Sequence and Series and mathematical Induction

Notes

i. The sum of first n natural number is $\frac{n(n+1)}{2}$

i.e. $1+2+3+\dots+n = \frac{n(n+1)}{2}$

ii. ~~The sum of first (m+1) natural number is, (m+1)(m~~

NOTES

1. If a is the first term & d be the common difference, then nth term of A.P. is

$$t_n = a+(n-1)d$$

If S_n be the sum of n-terms then,

$$t_1+t_2+t_3+\dots t_n = \sum_{n=1}^{n} t_k = \Sigma t_n$$

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

2. If a is the first term and r be the common ratio, then nth terms of G.P. is

$$t_n = ar^{n-1}$$

and sum of n terms is

$$S_n = t_1+t_2+\dots+t_n = \Sigma t_n$$
$$= \frac{a(r^n-1)}{r-1}$$

i.e. $S_n = \frac{a(r^n-1)}{r-1}$

* The sum of first n natural numbers is $\frac{n(n+1)}{2}$

i.e. $1+2+3+\ldots+n = \frac{n(n+1)}{2}$

or;

$\sum_{K=1}^{n} K = \sum n = \frac{n(n+1)}{2}$

* The sum of squares of first n natural number is $\frac{n(n+1)(2n+1)}{6}$

i.e. $1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

i.e. $\sum_{K=1}^{n} K^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$

* The sum of cubes of first n natural numbers is $\left[\frac{n(n+1)}{2}\right]^2$

i.e. $1^3+2^3+3^3+\ldots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$

i.e. $\sum n^3 = \left(\frac{n(n+1)}{2}\right)^2$

or, $\sum_{K=1}^{n} K^3 = \left(\frac{n(n+1)}{2}\right)^2$

$\circledast$ $\Sigma(a_n+b_n) = \Sigma a_n + \Sigma b_n$

$\Sigma(4n^2+n) = \Sigma 4n^2 + \Sigma n$

$\Sigma c a_n = c\Sigma a_n$

$\circledast$ $\Sigma 5 = \sum_{k=1}^{n} 5$

$= 5+5+5+\dots+5$

$= n.5$

$= 5n$

$\circledast$ $\Sigma 1 = 1+1+1+\dots+1 = n$

$\circledast$ $\Sigma a = na$

Exercise 6.1

1. Find the n^{th} term and then the sum of the first n terms of each of the following series.

a. $1.3+2.4+3.5+\dots$

Solution

The given series is,

$1.3+2.4+3.5+\dots$

n^{th} term of $1,2,3,\dots = a+(n-1)d$

$= 1+(n-1)1$

$= n$ $[\because t_n = a+(n-1)d]$

n^{th} term of $3,4,5\dots = a+(n-1)d$

$= 3+(n-1)1$

$= 2+n$

$= n+2$

n^{th} term of given series is

$t_n = n(n+2)$

$= n^2+2n$

If Sn be the sum of first n-terms, then.

$S_n = \Sigma t_n$ (or $\sum_{k=1}^{n} t_k$)

$= \Sigma(n^2+2n)$

$= \Sigma n^2 + \Sigma 2n$

$= \Sigma n^2 + 2\Sigma n$

$= (1^2+2^2+3^2+\cdots+n^2) + 2(1+2+3+\cdots+n)$

$= \frac{n(n+1)(2n+1)}{6} + 2.\frac{n(n+1)}{2}$

$= n(n+1)\left[\frac{2n+1}{6} + 1\right]$

$= n(n+1)\frac{2n+1+6}{6}$

$S_n = \frac{1}{6} n(n+1)(2n+7)$

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b $1+4+9+16+\cdots$

Solution

The given series is,

$1+4+9+16+\cdots$

Let t_n be the nth term and S_n be the sum of n terms;

Then,

$S_n = 1+4+9+16+\cdots+t_n$

$-S_n = \quad 1+4+9+\cdots+t_{n-1}+t_n$

$0 = 1+3+5+7+\cdots+(t_n-t_{n-1})-t_n$

So, $t_n = 1+3+5+7+\cdots+(t_n-t_{n-1})$

or, $t_n = 1+3+5+7+\cdots$ + nth terms

Now,

$S_n = \frac{n}{2}[2a+(n-1)d]$

$= \frac{n}{2}[2+(n-1)2]$

$= \frac{n}{2}(2+2n-2)$

$= n^2$

Now,

$S_n = \Sigma n^2$

$= 1^2+2^2+3^2+ \cdots + n^2$

$= \frac{n(n+1)(2n+1)}{6}$

c. $1.3+3.5+5.7+\cdots$

Solution

The given series is

$1.3+3.5+5.7+\cdots$

n^{th} term of $1,3,5\cdots = a+(n-1)d = 1+(n-1)2$

$= 2n-1$

n^{th} term of $3,5,7\cdots = a+(n-1)d = 3+(n-1)2$

$= 2n+1$

$t_n = (2n-1)(2n+1)$

$= 4n^2+2n-2n-1$

$= 4n^2-1$

$S_n = \Sigma t_n$

$= \Sigma(4n^2-1)$

$= \Sigma 4n^2 - \Sigma 1$

$= \frac{4n(n+1)(2n+1)}{6} - n$

$= \frac{2n(n+1)(2n+2) - 3n}{3}$

$= \frac{(2n^2+2n)(2n+1)-3n}{3}$

$= \frac{4n^3+2n^2+4n^2+2n-3n}{3}$

$= \frac{4n^3+6n^2-n}{3}$

$$= \frac{n(4n^2+6n-1)}{3}$$

d. $1.2.3 + 2.3.4 + 3.4.5 + \cdots$

Solution

nth term of $1,2,3 = 1+(n-1)1 = 1+n-1 = n$

nth term of $2,3,4 = 2+(n-1)1 = 2+n-1 = n+1$

nth term of $3,4,5 = 3+(n-1)1 = 3+n-1 = \cancel{n} = n+2$

$$t_n = n(n+1)(n+2)$$
$$= n(n^2+2n+n+2)$$
$$= n^3+2n^2+n^2+2n$$
$$= n^3+3n^2+2n$$

$$S_n = \Sigma t_n$$
$$= \Sigma(n^3+3n^2+2n)$$
$$= \Sigma n^3 + \Sigma 3n^2 + \Sigma 2n$$
$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$
$$= \left(\frac{n(n+1)}{2}\right)\left[\frac{n(n+1)}{2} + (2n+1) + 2\right]$$
$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)+4n+2+4}{2}\right]$$
$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)+4n+2+4}{2}\right]$$
$$= \frac{n(n+1)}{2}\left[\frac{n^2+n+4n+6}{2}\right]$$
$$= \frac{n(n+1)}{2}\left[\frac{n^2+n+4n+6}{2}\right]$$
$$= \frac{n(n+1)}{2}\left[\frac{n^2+5n+6}{2}\right]$$
$$= \frac{n(n+1)}{2}\left[\frac{n^2+3n+2n+6}{2}\right]$$
$$= \frac{n(n+1)}{2}\left[\frac{(n+3)(n+2)}{2}\right]$$
$$= \frac{n(n+1)(n+3)(n+2)}{4}$$

e. $1+(1+2)+(1+2+3)+\dots$

Solution

The given series is,

$1+(1+2)+(1+2+3)+\dots$

Here;

$t_1 = 1$

$t_2 = 1+2$

$t_3 = 1+2+3$

$t_n = 1+2+3+\dots+n$

Now,

$$S_n = \Sigma t_n$$

$$= \Sigma\left[\frac{n(n+1)}{2}\right]$$

$$= \Sigma\left(\frac{n^2+n}{2}\right)$$

$$= \frac{1}{2}\left[\Sigma n^2 + \Sigma n\right]$$

$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{1}{2}\cdot\frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right]$$

$$= \frac{n(n+1)}{4}\left[\frac{2n+1+3}{3}\right] = \frac{n(n+1)}{4}\left[\frac{2n+4}{3}\right]$$

$$= \frac{n(n+1)}{4}\left[\frac{2(n+2)}{3}\right]$$

$$= \frac{n(n+1)(n+2)}{6}$$

2. Sum to n terms of the following series.

a. $\cancel{(x+a)}\ \cancel{(x}\ (x+a)+(x^2+2a)+(x^3+3a)+ \cdots$

Solution

The given series is,

$(x+a)+(x^2+2a)+(x^3+3a)+\cdots$

$= (x+x^2+x^3+\cdots+x^n)+(a+2a+3a+\cdots na)$

$= \dfrac{x(x^n-1)}{x-1} + \dfrac{n}{2}\left[2a+(n-1)a\right]$

$= \dfrac{x(x^n-1)}{x-1} + \dfrac{n(2a+an-a)}{2}$

$= \dfrac{x(x^n-1)}{x-1} + \dfrac{n(2+n-1)a}{2}$

$= \dfrac{x(x^n-1)}{x-1} + \dfrac{n(n+1)a}{2}$

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b. $5+55+555+\cdots$ to n terms

Solution

The given series is;

$5+55+555+\cdots$ to n terms

$= 5(1+11+111+\cdots$ to n terms)

$= \dfrac{5}{9}(9+99+999+\cdots$ to n terms)

$= \dfrac{5}{9}\left[(10+100+1000+\cdots)-(1+1+1+\cdots)\right]$

Here, $a=10$; $r=10$

$S_n = \dfrac{a(r^n-1)}{r-1} = \dfrac{5}{9}\left[\dfrac{10(10^n-1)}{10-1} - n\right]$

$= \dfrac{5}{9}\left[\dfrac{10(10^n-1)}{9} - n\right]$

c. $0.3 + 0.33 + 0.333 + \cdots$ to n terms.

Solution

The given series is,

$0.3 + 0.33 + 0.333 + \cdots$

$= \frac{3}{10} + \frac{33}{100} + \frac{333}{1000} + \cdots$

$= 3\left[\frac{1}{10} + \frac{11}{100} + \frac{111}{1000} + \cdots\right]$

$= \frac{3}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \cdots\right]$

$= \frac{3}{9}\left[\frac{10-1}{10} + \frac{100-1}{100} + \frac{1000-1}{1000} + \cdots\right]$

$= \frac{3}{9}\left[\left(1-\frac{1}{10}\right) + \left(1-\frac{1}{100}\right) + \left(1-\frac{1}{1000}\right) + \cdots\right]$

$= \frac{3}{9}\left[(1+1+1+\cdots+n) - \left[\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots \text{to } n \text{ terms}\right)\right.$

Here, $a = \frac{1}{10}$, $r = \frac{1}{10}$

So,

$S_n = \frac{3}{9}\left[n - \frac{a(r^n - 1)}{r^n - 1}\right]$

$= \frac{3}{9}\left[n - \frac{1/10\left[(1/10)^n - 1\right]}{1/10 - 1}\right]$

$= \frac{3}{9}\left[n - \frac{1/10\left[(1/10)^n - 1\right]}{-9/10}\right]$

$= \frac{3}{9}\left[n - \frac{1}{9}\left\{1 - \left(\frac{1}{10}\right)^n\right\}\right]$

d. $1 \times n + 2 \times (n-1) + 3 \times (n-2) + \dots$

Solution

The given series is,

$1 \times n + 2 \times (n-1) + 3 \times (n-2) + \dots$

k^{th} term of $1, 2, 3 = \cancel{a+} \dots = a + (n-1)d = 1 + (k-1)1$
$= k$

k^{th} term of $n, n-1, n-2, \dots = n + (k-1) \cdot -1$
$= n - k + 1$

$\therefore$ kth term of given series $= k(n-k+1)$
$= kn - k^2 + k = nk + k - k^2$
$= (n+1)k - k^2$

Now,

$$S_n = \sum_{k=1}^{n} \left[(n+1)k - k^2 \right]$$

$$= \sum_{k=1}^{n} (n+1) \cdot k - \sum_{k=1}^{n} k^2$$

$$= (n+1)(1+2+3+\dots+n) - (1^2+2^2+3^2+\dots+n^2)$$

$$= (n+1) \cdot \frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2} \left[(n+1) - \left(\frac{2n+1}{3} \right) \right]$$

$$= \frac{n(n+1)}{2} \left[\frac{3n+3-2n-1}{3} \right]$$

$$= \frac{n(n+1)}{2} \left[\frac{n+2}{3} \right]$$

$$= \cancel{n(n+2)} \; \frac{n(n+1)(n+2)}{6}$$

e. $1+3+6+10+\cdots$

Solution

The given series is : $1+3+6+10+\cdots$

Let t_n be the ~~sum~~ nth term and S_n be the sum of n terms.

Then,

$$S_n = 1+3+6+10+\cdots+t_n$$
$$-S_n = \quad 1+3+6+\cdots+t_{n-1}+t_n$$
$$0 = 1+2+3+4+\cdots+(t_n-t_{n-1})-t_n$$

or; $t_n = 1+2+3+4+\cdots+t_n-t_{n-1}$

or; $t_n = 1+2+3+4+\cdots+n^{th}$ term

Now,

$$t_n \; \cancel{S_n} = \frac{n}{2}[2a+(n-1)d]$$
$$= \frac{n}{2}[2+(n-1).1]$$
$$= \frac{n}{2}(2+n-1)$$
$$= \frac{2n+n^2-n}{2}$$
$$= \frac{n^2+n}{2}$$
$$= \frac{n(n+1)}{2}$$

Now,

$$S_n = \Sigma t_n$$
$$= \Sigma\left[\frac{n(n+1)}{2}\right]$$
$$= \frac{1}{2}\left[\Sigma n^2 + \Sigma n\right]$$
$$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$
$$= \frac{1}{2}\cdot\frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right]$$

$= \frac{n(n+1)}{4}\left[\frac{2(n+2)}{3}\right]$

$= \frac{n(n+1)(n+2)}{6}$

6. $3+6+11+18+\cdots$

Solution

The given series is,

$3+6+11+18+\cdots$

Let t_n be the nth term and S_n be the sum of n terms.

Then,

$S_n = 3+6+11+18+\cdots+t_n$

$-S_n = \quad 3+6+11+\cdots+t_{n-1}+t_n$

$0 = 3+3+5+7+\cdots+(t_n-t_{n-2})-t_n$

or; $t_n = 3+3+5+7+\cdots+t_n-t_{n-1}$

or; $t_n = 3+3+5+7+\cdots+n$ terms

or; $t_n = 3+\frac{(n-1)}{2}(6+2n-4)$

or; $t_n = 3+\frac{(n-1)(2+2n)}{2}$

or; $t_n = 3+\frac{(n-1)2(n+1)}{2}$

or; $t_n = 3+n^2-1$

$S_n = \Sigma t_n$

$= \Sigma(3+n^2-1)$

$= \Sigma 3+\Sigma n^2-\Sigma 1$

$= 3n+\frac{n(n+1)(2n+1)}{6}-n$

$= \frac{3n+(n^2+n)(2n+1)}{6}-n$

$= \frac{18n+2n^3+2n^2+n^2+n-6n}{6}$

$= \frac{2n^3+3n^2+13n}{6}$

$= \frac{n(2n^2+3n+13)}{6}$