

Relation between roots & coefficient

Consider a quadratic equation,
 $ax^2 + bx + c = 0$ — (1), $a \neq 0$

The roots of (1) are:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \alpha \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \beta \text{ (let)}$$

Sum of roots

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a}$$

$$\therefore \alpha + \beta = \frac{-b}{a}$$

i.e. Sum of roots = $-\frac{b}{a}$

and,

$$\text{Product of roots} = \alpha \cdot \beta$$

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$= \frac{c}{a}$$

$$\therefore \alpha \cdot \beta = \frac{c}{a}$$

Note

From ①

$$ax^2 + bx + c = 0, \quad a \neq 0$$

$$\text{or: } x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\text{or: } x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$\text{or: } x^2 - (\alpha + \beta)x + \alpha \cdot \beta = 0$$

Note

If α and β be two roots of quadratic equation then quadratic equation will be

$$x^2 - (\alpha + \beta)x + \alpha \cdot \beta = 0$$

$$\text{i.e. } x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

Exercise 5.2

1. Form a quadratic equation whose roots are.

a. $3, -5$

Solution

Let $\alpha = 3$ and $\beta = -5$

The required equation is:

$$x^2 - (\alpha + \beta)x + \alpha \cdot \beta = 0$$

$$\text{or; } x^2 - (3 - 5)x + 3 \times (-5) = 0$$

$$\text{or; } x^2 + 2x - 15 = 0$$

b. $2, \frac{1}{2}$

Solution

Let $\alpha = 2$, and $\beta = \frac{1}{2}$

The required equation is:

$$x^2 - (\alpha + \beta)x + \alpha \cdot \beta = 0$$

$$\text{or; } x^2 - \left(2 + \frac{1}{2}\right)x + \left(2 \times \frac{1}{2}\right) = 0$$

$$\text{or; } x^2 - \frac{5}{2}x + 1 = 0$$

$$\text{or; } 2x^2 - 5x + 2 = 0$$

c. $2-3i, 2+3i$

Solution

Let $\alpha = 2-3i$ and $\beta = 2+3i$

The required eqn is

$$x^2 - (\alpha + \beta)x + \alpha \cdot \beta = 0$$

$$\text{or; } x^2 - (2-3i + 2+3i)x + (2-3i)(2+3i) = 0$$

$$\text{or; } x^2 - 4x + (2)^2 - (3i)^2 = 0$$

$$\text{or; } x^2 - 4x + 4 - 9i^2 = 0$$

$$\text{or; } x^2 - 9i^2 = 0 \quad x^2 - 4x + 4 + 9 = 0$$

$$\text{or; } x^2 + 9 = 0 \quad \therefore x^2 - 4x + 13 = 0$$

2. Note:

1. The irrational roots of quadratic equations always occur in ~~conjugate~~ conjugate pair.
i.e.

if $a + \sqrt{b}$ is one root, then
 $a - \sqrt{b}$ is another root.

2. The imaginary roots of quadratic equation always occur in conjugate pair. i.e. if $p + iq$ be one root then another root is $p - iq$.

3.
$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$
$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

2. From a quadratic equation whose one root is.

a) $3 - \sqrt{5}$

Solution

$3 - \sqrt{5}$ is one root then $3 + \sqrt{5}$ is another root.

Let $\alpha = 3 - \sqrt{5}$ and $\beta = 3 + \sqrt{5}$

The required equation is,

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

or, $x^2 - (3 - \sqrt{5} + 3 + \sqrt{5})x + (3)^2 - (\sqrt{5})^2 = 0$

or, $x^2 - 6x + 9 - 5 = 0$

or, $x^2 - 6x + 4 = 0$

b) $2i$

Solution

$2i$ is one root then $-2i$ is another root.

Let $\alpha = 2i$ and $\beta = -2i$

The required equation is,

$$x^2 - (\alpha + \beta)x + \alpha \cdot \beta = 0$$

$$\text{or; } x^2 - (2i - 2i)x + 2i \cdot (-2i) = 0$$

$$\text{or; } x^2 - 4i^2 = 0$$

$$\text{or; } x^2 + 4 = 0$$

c) $i\sqrt{3} + 1$

Solution

$i\sqrt{3} + 1$ is one root then $-i\sqrt{3} + 1$ is another root.

Let $\alpha = 1 + i\sqrt{3}$ and $\beta = 1 - i\sqrt{3}$

The required equation is,

$$x^2 - (\alpha + \beta)x + \alpha \cdot \beta = 0$$

$$\text{or; } x^2 - (1 + i\sqrt{3} + 1 - i\sqrt{3})x + (1 + i\sqrt{3})(1 - i\sqrt{3}) = 0$$

$$\text{or; } x^2 - 2x + (1)^2 - (i\sqrt{3})^2 = 0$$

$$\text{or; } x^2 - 2x + 1 - 3i^2 = 0$$

$$\text{or; } x^2 - 2x + 1 + 3 = 0$$

$$\text{or; } x^2 - 2x + 4 = 0$$

$$d) \frac{1}{3+\sqrt{5}}$$

$$= \frac{1}{3+\sqrt{5}} \times \frac{3-\sqrt{5}}{3-\sqrt{5}}$$

$$= \frac{3-\sqrt{5}}{(3)^2 - (\sqrt{5})^2}$$

$$= \frac{3-\sqrt{5}}{9-5}$$

$$= \frac{3-\sqrt{5}}{4}$$

$$= \frac{3}{4} - \frac{\sqrt{5}}{4}$$

$\frac{3-\sqrt{5}}{4}$ is one root, then $\frac{3+\sqrt{5}}{4}$ is another root.

$$\text{Let } \alpha = \frac{3-\sqrt{5}}{4} \text{ and } \beta = \frac{3+\sqrt{5}}{4}$$

The required equation is

$$x^2 - (\alpha + \beta)x + \alpha \cdot \beta = 0$$

$$\text{or; } x^2 - \left(\frac{3-\sqrt{5}}{4} + \frac{3+\sqrt{5}}{4} \right)x + \left(\frac{3-\sqrt{5}}{4} \right) \left(\frac{3+\sqrt{5}}{4} \right) = 0$$

$$\text{or; } x^2 - \frac{3}{2}x + \left(\frac{3}{4} \right)^2 - \left(\frac{\sqrt{5}}{4} \right)^2 = 0$$

$$\text{or; } x^2 - \frac{3}{2}x + \frac{9}{16} - \frac{5}{16} = 0$$

$$\text{or; } x^2 - \frac{3}{2}x + \frac{1}{4} = 0$$

$$\text{or; } 4x^2 - 6x + 1 = 0$$

e. $\frac{1}{3i}$

Solution

$\frac{1}{3i}$ is one root, then $-\frac{1}{3i}$ is another root

Let $\alpha = \frac{1}{3i}$ and $\beta = -\frac{1}{3i}$

The required equation is.

$$x^2 - (\alpha + \beta)x + \alpha \cdot \beta = 0$$

$$\text{or; } x^2 - \left(\frac{1}{3i} - \frac{1}{3i}\right)x + \frac{1}{3i} \cdot \left(-\frac{1}{3i}\right) = 0$$

$$\text{or; } x^2 - \frac{1}{9i^2} = 0$$

$$\text{or; } x^2 + \frac{1}{9} = 0$$

$$\text{or; } 9x^2 + 1 = 0$$

$$\text{or; } 9x^2 + 1 = 0$$

3. Form a quadratic equation whose roots are.

a) Square the roots of $4x^2 + 8x - 5 = 0$.

Solution

The given equation is:

$$4x^2 + 8x - 5 = 0 \quad \text{--- (1)}$$

Comparing (1) with $ax^2 + bx + c = 0$, we get,

$$a = 4, b = 8, c = -5$$

Let, α and β be two roots of ①

$$\text{then, } \alpha + \beta = \frac{-8}{4} = -2$$

$$\alpha \cdot \beta = \frac{-5}{4}$$

By given condition, α^2 and β^2 be the roots of the required equation.

Now;

$$\begin{aligned}\text{Sum of roots} &= \alpha^2 + \beta^2 \\ &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (-2)^2 - 2 \cdot \frac{-5}{4}\end{aligned}$$

$$= 4 + \frac{10}{4}$$

$$= \frac{16 + 10}{2}$$

$$= \frac{26}{2}$$

$$= 13$$

$$\begin{aligned}\text{Product of roots} &= \alpha^2 \cdot \beta^2 \\ &= (\alpha \cdot \beta)^2 \\ &= \left(\frac{-5}{4}\right)^2\end{aligned}$$

$$= \frac{25}{16}$$

Hence, the required equation is;

$$\begin{aligned}x^2 - (\text{sum of roots})x + (\text{product of roots}) &= 0 \\ \text{or; } x^2 - \frac{13}{2}x + \frac{25}{16} &= 0\end{aligned}$$

$$\text{or; } 16x^2 - 104x + 25 = 0$$

b) Reciprocal of the roots of $3x^2 - 5x - 2 = 0$
Solution

The given equation is $3x^2 - 5x - 2 = 0$ — (1)

Let α and β be two roots of (1)

$$\text{then, } \alpha + \beta = \frac{5}{3}$$

$$\alpha \cdot \beta = \frac{-2}{3}$$

By given condition, $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are be the roots of the required equation.

Now;

$$\text{Sum of roots} = \frac{1}{\alpha} + \frac{1}{\beta}$$

$$= \frac{\beta + \alpha}{\alpha \beta}$$

$$= \frac{5}{3}$$

$$\frac{-2}{3}$$

$$= \frac{5}{-2}$$

$$\text{Product of roots} = \frac{1}{\alpha} \cdot \frac{1}{\beta}$$

$$= \frac{1}{\alpha \beta}$$

$$= \frac{1}{\frac{-2}{3}}$$

$$= \frac{3}{-2}$$

Hence, the required equation is,

$$x^2 - (\text{Sum of roots})x + (\text{product of roots}) = 0$$

or; $x^2 + \frac{5}{2}x - \frac{3}{2} = 0$

or; $2x^2 + 5x - 3 = 0$

c) m -times the roots of $x^2 - bx + c = 0$

Solution

The given equation is $x^2 - bx + c = 0$ --- ①

Let α and β be the two roots of ①

Then,

$$\therefore \alpha + \beta = \frac{b}{1} = b$$

$$\therefore \alpha \cdot \beta = \frac{c}{1} = c$$

By the given condition, $m \cdot \alpha$ and $m \cdot \beta$ be the roots of the equation.

Now;

$$\begin{aligned} \text{Sum of roots} &= m \cdot \alpha + m \cdot \beta \\ &= m(\alpha + \beta) \\ &= m \cdot b \end{aligned}$$

$$\begin{aligned} \text{Product of roots} &= m \cdot \alpha \cdot m \cdot \beta \\ &= m^2 \alpha \beta \\ &= m^2 c \end{aligned}$$

Hence, the required equation is,

$$x^2 - (\text{Sum of roots})x + (\text{product of roots}) = 0$$

or; $x^2 - m \cdot bx + m^2 c = 0$

d. greater by h than the roots of $x^2 - px + q = 0$.
Solution

The given equation is $x^2 - px + q = 0$ --- (1)

Let α and β be the two roots of (1)

Then,

$$\alpha + \beta = \frac{p}{1} = p$$

$$\alpha \cdot \beta = \frac{q}{1} = q$$

By the given condition, $\alpha + h$, $\beta + h$ be the roots of the equation.

Now,

$$\begin{aligned}\text{Sum of roots} &= \alpha + h + \beta + h \\ &= (\alpha + \beta) + 2h \\ &= p + 2h\end{aligned}$$

$$\begin{aligned}\text{Product of roots} &= (\alpha + h) \cdot (\beta + h) \\ &= \alpha\beta + \alpha h + \beta h + h^2 \\ &= q + h(\alpha + \beta) + h^2 \\ &= q + ph + h^2\end{aligned}$$

Hence, the required equation is,

$$\begin{aligned}x^2 - (\text{sum of roots})x + (\text{product of roots}) &= 0 \\ \text{or; } x^2 - (p + 2h)x + (q + ph + h^2) &= 0\end{aligned}$$

4. If one root of the equation $ax^2+bx+c=0$ is thrice the other, then show that $3b^2=16ac$.

Solution

The given equation is

$$ax^2+bx+c=0$$

If α be the one root, then by the given condition another root will be 3α .

Now;

Sum of roots

$$\alpha + 3\alpha = \frac{-b}{a}$$

$$\Rightarrow 4\alpha = \frac{-b}{a}$$

$$\therefore \alpha = \frac{-b}{4a}$$

Product of roots

$$\alpha \cdot 3\alpha = \frac{c}{a}$$

$$\Rightarrow 3\alpha^2 = \frac{c}{a}$$

$$\Rightarrow 3\left(\frac{-b}{4a}\right)^2 = \frac{c}{a}$$

$$\Rightarrow \frac{3b^2}{16a^2} = \frac{c}{a}$$

$$\Rightarrow 3b^2 = \frac{16a^2c}{a}$$

i.e. $3b^2 = 16ac$ proved

5. If the roots of the equation $ax^2+bx+c=0$ are in the ratio $p:q$, show that $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{b}{a}} = 0$

Solution

Given equation is:

$$ax^2+bx+c=0 \quad \text{--- (1)}$$

Let α and β be the two roots of (1)
then,

$$\alpha + \beta = -\frac{b}{a}$$

$$\alpha \cdot \beta = \frac{c}{a}$$

By given condition,

$$\alpha : \beta = p : q$$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{p}{q} = k \text{ (let)}$$

$$\Rightarrow \alpha = pk, \beta = qk.$$

$$\text{i.e. } p = \frac{\alpha}{k}, q = \frac{\beta}{k}$$

Now,

$$\text{LHS} = \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{b}{a}}$$

$$= \frac{\sqrt{p}}{\sqrt{q}} + \frac{\sqrt{q}}{\sqrt{p}} + \sqrt{\frac{b}{a}}$$

$$L = \frac{p+q}{\sqrt{pq}} + \sqrt{\frac{b}{a}}$$

$$= \frac{\frac{\alpha}{k} + \frac{\beta}{k}}{\sqrt{\frac{\alpha}{k} \cdot \frac{\beta}{k}}} + \sqrt{\frac{b}{a}}$$

$$= \frac{\frac{\alpha + \beta}{k}}{\frac{\sqrt{\alpha\beta}}{k}} + \sqrt{\frac{b}{a}}$$

$$= \frac{\alpha + \beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{b}{a}}$$

$$= \frac{-\frac{b}{a}}{\sqrt{\frac{b}{a}}} + \sqrt{\frac{b}{a}}$$

$$= \frac{-\frac{b}{a} + \frac{b}{a}}{\sqrt{\frac{b}{a}}}$$

$$= 0$$

= RHS proved.

6. If α, β be the roots of the equation $px^2 + qx + q = 0$, show that:

a) $\frac{1}{\alpha} + \frac{1}{\beta} + 1 = 0$

Solution

Here, the given equation is :

$$px^2 + qx + q = 0$$

Sum of roots i.e. $\alpha + \beta = \frac{-q}{p}$

Product of roots i.e. $\alpha \cdot \beta = \frac{q}{p}$

Now,

$$\text{LHS} = \frac{1}{\alpha} + \frac{1}{\beta} + 1$$

$$= \frac{\beta + \alpha}{\alpha \cdot \beta} + 1$$

$$= \frac{-\frac{q}{p}}{\frac{q}{p}} + 1$$

$$\frac{q}{p}$$

$$= \frac{-q}{q} + 1$$

$$= -1 + 1$$

$$= 0$$

$$= \text{RHS proved.}$$

$$b) \sqrt{\frac{\alpha}{p}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}} = 0$$

Solution

The given equation is ;

$$px^2 + qx + q = 0$$

$$\text{Sum of roots i.e. } \alpha + \beta = -\frac{q}{p}$$

$$\text{Product of roots i.e. } \alpha \cdot \beta = \frac{q}{p}$$

Now;

$$\text{LHS} = \sqrt{\frac{\alpha}{p}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}}$$

$$= \frac{\sqrt{\alpha}}{\sqrt{p}} + \frac{\sqrt{\beta}}{\sqrt{\alpha}} + \sqrt{\frac{q}{p}}$$

$$= \frac{\alpha + \beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{q}{p}}$$

$$= \frac{-\frac{q}{p}}{\sqrt{\frac{q}{p}}} + \sqrt{\frac{q}{p}}$$

$$= \frac{-\frac{q}{p} + \frac{q}{p}}{\sqrt{\frac{q}{p}}}$$

$$= 0$$

= RHS proved.

7. If one root of the equation $ax^2+bx+c=0$ is square of other, prove that $b^3+a^2c+ac^2=3abc$.

Solution

The given equation is :

$$ax^2+bx+c=0 \text{ --- (1)}$$

Let α be one root then by given condition, α^2 is another root.

Now,

$$\alpha + \alpha^2 = -\frac{b}{a}$$

$$\text{or; } \alpha(1+\alpha) = -\frac{b}{a} \text{ --- (*)}$$

And,

$$\alpha \cdot \alpha^2 = \frac{c}{a}$$

$$\text{or; } \alpha^3 = \frac{c}{a} \text{ --- (**)}$$

From (*)

$$\alpha(1+\alpha) = -\frac{b}{a}$$

$$\Rightarrow (\alpha(1+\alpha))^3 = \left(-\frac{b}{a}\right)^3$$

$$\Rightarrow \alpha^3 \cdot (1+\alpha)^3 = \frac{-b^3}{a^3}$$

$$\Rightarrow \frac{c}{a} [1^3 + \alpha^3 + 3\alpha(1+\alpha)] = \frac{-b^3}{a^3}$$

$$\Rightarrow \frac{c}{a} \left[1^3 + \frac{c}{a} + 3 \cdot \left(-\frac{b}{a}\right)\right] = \frac{-b^3}{a^3}$$

$$\Rightarrow \frac{(ac + c^2 - 3bc)}{a^2} = \frac{-b^3}{a^2}$$

$$\Rightarrow a(ac + c^2 - 3bc) = -b^3$$

$$\Rightarrow a^2c + ac^2 - 3abc = -b^3$$

$$\text{i.e. } a^2c + ac^2 + b^3 = 3abc. \text{ proved.}$$

8. Let α, β be the roots of the equation $x^2 + px + q = 0$, then find the equation whose roots are.

a) $\alpha\beta^{-1}, \alpha^{-1}\beta$

Solution

Given equation is

$$x^2 + px + q = 0 \text{ ————— ①}$$

Since α and β are roots of ①, so,

$$\alpha + \beta = -p \text{ and } \alpha \cdot \beta = q.$$

Now,

Roots of required equation are $\alpha\beta^{-1}$ and $\alpha^{-1}\beta$

Sum of roots

$$\alpha\beta^{-1} + \alpha^{-1}\beta$$

$$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(-p)^2 - 2q}{q}$$

$$= \frac{p^2 - 2q}{q}$$

Product of roots

$$\alpha \beta^{-1} \cdot \alpha^{-1} \beta$$

$$= \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha}$$

$$= 1$$

The required equation is.

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$\text{i.e. } x^2 - \left(\frac{p^2 - 2q}{q}\right)x + 1 = 0$$

b) $(\alpha - \beta)^2, (\alpha + \beta)^2$

Solution

Given equation is

$$x^2 + px + q = 0 \quad \text{--- ①}$$

Since α and β are roots of ①

$$\alpha + \beta = -p \text{ and } \alpha \cdot \beta = q.$$

Now,

Roots of required equation are $(\alpha - \beta)^2$ & $(\alpha + \beta)^2$

Sum of roots

$$(\alpha - \beta)^2 + (\alpha + \beta)^2$$

$$= \alpha^2 - 2\alpha\beta + \beta^2 + \alpha^2 + 2\alpha\beta + \beta^2$$

$$= 2\alpha^2 + 2\beta^2$$

$$= 2(\alpha^2 + \beta^2)$$

$$= 2[(\alpha + \beta)^2 - 2\alpha\beta]$$

$$= 2[(-p)^2 - 2q]$$

$$= 2[p^2 - 2q]$$

$$= 2p^2 - 4q.$$

Product of roots

$$(\alpha - \beta)^2 \cdot (\alpha + \beta)^2$$

$$= (\alpha^2 - 2\alpha\beta + \beta^2) (\alpha^2 + 2\alpha\beta + \beta^2)$$

$$= (\alpha^2 + \beta^2 - 2\alpha\beta) (\alpha^2 + \beta^2 + 2\alpha\beta)$$

$$= \cancel{8} (\alpha + \beta)^2 - 4\alpha\beta (\alpha + \beta)^2$$

$$= (p^2 - 4q) (p^2)$$

$$= (p^2 - 4q) p^2$$

$$= p^4 - 4p^2q$$

The required equation is :

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

$$\text{or; } x^2 - (2p^2 - 4q)x + (p^4 - 4p^2q) = 0$$

$$\text{or; } x^2 - 2(p^2 - 2q)x + p^2(p^2 - 4q) = 0$$

c) $\alpha^2 \beta^{-1}, \beta^2 \alpha^{-1}$

Solution

Given equation is

$$x^2 + px + q = 0 \quad \text{--- (1)}$$

Since α and β are roots of (1). So,

$$\alpha + \beta = -p \quad \text{and} \quad \alpha \cdot \beta = q$$

Now,

Roots of ^{required} equation are $\alpha^2 \beta^{-1}$ and $\beta^2 \alpha^{-1}$

Sum of roots

$$\alpha^2 \beta^{-1} + \beta^2 \alpha^{-1}$$

$$= \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$$

$$= \frac{\alpha^3 + \beta^3}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{\alpha\beta}$$

$$= \frac{(-p)^3 - 3 \cdot (-p) \cdot q}{q}$$

$$= \frac{-p^3 + 3pq}{q}$$

Product of roots

$$\alpha^2 \beta^{-1} \cdot \beta^2 \alpha^{-1}$$

$$= \frac{\alpha^2}{\beta} \times \frac{\beta^2}{\alpha}$$

$$= \alpha \times \beta$$

$$= \alpha\beta$$

$$= q.$$

The required equation is

$$\lambda^2 - (\text{sum of roots})\lambda + (\text{product of roots}) = 0$$

$$\text{or; } \lambda^2 - \left(\frac{-p^3 + 3pq}{q} \right) \lambda + q = 0$$

$$\text{or; } q\lambda^2 + (p^3 - 3pq)\lambda + q^2 = 0$$

9. Find the value of K so that the equation.

a. $2x^2 + Kx - 15 = 0$ has one root 3.

Solution

Let α be the another root.

Given equation is

$$2x^2 + Kx - 15 = 0$$

Now;

Product of roots

$$\alpha \cdot 3 = \frac{-15}{2}$$

$$\text{or; } \alpha = \frac{-15}{2} \times \frac{1}{3}$$

$$\therefore \alpha = \frac{-15}{6}$$

Sum of roots

$$\alpha + 3 = \frac{-K}{2}$$

$$\text{or; } \frac{-15}{6} + 3 = \frac{-K}{2}$$

$$\text{or; } \frac{1}{2} = \frac{-K}{2}$$

$$\text{or; } \therefore K = -1$$

b. $3x^2 + kx - 2 = 0$ has two roots whose sum is 6.
Solution

Here;

Given equation is,

$$3x^2 + kx - 2 = 0$$

Let α and β be the roots of the equation.

Sum of roots

$$\alpha + \beta = -\frac{k}{3}$$

$$\text{or; } 6 = -\frac{k}{3}$$

$$\therefore k = -18$$

c. $2x^2 + (4-k)x - 17 = 0$ has roots equal but opposite in sign.

Solution

Given equation is

$$2x^2 + (4-k)x - 17 = 0$$

Let α be one root then by given condition $-\alpha$ be another root.

Now,

Sum of roots

$$\alpha + (-\alpha) = -\frac{4+k}{2}$$

$$\text{or; } 0 = -\frac{4+k}{2}$$

$$\text{or; } 0 = -4+k$$

$$\therefore k = 4$$

d. $7kx^2 - 12x - 21 = 0$ has reciprocal roots.

Solution

Let α be one root then by the given condition

$\frac{1}{\alpha}$ is another root.

α

Here;

Given equation is .

$$7kx^2 - 12x - 21 = 0$$

Now;

Product of roots

$$\alpha \cdot \frac{1}{\alpha} = \frac{-21}{7k}$$

$$\text{or } 1 = \frac{-21}{7k}$$

$$\text{or } 7k = -21$$

$$\therefore k = -3$$

e. $x^2 - kx + 1 = 0$ has a root square of another.

Solution

Given equation is

$$x^2 - kx + 1 = 0$$

Let α be one root then by given condition

α^2 be another root.

Now;

Product of roots

$$\alpha \cdot \alpha^2 = 1$$

$$\text{or } \alpha^3 = 1$$

$$\therefore \alpha = 1$$

Sum of roots

$$\alpha + \alpha^2 = \frac{-(-K)}{1} = K.$$

$$\Rightarrow \alpha(1+\alpha) = K$$

$$\Rightarrow 1(1+1) = K$$

$$\therefore K = 2$$

10. The sum of the roots of a quadratic equation is 1 and the sum of their square is 13, find the equation.

Solution.

Let the roots of the quadratic equation be α and β .

Then, by given condition,

$$\alpha + \beta = 1 \quad \text{--- (1)}$$

$$\alpha^2 + \beta^2 = 13$$

$$\text{or, } (\alpha + \beta)^2 - 2\alpha\beta = 13$$

$$\text{or, } (1)^2 - 2\alpha\beta = 13$$

$$\text{or, } -2\alpha\beta = 12$$

$$\therefore \alpha\beta = -6$$

The required equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$\therefore x^2 - x - 6 = 0$$

11. If the roots of the equation $x^2 + px + q = 0$ are in the same ratio as those of the equation $x^2 + lx + m = 0$. Show that $p^2 m = l^2 q$.

Solution

Given equation is;

$$x^2 + px + q = 0 \quad \dots \quad (1)$$

Let α and β be two roots of (1)
then,

$$\alpha + \beta = -p \quad \dots \quad (2)$$

$$\alpha \cdot \beta = q \quad \dots \quad (3)$$

By given condition

$K\alpha$ and $K\beta$ are the roots of the equation
 $x^2 + lx + m = 0$

Now;

$$K\alpha + K\beta = -l$$

$$\Rightarrow K(\alpha + \beta) = -l$$

$$\Rightarrow K = \frac{-l}{\alpha + \beta}$$

$$= \frac{-l}{-b}$$

$$= \frac{l}{b}$$

$$\therefore K = \frac{l}{p}$$

and,

$$K\alpha \cdot K\beta = m$$

$$\Rightarrow K^2 = \frac{m}{\alpha \beta}$$

$$\Rightarrow K^2 = \frac{m}{q}$$

$$\Rightarrow \left(\frac{l}{p}\right)^2 = \frac{m}{q}$$

$$\Rightarrow \frac{d^2}{p^2} = \frac{m}{q}$$

i.e. $d^2 q = p^2 m$ proved.

12. If the ratio of roots of the equations $lx^2 + mx + n = 0$ be equal to that of the roots $l_1x^2 + m_1x + n_1 = 0$ prove that $\frac{m^2}{m_1^2} = \frac{ln}{l_1n_1}$

Solution

Given equation is

$$lx^2 + mx + n = 0 \text{ --- (1)}$$

Let α and β be two roots of (1)
then,

$$\alpha + \beta = -\frac{m}{l} \quad \text{and} \quad \alpha \cdot \beta = \frac{n}{l}$$

By given condition

$K\alpha$ and $K\beta$ are the roots of the equation $l_1x^2 + m_1x + n_1 = 0$

Now,

$$K\alpha + K\beta = -\frac{m_1}{l_1}$$

$$\Rightarrow K(\alpha + \beta) = -\frac{m_1}{l_1}$$

$$\Rightarrow K \cdot \frac{\alpha + \beta}{\alpha + \beta} = \frac{-m_1}{l_1}$$

$$\Rightarrow K \cdot \frac{-\frac{m}{l}}{-\frac{m}{l}} = \frac{-m_1}{l_1}$$

$$\Rightarrow K \cdot \frac{m}{l} = \frac{m_1}{l_1}$$

$$\Rightarrow K d_1 = \frac{\Delta m_1}{m}$$

$$\Rightarrow K = \frac{\Delta m_1}{m} \times \frac{1}{d_1}$$

$$\Rightarrow K = \frac{\Delta m_1}{\Delta m}$$

and.

$$K\alpha \cdot K\beta = \frac{n_1}{d_1}$$

$$\Rightarrow K^2 = \frac{n_1}{d_1} \times \frac{1}{\alpha\beta}$$

$$\Rightarrow K^2 = \frac{n_1}{d_1} \times \frac{1}{\frac{n}{d}}$$

$$\Rightarrow K^2 = \frac{n_1}{d_1} \times \frac{d}{n}$$

$$\Rightarrow \left(\frac{\Delta m_1}{\Delta m} \right)^2 = \frac{n_1}{d_1} \times \frac{d}{n}$$

$$\Rightarrow \frac{\Delta^2 m_1^2}{d_1^2 m^2} = \frac{n_1}{d_1} \times \frac{d}{n}$$

$$\Rightarrow \frac{\Delta m_1^2}{d_1 m^2} = \frac{n_1}{n}$$

$$\Rightarrow \frac{\Delta n}{\Delta n_1} = \frac{m^2}{m_1^2} \quad \text{proved}$$