

chapter-5

QUADRATIC EQUATIONS

Polynomial Functions

A function $f(x)$ is defined by $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$, ($a_n \neq 0$)

Or,

$$= a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + a_n x^n$$

Where, $a_0, a_1, a_2, \dots, a_n$ are constants and n is positive integer is called polynomial function of degree, n .

Polynomial Equation

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ ($a_n \neq 0$) be polynomial function of degree n . Then,

if $f(x) = 0$ for some value of x then the expression $f(x) = 0$ is called polynomial.

Equation of degree n

Example:

i) $x^2 - x - 6 = 0$, polynomial eqn of degree 2.

ii) $x^7 + 5x^4 + 9x + 1 = 0$, polynomial equation of degree 7.

Solution Equation

The values of x which satisfy the given equation are called solution.

Example: $x + 2 = 0$

$$f(x) = x + 2$$

$$f(x) = 0$$

Here, $f(-2) = 0$

$\therefore x = -2$ is solution.

If $f(x) = 0$, then $x = a$ is called root (solution) of equation $f(x) = 0$.

Note

1. If $f(x) = 0$ for all values of x , then the expression $f(x) = 0$ is called an identity.
2. If $f(x) = 0$, for some values of x , then the expression $f(x) = 0$ is called an equation.

Example

- i. $(x+2)^2 - (x^2 + 4x + 4) = 0$ which is true for all values of x .
- ii. $x+2=7$, which is true for some values of x (i.e. when $x=5$). So, it is an equation.
- iii. All identities are equation but converse may not be true.

Quadratic Function

A function $f(x)$ defined by $f(x) = ax^2 + bx + c$, $a \neq 0$ where, a, b, c are constants is called quadratic function. (second degree of polynomial function)

Quadratic Equation

An equation of the form:

$$ax^2 + bx + c = 0 \text{ and } a \neq 0$$

where, a, b, c are constants is called quadratic equation or second degree polynomial equation.

Note

The quadratic equation has two roots. The two roots are:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Proof: Consider a quadratic equation.

$$ax^2 + bx + c = 0, a \neq 0$$

$$\text{or; } x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\text{or; } x^2 + 2x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{2c}{a} = 0$$

$$\text{or; } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\text{or; } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\text{or; } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\text{or; } \left(x + \frac{b}{2a}\right)^2 = \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2$$

$$\Rightarrow \left(x + \frac{b}{2a}\right) = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Nature of the roots of the equation

The expression $b^2 - 4ac$ is called discriminant of the quadratic equation. It determines the nature of the roots of the quadratic equation.

We have following possibilities:

- i) If $b^2 - 4ac > 0$, then roots are real and distinct.
- ii) If $b^2 - 4ac = 0$, then the roots are ^{real} imaginary and equal.
- iii) If $b^2 - 4ac < 0$, then roots are imaginary and distinct.
- iv) If a, b, c are rational and $(b^2 - 4ac)$ is perfect square then roots are rational.

Determine the nature of the following equation.

$$x^2 - 5x + 6 = 0$$

Here, $a = 1, b = -5, c = 6$

$$b^2 - 4ac = (-5)^2 - 4 \cdot 1 \cdot 6 = 25 - 24 = 1 > 0$$

Roots are real & distinct

Fundamental Theorem of Algebra

Every equation $f(x) = 0$ has at least one root, real or imaginary.

Note: ① Every polynomial equation of degree n has exactly n roots

② The quadratic equation has two roots.

Exercise 5.1

1. Determine the nature of the roots of the following equations.

a) $x^2 - 12x + 40 = 0$

Here, $a = 1$, $b = -12$, $c = 40$

$$b^2 - 4ac = (-12)^2 - 4 \cdot 1 \cdot 40$$

$$= 144 - 160$$

$$= -16 > 0 \quad -16 < 0$$

$\therefore$ Roots are imaginary and unequal

b) $x^2 - 4x - 3 = 0$

Here, $a = 1$, $b = -4$, $c = -3$

$$b^2 - 4ac = (-4)^2 - 4 \cdot 1 \cdot (-3)$$

$$= 16 + 12$$

$$= 28 > 0$$

$\therefore$ Roots are real and unequal. ^{Roots are} ~~not~~ also irrational

c) $2x^2 - 12x + 18 = 0$

Here, $a = 2$, $b = -12$, $c = 18$

$$b^2 - 4ac = (-12)^2 - 4 \cdot 2 \cdot 18$$

$$= 144 - 144$$

$$= 0 = 0$$

$\therefore$ Roots are real & equal.

Roots are rational.

d) $4x^2 + 8x - 5 = 0$
Here, $a = 4, b = 8, c = -5$

$$b^2 - 4ac = (8)^2 - 4 \cdot 4 \cdot (-5)$$

$$= 64 + 80$$
$$= 144 > 0$$

$\therefore$ Roots are real and unequal and rational.

e) $x^2 - 16 = 0$

Here;

$a = 1, b = 0, c = -16$

$$b^2 - 4ac = (0)^2 - 4 \cdot 1 \cdot (-16)$$

$$= 0 + 64$$

$$= 64 > 0$$

$\therefore$ Roots are real, unequal and rational.

2. For what value of the p will the equation $5x^2 - px + 45 = 0$ have the equal roots?

Solution

Given equation is

$$5x^2 - px + 45 = 0 \quad \dots \quad (1)$$

Comparing (1) with $ax^2 + bx + c = 0$

We get,

$$a = 5, b = -p, c = 45$$

Since, roots are real and unequal,

$$b^2 - 4ac = 0$$

$$\Rightarrow (-p)^2 - 4 \cdot 5 \cdot 45 = 0$$

$$\Rightarrow p^2 - 900 = 0$$

$$\Rightarrow p = \pm 30$$

3. Find the value of k so that the equation:

a) $x^2 + (k+2)x + 2k = 0$ has equal roots.

Solution

Here,

Given equation is,

$$x^2 + (k+2)x + 2k = 0 \quad \text{--- (1)}$$

Comparing (1) with $ax^2 + bx + c = 0$, we get,

$$a = 1, b = (k+2), c = 2k$$

Since the equation has equal roots,

$$b^2 - 4ac = 0$$

$$\text{or; } (k+2)^2 - 4 \cdot 1 \cdot 2k = 0$$

$$\text{or; } (k+2)^2 - 8k = 0$$

$$\text{or; } k^2 + 2 \cdot 4k + 4 - 8k = 0$$

$$\text{or; } k^2 - 4k + 4 = 0$$

$$\text{or, } \cancel{k^2 - 4k = -4}$$

$$\text{or; } \cancel{k(k-4) = -4}$$

$$\therefore \cancel{k = \pm 2}$$

$$\text{or; } k^2 - 2k - 2k + 4 = 0$$

$$\text{or; } k(k-2) - 2(k-2) = 0$$

$$\text{or; } (k-2)^2 = 0$$

$$\text{or; } k-2 = 0$$

$$\therefore k = 2$$

b) $x^2 - (2k-1)x - (k-1) = 0$ has equal roots.

Solution

Given equation is;

$$x^2 - (2k-1)x - (k-1) = 0 \quad \dots \textcircled{1}$$

Comparing $\textcircled{1}$ with $ax^2 + bx + c = 0$

$$a = 1, b = -2k+1, c = -k+1$$

Since the equation has equal roots,

$$b^2 - 4ac = 0$$

$$\text{or}; (-2k+1)^2 - 4 \cdot 1 \cdot (-k+1) = 0$$

$$\text{or}; (-2k+1)^2 + 4k - 4 = 0$$

$$\text{or}; 4k^2 - 4k + 1 + 4k - 4 = 0$$

$$\text{or}; 4k^2 - 3 = 0$$

$$\text{or}; 4k^2 = 3$$

$$\text{or}; k^2 = \frac{3}{4}$$

$$\therefore k = \pm \frac{\sqrt{3}}{2}$$

4. If the equation $(1+m^2)x^2 + 2mcx + c^2 - a^2 = 0$ has equal roots, show that $c^2 = a^2(1+m^2)$

Solution

Given equation is

$$(1+m^2)x^2 + 2mcx + c^2 - a^2 = 0 \quad \dots \textcircled{1}$$

Comparing $\textcircled{1}$ with $ax^2 + bx + c = 0$

$$a = (1+m^2), b = 2mc, c = c^2 - a^2$$

Since, the equation has equal roots,

$$b^2 - 4ac = 0$$

$$\text{or; } (2mc)^2 - 4 \cdot (1+m^2) \cdot (c^2 - a^2) = 0$$

$$\text{or; } 4m^2c^2 - (4+4m^2) \cdot (c^2 - a^2) = 0$$

$$\text{or; } 4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4m^2a^2 = 0$$

$$\text{or; } 4a^2 + 4m^2a^2 - 4c^2 = 0$$

$$\text{or; } 4(a^2 + m^2a^2 - c^2) = 0$$

$$\text{or; } a^2 + m^2a^2 - c^2 = 0$$

$$\text{or; } c^2 = a^2 + m^2a^2$$

$$\text{or; } c^2 = a^2(1+m^2).$$

proved.

5. Show that the roots of the equation $(a^2 - bc)x^2 + 2(b^2 - ca)x + (c^2 - ab) = 0$ will be equal if either $b = 0$ or $a^3 + b^3 + c^3 - 3abc = 0$.

Solution

Given equation is

$$(a^2 - bc)x^2 + 2(b^2 - ca)x + (c^2 - ab) = 0 \quad \dots \textcircled{1}$$

Comparing $\textcircled{1}$ with $ax^2 + bx + c = 0$,

$$a = (a^2 - bc), \quad b = 2(b^2 - ca), \quad c = (c^2 - ab)$$

Since, the equation has equal roots,

$$b^2 - 4ac = 0$$

$$\text{or; } 4(b^2 - ca)^2 - 4(a^2 - bc)(c^2 - ab) = 0$$

$$\text{or; } 4(b^2 - ca)^2 - 4(a^2c^2 - a^3b - bc^3 + ab^2c) = 0$$

$$\text{or; } 4b^4 - 8b^2ca + 4c^2a^2 - 4a^2c^2 - 4a^3b - 4bc^3 + 4ab^2c = 0$$

$$\text{or; } 4(b^4 - 2b^2ca + c^2a^2) - 4a^2c^2 - 4a^3b - 4bc^3 + 4ab^2c = 0$$

$$\text{or; } 4b^4 - 8b^2ca + 4c^2a^2 - 4a^2c^2 - 4a^3b - 4bc^3 + 4ab^2c = 0$$

$$\text{or; } 4b^4 - 4ab^2c - 4a^3b - 4bc^3 = 0$$

$$\text{or; } b^4 - ab^2c - a^3b - bc^3 = 0$$

$$\text{or; } b(b^3 - abc - a^3 - c^3) = 0$$

$$\text{or; } b^3$$

5. Show that the roots of the equation $(a^2 - bc)x^2 + 2(b^2 - ca)x + (c^2 - ab) = 0$ will be equal if either $b = 0$ or $a^3 + b^3 + c^3 - 3abc = 0$

Solution

Here;

Given equation is:

$$(a^2 - bc)x^2 + 2(b^2 - ca)x + (c^2 - ab) = 0 \quad \dots \textcircled{1}$$

Comparing $\textcircled{1}$ with $ax^2 + bx + c = 0$

$$a = (a^2 - bc), \quad b = 2(b^2 - ca), \quad c = c^2 - ab$$

Since, the equation has equal roots.

$$b^2 - 4ac = 0$$

$$\text{or; } [2(b^2 - ca)]^2 - 4(a^2 - bc)(c^2 - ab) = 0$$

$$\text{or; } (b^2 - ca)^2 - (a^2 - bc)(c^2 - ab) = 0$$

$$\text{or; } b^4 - 2b^2ca + c^2a^2 - (a^2c^2 - a^3b - bc^3 + ab^2c) = 0$$

$$\text{or; } b^4 - 2b^2ca + c^2a^2 - a^2c^2 + a^3b + bc^3 - ab^2c = 0$$

$$\text{Or; } b^4 - 2b^2ca + a^3b + bc^3 - ab^2c = 0$$

$$\text{or; } b^4 - ab^2c + a^3b + bc^3 = 0$$

$$\text{or; } b(b^3 - 3abc + a^3 + c^3) = 0$$

$$\text{or; } b^3 - 3abc + a^3 + c^3 = 0$$

Either,

$$b = 0$$

Or,

$$a^3 + b^3 + c^3 - 3abc = 0$$

proved

6. Prove that the roots of the equation $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ are real. Also, prove that the roots are equal if $a=b=c$.

Solution

Given equation is,

$$(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$$

$$\text{or } x^2 - bx - ax + ab + x^2 - cx - bx + bc + x^2 - ax - cx + ac = 0$$

$$\text{or } 3x^2 - 2bx - 2ax - 2cx + ab + bc + ac = 0$$

$$\text{or } 3x^2 - 2x(a+b+c) + ab + bc + ac = 0 \dots \dots \textcircled{1}$$

Comparing $\textcircled{1}$ with $ax^2 + bx + c = 0$

$$a = 3, b = -2(a+b+c), c = ab + bc + ac$$

Now;

$$b^2 - 4AC = [-2(a+b+c)]^2 - 4 \cdot 3 \cdot (ab + bc + ac)$$

$$= 4(a+b+c)^2 - 12(ab + bc + ac)$$

$$= 4[a^2 + b^2 + c^2 + 2(ab + bc + ca)] - 12(ab + bc + ac)$$

$$= 4[a^2 + 2ab + b^2 + 2(ac + bc) + c^2] - 12(ab + bc + ac)$$

$$= 4(a^2 + 2ab + b^2 + 2ac + 2bc + c^2) - 12(ab + bc + ac)$$

$$= 4a^2 + 8ab + 4b^2 + 8ac + 8bc + 4c^2 - 12ab - 12bc - 12ac$$

$$= 4a^2 + 4b^2 + 4c^2 - 4ab - 4bc - 4ac$$

$$= 2(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac)$$

$$= 2(a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2)$$

$$= 2[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0$$

$$\therefore b^2 - 4AC \geq 0$$

Hence, roots are real.

Again;

When, $a, b \in \mathbb{C}$

$$\begin{aligned} B^2 - 4AC &= 2[(a-b)^2 + (b-c)^2 + (c-a)^2] \\ &= 2(0+0+0) \\ &= 2 \times 0 \\ &= 0 \end{aligned}$$

$\therefore$ Roots are equal.

7. If the roots of the quadratic equation $(a^2+b^2)x^2 - 2(ac+bd)x + (c^2+d^2) = 0$ are equal then prove that

$$\frac{a}{b} = \frac{c}{d}$$

Solution

Here;

Given eqn is:

$$(a^2+b^2)x^2 - 2(ac+bd)x + (c^2+d^2) = 0 \quad \text{--- ①}$$

Comparing ① with $Ax^2 + Bx + C = 0$

$$A = (a^2+b^2), B = -2(ac+bd), C = (c^2+d^2)$$

Since, the equation has equal roots,

$$B^2 - 4AC = 0$$

$$\Rightarrow 4(ac+bd)^2 - 4(a^2+b^2)(c^2+d^2) = 0$$

$$\Rightarrow 4(ac+bd)(ac+bd) - (a^2+b^2)(c^2+d^2) = 0$$

$$\Rightarrow a^2c^2 + 2abcd + b^2d^2 - (a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2) = 0$$

$$\Rightarrow a^2c^2 + 2abcd + b^2d^2 - a^2c^2 - a^2d^2 - b^2c^2 - b^2d^2 = 0$$

$$\Rightarrow -a^2d^2 + 2abcd - b^2c^2 = 0$$

$$\Rightarrow -(a^2d^2 - 2abcd + b^2c^2) = 0$$

$$\Rightarrow -(ad-bc)^2 = 0$$

$$\Rightarrow (ad-bc)^2 = 0$$

$$\Rightarrow ad-bc = 0$$

$$\Rightarrow ad = bc$$

$$\Rightarrow a = \frac{bc}{d}$$

$$\Rightarrow \frac{a}{b} = \frac{c}{d} \text{ proved.}$$

8. If a, b, c are rational and $a+b+c=0$, show that the roots of $(b+c-a)x^2 + (c+a-b)x + (a+b-c) = 0$ are rational.

Solution

We have,

$$a+b+c=0$$

and Given equation is

$$(b+c-a)x^2 + (c+a-b)x + (a+b-c) = 0$$

Comparing with $Ax^2 + Bx + C = 0$, we get,

$$A = (b+c-a), B = (c+a-b), C = (a+b-c)$$

Now;

$$\begin{aligned} B^2 - 4AC &= (c+a-b)^2 - 4(b+c-a)(a+b-c) \\ &= (c+a+b-2b)^2 - 4(a+b+c-2a)(a+b+c-2c) \\ &= (0-2b)^2 - 4(0-2a)(0-2c) \quad [\because a+b+c=0] \\ &= (-2b)^2 - 4(-2a)(-2c) \\ &= 4b^2 - 16ac \\ &= 4(b^2 - 4ac) \\ &= 4[(-a+c)^2 - 4ac] \quad [\because b = -(a+c)] \\ &= 4[(a+c)^2 - 4ac] \\ &= 4(a-c)^2 \\ &= (2(a-c))^2 \end{aligned}$$

which is perfect square.

This shows that roots of given equation are rational.

9. Prove that the roots of the equation $(x-a)(x-b) = k^2$ are real for all value of k .

Solution

Here the given equation is,

$$(x-a)(x-b) = k^2$$

$$\text{or, } x^2 - bx - ax + ab = k^2$$

$$\text{or, } x^2 - bx - ax + ab - k^2 = 0$$

$$\text{or, } x^2 - (a+b)x + (ab - k^2) = 0 \quad \dots \text{--- (1)}$$

Comparing (1) with $Ax^2 + Bx + C = 0$, we get,
 $A = 1$, $B = -(a+b)$, $C = (ab - k^2)$

Now,

$$B^2 - 4AC = [-(a+b)]^2 - 4 \cdot 1 \cdot (ab - k^2)$$

$$= (a+b)^2 - 4(ab - k^2)$$

$$= a^2 + 2ab + b^2 - 4ab + 4k^2$$

$$= (a-b)^2 + 4k^2 \geq 0 \text{ which is true for all values of } k.$$

$\therefore$ Roots of the equation are real.

10. Show that the roots of the quadratic equation $(b-c)x^2 + 2(c-a)x + (a-b) = 0$ are always real.

Solution

Given equation is

$$(b-c)x^2 + 2(c-a)x + (a-b) = 0 \quad \dots \text{--- (1)}$$

Comparing (1) with $Ax^2 + Bx + C = 0$, we get.

$$A = (b-c), B = 2(c-a), C = (a-b)$$

Now,

Since the roots of the given equation are

$$B^2 - 4AC = [2(c-a)]^2 - 4 \cdot (b-c) \cdot (a-b)$$

$$= 4(c-a)^2 - 4(ba - b^2 - ca + bc)$$

$$= 4[c^2 - 2ac + a^2 - (ba - b^2 - ca + bc)]$$

$$= 4[c^2 - 2ac + a^2 - ba + b^2 + ac - bc]$$

$$= 4[c^2 - ac + a^2 - ba + ac - bc + b^2]$$

$$= 4[c^2 - 2ab + a^2] - 4(ab - ac - b^2 + bc)$$

$$= 4[c^2 - 2abc + a^2 - Cab - ac - b^2 + bc]$$

$$= 4[c^2 - 2ac + a^2 - ab + ac + b^2 - bc]$$

$$= 4[c^2 - ac + a^2 - ab + b^2 - bc]$$

$$= 2[2c^2 - 2ac + 2a^2 - 2ab + 2b^2 - 2bc]$$

$$= 2[c^2 + c^2 + a^2 + a^2 + b^2 + b^2 - 2ac - 2ab - 2bc]$$

$$= 2[(c^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (a^2 - 2ac + a^2)]$$

$$= 2[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0$$

$\therefore B^2 - 4AC \geq 0$ Hence, roots are always real.

11. Show that the roots of the equation $x^2 + (2m-1)x + m^2 = 0$ are real if $m \leq \frac{1}{4}$.

Solution

Given equation is

$$x^2 + (2m-1)x + m^2 = 0 \quad \dots \textcircled{1}$$

Comparing $\textcircled{1}$ with $Ax^2 + Bx + C = 0$, we get,

$$A=1, B=(2m-1), C=m^2$$

We know that the roots are real only iff

$$B^2 - 4AC \geq 0$$

$$\Leftrightarrow (2m-1)^2 - 4 \cdot 1 \cdot m^2 \geq 0$$

$$\Leftrightarrow 4m^2 - 4m + 1 - 4m^2 \geq 0$$

$$\Leftrightarrow -4m + 1 \geq 0$$

$$\Leftrightarrow 1 \geq 4m$$

$$\Leftrightarrow \frac{1}{4} \geq m$$

$$\text{i.e. } m \leq \frac{1}{4}$$

proved.

12. Show that the roots of the equation $x^2 + 4abx + (a^2 + 2b^2)^2 = 0$ are imaginary.

Solution

Here, the given equation is

$$x^2 + 4abx + (a^2 + 2b^2)^2 = 0 \quad \text{--- (1)}$$

Comparing (1) with $Ax^2 + Bx + C = 0$, we get,

$$A = 1, B = 4ab, C = (a^2 + 2b^2)^2$$

Now,

$$B^2 - 4AC = (4ab)^2 - 4 \cdot 1 \cdot (a^2 + 2b^2)^2$$

$$= 16a^2b^2 - 4(a^4 + 2a^2b^2 + 4b^4)$$

$$= 16a^2b^2 - 4a^4 - 8a^2b^2 - 16b^4$$

$$= -4a^4 - 16b^4$$

$$= -4(a^4 + 4b^4) \leq 0$$

Hence, the roots are imaginary.

13. If the roots of the equation $qx^2 + 2px + 2q = 0$ are real and unequal, prove that the roots of the equation $(p+q)x^2 + 2qx + (p-q) = 0$ are imaginary.

Solution

Here, the given equation is

$$qx^2 + 2px + 2q = 0 \quad \dots \textcircled{1}$$

Comparing $\textcircled{1}$ with $Ax^2 + Bx + C = 0$ we get,

$$A = q, B = 2p, C = 2q.$$

Now;

$$B^2 - 4AC = (2p)^2 - 4 \cdot q \cdot 2q$$

Since, the roots are real and unequal,

$$B^2 - 4AC > 0$$

$$\text{or; } (2p)^2 - 4 \cdot q \cdot 2q > 0$$

$$\text{or; } 4p^2 - 8q^2 > 0 \quad \dots \textcircled{II}$$

$$\text{or; } 4(p^2 - 2q^2) > 0$$

Now;

Another equation is

$$(p+q)x^2 + 2qx + (p-q) = 0$$

Comparing with $Ax^2 + Bx + C = 0$, we get,

$$A = (p+q), B = 2q, C = (p-q)$$

$$B^2 - 4AC = (2q)^2 - 4 \cdot (p+q) \cdot (p-q)$$

$$= (2q)^2 - 4(p^2 - q^2)$$

$$= 4q^2 - 4p^2 + 4q^2$$

$$= -4p^2 + 8q^2$$

$$= -(4p^2 - 8q^2) < 0 \quad [\text{since, } 4p^2 - 8q^2 < 0]$$

$\therefore$ The roots are imaginary.

14. If the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ be equal then show that a, b, c are in H.P. (i.e. $b(a+c) = 2ac$)

Solution

Given equation is:

$$a(b-c)x^2 + b(c-a)x + c(a-b) = 0 \quad \text{--- (1)}$$

Comparing (1) with $Ax^2 + Bx + C = 0$ we get,
 $A = a(b-c)$, $B = b(c-a)$, $C = c(a-b)$

Since, roots are equal. So,

$$B^2 - 4AC = 0$$

$$\Rightarrow (b(c-a))^2 - 4 \cdot a(b-c) \cdot c(a-b) = 0$$

$$\Rightarrow (bc-ab)^2 - 4ac(ab-b^2-ac+bc) = 0$$

$$\Rightarrow b^2c^2 - 2ab^2c + a^2b^2 - 4a^2bc + 4ab^2c + 4a^2c^2 - 4abc^2 = 0$$

$$\Rightarrow b^2c^2 + 2ab^2c + a^2b^2 - 4abc(a+c) + 4a^2c^2 = 0$$

$$\Rightarrow (bc)^2 + 2 \cdot (bc) \cdot (ab) + (ab)^2 - 4abc(a+c) + 4a^2c^2 = 0$$

$$\Rightarrow (bc+ab)^2 - 2 \cdot 2ac(ab+bc) + (2ac)^2 = 0$$

$$\Rightarrow [(bc+ab) - 2ac]^2 = 0$$

$$\Rightarrow (bc+ab) - 2ac = 0$$

$$\Rightarrow b(c+a) - 2ac = 0$$

$$\Rightarrow b(c+a) = 2ac$$

$$\therefore b = \frac{2ac}{a+c}$$

i.e. a, b, c are in H.P.

Proved