Cross Multiplication Method

Consider two equation,

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

$$\begin{array}{ccccc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \end{array}$$

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad y = \frac{c_1a_2 - a_1c_2}{a_1b_2 - a_2b_1}$$

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

~~Consider~~ Condition for the two quadratic equations to have one common root

Consider two quadratic equation

$$ax^2 + bx + c = 0$$
$$a_1x^2 + b_1x + c_1 = 0$$

Let α be the common root. Then,

$$a\alpha^2 + b\alpha + c = 0 \quad \text{—— (1)}$$

$$a_1\alpha^2 + b_1\alpha + c_1 = 0 \quad \text{—— (2)}$$

Solving ① and ② by using cross-multiplication

$$\frac{\alpha^2}{bc_1 - b_1c} = \frac{\alpha}{a_1c - ac_1} = \frac{1}{ab_1 - a_1b}$$

Taking 1st two ratios

$$\frac{\alpha^2}{bc_1 - b_1c} = \frac{\alpha}{a_1c - ac_1}$$

$$\Rightarrow \alpha = \frac{bc_1 - b_1c}{a_1c - ac_1} \quad ---- \quad (*)$$

Taking last two ratios

$$\frac{\alpha}{a_1c - ac_1} = \frac{1}{ab_1 - a_1b}$$

$$\Rightarrow \alpha = \frac{a_1c - ac_1}{ab_1 - a_1b} \quad --- \quad (*)$$

Exercise 5.3

1. Show that the following pair of equations has a common root.

a. $2x^2 + x - 3 = 0$, $3x^2 - 4x + 1 = 0$

Solution

Here, $a = 2$, $b = 1$, $c = -3$; $a_1 = 3$, $b_1 = -4$, $c_1 = 1$

Now,

$$(ab_1 - a_1b)(bc_1 - b_1c) = (ca_1 - c_1a)^2$$

$$\Rightarrow (2\times(-4) - 3\times1)(1\times1 - (-4)\times(-3)) = (-3\times3 - 1\times2)^2$$

$\Rightarrow (-8-3)(1-12) = (-\cancel{8}-2)^2$

$\Rightarrow (-11)(-11) = \cancel{(-8)^2}\ (-11)^2$

$\Rightarrow 121 = 121$

True

$\therefore$ The given equation has a common root.

b. $3x^2 - 8x + 4 = 0,\ 4x^2 - 7x - 2 = 0$

Solution

Here, $a = 3, b = -8, c = 4, a_1 = 4, b_1 = -7, c_1 = -2$

Now,

$(ab_1 - a_1b)(bc_1 - b_1c) = (ca_1 - c_1a)^2$

$\Rightarrow (3\times(-7) - 4\times(-8))(-8\times(-2) - (-7)\times 4) = (4\times4 - (-2)\times3)^2$

$\Rightarrow (-21+32)(16+28) = \cancel{(16+28)^2}\ (16+6)^2$

$\Rightarrow 11 \times 44 = \cancel{(18)^2}\ (22)^2$

$\Rightarrow 484 = 484$

True

$\therefore$ The given equation has a common root.

2. Determine the value of m for which the equations $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.

Solution.

Here, $a = 3, b = 4m, c = 2, a_1 = 2, b_1 = 3, c_1 = -2$

Since the equations have a common root,

$(ab_1 - a_1b)(bc_1 - b_1c) = (ca_1 - c_1a)^2$

$\Rightarrow (3\times3 - 2\times4m)(4m\times(-2) - 3\times2) = (2\times2 - (-2)\times3)^2$

$\Rightarrow (9-8m)(-8m-6) = (4+6)^2$

$\Rightarrow -72m - 54 + 64m^2 + 48m = 100$

$\Rightarrow 64m^2 - 24m - 154 = 0$

$\Rightarrow$ ~~$16m^2 - 6m$~~

$\therefore m = \frac{7}{4}, -\frac{11}{8}$

3. Find the value of p so that each pair of equations may have a common root.

a) $4x^2 + px - 12 = 0,\ 4x^2 + 3px - 4 = 0$

Solution

Here; $a = 4, b = p, c = -12, a_1 = 4, b_1 = 3p, c_1 = -4$

Since the equation has common root.

$(ab_1 - a_1b)(bc_1 - b_1c) = (ca_1 - c_1a)^2$

$\Rightarrow (4\times 3p - 4\times p)(p\times(-4) - 3p\times(-12)) = (-12\times 4 - (-4)\times 4)^2$

$\Rightarrow (12p - 4p)(-4p + 36p) = (-48 + 16)^2$

$\Rightarrow 8p \times 32p = (-32)^2$

$\Rightarrow 256p^2 = 1024$

$\Rightarrow p^2 = 4$

$\therefore p = \pm 2$

b) $2x^2 + px - 1 = 0,\ 3x^2 - 2x - 5 = 0$

Solution

Here,

$a = 2, b = p, c = -1, a_1 = 3, b_1 = -2, c_1 = -5$

Since the equation has common roots,

$(ab_1 - a_1b)(bc_1 - b_1c) = (ca_1 - c_1a)^2$

$\Rightarrow (2\times(-2) - 3\times p)(p\times(-5) - (-2)\times(-1)) = (-1\times 3 - (-5)\times 2)^2$

$\Rightarrow (-4 - 3p)(\cancel{-15p} - 5p - 2) = (-3 + 10)^2$

$\Rightarrow 20p + 8 + 15p^2 + 6p = 49$

$\Rightarrow 15p^2 + 26p - 41 = 0$

$\therefore p = 1, -\frac{41}{15}$

4. If the quadratic equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root show that it must be either $\dfrac{pq'-p'q}{q-q'}$ or $\dfrac{q-q'}{p'-p}$

Solution

Here;

Given equations are;

$$x^2+px+q=0 \quad \text{and}$$
$$x^2+p'x+q'=0$$

$a=1,\ b=p,\ c=q,\ a_1=1,\ b_1=p',\ c_1=q'$

Since, the equations have common root,

$$(ab_1-a_1b)(bc_1-b_1c)=(ca_1-c_1a)^2$$
$$\Rightarrow (1\times p'-1\times p)(p\times q'-p'\times q)=(q\times 1-q'\times 1)^2$$
$$\Rightarrow (p'-p)(pq'-p'q)=(q-q')^2$$
$$\Rightarrow$$

Let α be the common root, then.

$$\alpha^2+p\alpha+q=0 \quad \text{---} \ (1)$$

and,

$$\alpha^2+p'\alpha+q'=0 \quad \text{---} \ (2)$$

Solving (1) & (2) by cross-multiplication method,

$$\frac{\alpha^2}{pq'-p'q}=\frac{\alpha}{q-q'}=\frac{1}{p'-p}$$

Taking 1st two ratios

$$\frac{\alpha^2}{pq'-p'q}=\frac{\alpha}{q-q'}$$

$\Rightarrow \alpha = \frac{pq' - p'q}{q - q'}$

Taking last two ratios

$\alpha = \frac{q - q'}{p' - p}$

Common ratios is either,

$\frac{pq' - p'q}{q - q'}$ or $\frac{q - q'}{p' - p}$ proved

5. If the equations $x^2 + qx + pr = 0$ and $x^2 + rx + pq = 0$ have a common root, prove that $p + q + r = 0$

Solution

Given equations are:

$x^2 + qx + pr = 0$

$x^2 + rx + pq = 0$

Let α be the common root. Then,

$\alpha^2 + p\alpha + pr = 0$ ---- ①

and,

$\alpha^2 + r\alpha + pq = 0$ --- ②

Solving ① & ② by cross-multiplication method.

$\frac{\alpha^2}{qpq - pr^2} = \frac{\alpha}{pr - pq} = \frac{1}{r - q}$

Taking 1st two ratios,

$$\alpha = \frac{p(q^2 - r^2)}{p(r-q)} \quad \text{----} \quad (3)$$

Now,

Taking last two ratios,

$$\alpha = \frac{p(r-q)}{(r-q)}$$

or; $\alpha = p$. --- (4)

Now,

~~From (iii) & iv~~ From (3) & (4)

$$\frac{q^2 - r^2}{r-q} = p.$$

or; $\dfrac{(q+r)(q-r)}{(r-q)} = p$

or; $-(q+r) = p$

or; $p+q+r=0$

proved.

6. If $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have a common root show that their other roots will satisfy the equation $x^2 + ax + bc = 0$

Solution

Given equation

$$x^2 + bx + ca = 0$$

and, $x^2 + cx + ab = 0$

Let α be the common ratio

$$\alpha^2 + b\alpha + ca = 0 \quad \text{--- } (1)$$
$$\alpha^2 + c\alpha + ab = 0 \quad \text{--- } (2)$$

Solving (1) & (2) using cross-multiplication,

$$\frac{\alpha^2}{ab^2 - ac^2} = \frac{\alpha}{ca - ab} = \frac{1}{c-b}$$

Taking first two ratios,

$$\alpha = \frac{a(b^2 - c^2)}{a(c-b)}$$
$$= -(c+b) \quad \text{--- } (3)$$

Taking last two ratios,

$$\alpha = \frac{a(c-b)}{(c-b)}$$
$$= a.$$

From (3) & (4)

$$a = -(b+c)$$

Let β be other root of eqn (1) then,

$\alpha . \beta = ca$ $\qquad$ $\alpha + \beta = -b$

$\cancel{\Rightarrow \alpha\beta = ca}$ $\qquad$ $\Rightarrow \beta = -b - \alpha$

$$\therefore \beta = c$$

Let γ be other root of eqn (1) then,

$\alpha + \gamma = -c$

$\gamma = -c - a$

$\alpha . \gamma = ab$

$\gamma = \frac{ab}{a}$

$\therefore \gamma = b$

The equation whose roots are β and γ is,

$x^2 - (\beta + \gamma)x + \beta . \gamma = 0$

or; $x^2 - (b + c)x + bc = 0$

or; $x^2 + ax + bc = 0$

$\therefore x^2 + ax + bc = 0$

That shows that β and γ satisfy the equation $x^2 + ax + bc = 0$. proved.

7. If $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0$ have a common root then show that $a + 4b + 4c = 0$.

Solution

Given equation,

$ax^2 + 2bx + c = 0$

and, $ax^2 + 2cx + b = 0$

Let α be a common roots then,

$a\alpha^2 + 2b\alpha + c = 0$ --- (i)

and $a\alpha^2 + 2c\alpha + b = 0$ --- (ii)

Solving (i) & (ii) using cross multiplication.

$$\frac{\alpha^2}{2b^2 - 2c^2} = \frac{\alpha}{ac - ab} = \frac{1}{2ac - 2ab}$$

Taking first two roots,

$$\alpha = \frac{2(b^2-c^2)}{a(c-b)} = \frac{-2(b+c)}{a} \quad \text{---- (iii)}$$

Taking last two ratio: roots,

$$\alpha = \frac{ac-ab}{2ca-2ba} = \frac{(c-b)}{2(c-b)} = \frac{1}{2} \quad \text{—— (iv)}$$

From (iii) & (iv)

$$\frac{1}{2} = \frac{-2(b+c)}{a}$$

or; $a = -4b-4c$

or; $a+4b+4c=0$ proved.

8. If $x^2+px+q=0$ and $x^2+qx+p=0$ have a common root, prove that either $p=q$ or $p+q+1=0$

Solution

The given equation,

$$x^2+px+q=0$$

and $$x^2+qx+p=0$$

Let α be a common root.

$$\alpha^2+p\alpha+q=0 \quad \text{--- (i)}$$

and $$\alpha^2+q\alpha+p=0 \quad \text{--- (ii)}$$

Solving (i) and (ii) using cross multiplication

$$\frac{\alpha^2}{p^2-q^2} = \frac{\alpha}{q-p} = \frac{1}{q-p}$$

Taking first two ratios,

$$\alpha = \frac{p^2-q^2}{q-p} = -(q+p) \quad --- \text{(III)}$$

Taking last two ratios,

$$\alpha = \frac{q-p}{q-p} = 1 \quad --- \text{(IV)}$$

From (III) & (IV)

$$\frac{p^2-q^2}{q-p} = \frac{q-p}{q-p}$$

or; $p^2-q^2 = q-p$

or; $(p-q)(p+q) = (q-p)$

or; $(p-q)(p+q) = -(p-q)$

or; $(p-q)(p+q) + (p-q) = 0$

or; $(p-q)(p+q+1) = 0$

Either,

$p-q=0$

$\therefore p=q$

Or,

$p+q+1=0$

$\therefore p=q$ or $p+q+1=0$

proved.