$\therefore$ Point where tangent is parallel to x-axis is $(3, 9)$.

b) $f(x) = 2x^2 - 4x$ in $[0,2]$

Sol.n

Here,

$f(x) = 2x^2 - 4x \quad \text{--- (1)}$

1) Here $f(x)$ being polynomial is continuous on $[0,2]$.

2) Diff (1) w.r.t. x,

$f'(x) = 4x - 4 \quad \text{--- (2)}$

which is defined on $(0,2)$. So, $f(x)$ is differentiable on $(0,2)$.

3) When $x=0$, $f(0)=0$

when $x=2$, $f(2)=0$

$\therefore f(0) = f(2)$.

Since $f(x)$ holds all three conditions of Rolle's theorem. We can find point on curve where tangent is parallel to x-axis.

$f'(c) = 0$

or, $4c - 4 = 0$

$\therefore c = 1$

When $c = 1$,

$f(c) = -4\cdot 1 + 1\cdot 2 = \cancel{8-4}$

$= -2$.

$\therefore (1,-2)$ is the point where tangent is parallel to x-axis.

[Note copy for Notes].
~~Hell todo~~

Date |
Page |

3. Verify by mean value theorem for each function in given interval.

a) $f(x) = 3x^2 - 2$ in $[2,3]$
Sol-n
Here,
$f(x) = 3x^2 - 2$.

1) $f(x)$ being polynomial function is continuous on $[2,3]$.

2) Diff. w.r.t x, we get
$f'(x) = 6x$. which is defined on $(2,3)$.
So, $f(x)$ is differentiable on $(2,3)$.

As $f(x)$ holds both conditions of second value theorem,
we can find at least one point $c \in (2,3)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Here,
$f'(c) = 6c$ ·①
When $a = 2$, $f(2) = 10$
When $b = 3$, $f(3) = 25$

i.e. $6c = \frac{25-10}{3-2}$
or, $6c = 15$

$\therefore c = \frac{5}{2} \in (2,3)$.

Hence theorem is verified.

b) $f(x) = x(x-1)(x-2)$ in $[0, \frac{1}{2}]$

Soln

Here,

$$f(x) = x(x-1)(x-2)$$
$$= x(x^2 - 3x + 2)$$
$$= x^3 - 3x^2 + 2x \quad -(1)$$

i) $f(x)$ being polynomial is continuous on $[0, \frac{1}{2}]$.

ii) Diff (1) w.r.t. x,

$f'(x) = 3x^2 - 6x + 2$ which is defined on $(0, \frac{1}{2})$

So, $f(x)$ is differentiable on $(0, \frac{1}{2})$.

As $f(x)$ holds both condition of mean value theorem.

There exists a point $c \in (0, \frac{1}{2})$ such that:

$$f'(c) = \frac{f(1/2) - f(0)}{\frac{1}{2} - 0}$$

or, $3c^2 - 6c + 2 = \dfrac{\frac{3}{8} - 0}{1/2}$

Or, $3c^2 - 6c + 2 = \frac{3}{4}$

Or, $3c^2 - 6c + \frac{5}{4} = 0$

Or, $12c^2 - 24c + 5 = 0$

Or, $c = \frac{-(-24) \pm \sqrt{(-24)^2 - 4 \cdot 12 \cdot 5}}{2 \cdot 12}$

Or $c = \frac{24 \pm \sqrt{336}}{24}$

Or $c = \frac{24 \pm 4\sqrt{21}}{24}$

$\therefore c = \frac{6 \pm \sqrt{21}}{6}$.

Here

$c = \frac{6 + \sqrt{21}}{6} \notin [0, \frac{1}{2}]$.

$c = \frac{6 - \sqrt{21}}{6} \in [0, \frac{1}{2})$.

$\therefore$ Theorem is verified.

---

a) $f(x) = e^x$ in $[0,1]$

Soln

Here

$f(x) = e^x$

i) $f(x)$ being exponential function is continuous on $[0,1]$

ii) Diff ① w.r.t x,

$f'(x) = e^x$ which is defined on $(0,1)$.

So, $f(x)$ is differentiable on $(0,1)$

Date |
Page |

As f(x) holds both condition of meanvalue theorem.
we can find at least one point on $c \in (0,1)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or $$e^{c} = \frac{e^1 - e^0}{1-0}$$

or $$e^c = e - 1$$

Taking ln on bothsides,

$$c \ln e = \ln(e-1)$$

or $$c = \frac{\ln(e-1)}{\ln e}$$

$\therefore$ $$c = \ln(e-1) \quad [\because \ln e = 1].$$

Here
$c = \ln(e-1) = 0.54 \in [0,1]$
$\therefore$ Theorem is verified.

---

e) $f(x) = \sqrt{x^2-4}$ in $[2,4]$
Soln
$$f(x) = \sqrt{x^2-4} \quad -(1)$$

Here $x \in [2,4]$.

i) f(x) ~~be~~ is ~~defined~~ exists for all ~~x ∉ [2,2]~~
So, x is ~~defined~~ continuous for $x \in [2,4]$.

ii) Diff ~~f~~ (1) w.r.t. x,

$$f'(x) = \frac{1}{2\sqrt{x^2-4}} \cdot 2x$$

$$= \frac{x}{\sqrt{x^2-4}}$$ which is defined on (2,4).

So, f(x) is differentiable at (2,4).

Now,
f(x) holds both condition of meanvalue theorem
So, there exists at least a $c \in (2,4)$
such that

$$f'(c) = \frac{f(4)-f(2)}{4-2}$$

or, $$\frac{c}{\sqrt{c^2-4}} = \frac{\sqrt{12}-0}{2}$$

or $$\frac{c}{\sqrt{c^2-4}} = \sqrt{3}.$$

or, $c^2 = 3(c^2-4)$

or, $c^2 = 3c^2-12$

or, $12 = 2c^2$

or, $c^2 = 6$

$\therefore c = \pm\sqrt{6}$.

since,
$c = \sqrt{6} \in (2,4)$.
Theorem is verified.

---

4. Show that mean value theorem is not applicable to $f(x) = \frac{1}{x}$ in $(-1,1)$

Soln)
Here
$$f(x) = \frac{1}{x}$$

1) f(x) is not defined at $x=0$.
So, f(x) is not continuous on $[-1,1]$
So, theorem is not satisfied.

Date |
Page |

5 Find the point on $f(x) = (x-2)^2$ where tangent is parallel to chord joining (3,1) and (4,4).

Sol:n

Here,

$$f(x) = (x-2)^2$$

$$f'(x) = 2x-4.$$

Given points on the chord be:

(3,1) and (4,4).

Equ:n of tangent be,

$$y-1 = \frac{3}{1}(x-3)$$

$$\text{or, } y-1 = 3x-9$$

$$\therefore y = 3x-8 \quad \text{—(1)}$$

Now, Diff (1) w.r.t x,

$$\frac{dy}{dx} = 3.$$

Now,

Slope of $f(x)$ at point $(x,y) = 3$

$$\text{or, } f'(x) = 3$$

$$\text{or, } 2x-4 = 3$$

$$\text{or, } 2x = 7$$

$$\therefore x = \frac{7}{2}$$

When $x = \frac{7}{2}$

$$y = \left(\frac{7}{2}-2\right)^2$$

$$= \left(\frac{3}{2}\right)^2 = \frac{9}{4}.$$

$\therefore$ Req. points are $\left(\frac{7}{2}, \frac{9}{4}\right)$.

[Notes].

Date |
Page |

∴ Indeterminate forms

If a functions takes form of $\frac{0}{0}$ as $x \to a$, the value is said to be indeterminate

Other indeterminate forms are $\frac{\infty}{\infty}$, $\infty - \infty$, 0^0, 1^∞, ∞^0.

∵ L' Hospital Theorem

If $f(x)$ and $g(x)$ are any two functions having their derivatives $f'(x)$ and $g'(x)$ which are continuous at $x=a$ & if $f(a)=g(a)=0$,

then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

provided that $g'(a) \neq 0$.

This theorem can be further used provided that $f'(a)$ & $g'(a)$ are both zero.

Note: L' Hospital Rule can be Used only for $\frac{0}{0}$ & $\frac{\infty}{\infty}$ form.