## Chapter 8
## Trigonometry

~~Some basic formulae~~

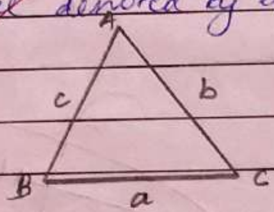
# Properties of Triangle

A triangle has 6 parts:

(a) The 3 sides and

(b) The 3 angles.

The sides opposite to the angle A, B, and C of triangle ABC are denoted by a, b and c respectively




Triangles

↓ Right angled (Pythagoras th.)

↓ Non-right angled ↳ acute or obtuse (sine/cosine or using laws)

# Sine law :-

In any triangle ABC, the sides are proportional to the sines of the opposite angle

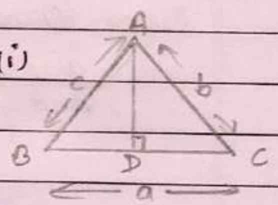
i.e. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

or

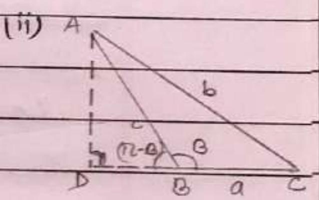
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof:-

Let ABC be a triangle, draw AD perpendicular to BC or BC produced to AD.




Then we have two cases: all angles acute and one angle is obtuse as shown in figures beside.




In figure (i) [ABC is acute triangle]

In $\Delta ADC$, $\sin C = \frac{AD}{AC}$

$\Rightarrow \sin C = \frac{AD}{b}$

$\therefore AD = b \sin C$ ——— (i)

In $\Delta ADB$, $\sin B = \frac{AD}{AB}$

$\Rightarrow \sin B = \frac{AD}{c}$

$\therefore AD = c \sin B$ ——— (ii)

Combining (i) and (ii),

$$b \sin C = c \sin B$$

$$\Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{——— (A)}$$

Similarly if we draw perpendicular from B to AC, we get

$$\frac{a}{\sin A} = \frac{c}{\sin C} \quad \text{——— (B)}$$

From (A) and (B)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In figure (ii)

In $\triangle ADC$, $\sin C = \frac{AD}{AC}$

$$\Rightarrow \sin C = \frac{AD}{b}$$

$$\Rightarrow AD = \cancel{AD}\, b \sin C \quad \text{——— (iii)}$$

In $\triangle ADB$, $\sin \angle ABD = \frac{AD}{AB}$

$$\Rightarrow \sin(\pi - B) = \frac{AD}{c}$$

$$\Rightarrow \sin B = \frac{AD}{c}$$

$$\Rightarrow AD = c \sin B \quad \text{——— (iv)}$$

From (iii) and (iv)

$$b \sin C = c \sin B$$

$$\Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C} \text{ ——— (v)}$$

Similarly, by drawing perpendicular from B to AC, we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C} \text{ ——— (vi)}$$

From (v) and (vi)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

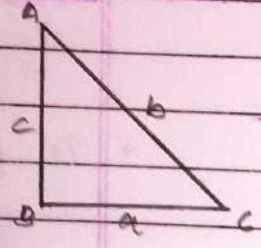
Hence, in either cases

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

(Proved)

Note:-

For right-angled triangle,




since

$$\angle B = 90^\circ$$

$$\therefore \sin B = 1 \text{ ——— (a)}$$

and $\sin A = \frac{BC}{AC}$

$$\Rightarrow \sin A = \frac{a}{b}$$

$$\Rightarrow b = \frac{a}{\sin A} \text{ ——— (*)}$$

Similarly,

$$\sin C = \frac{AB}{AC} = \frac{c}{b}$$

$$\Rightarrow b = \frac{c}{\sin C} \quad \text{———} (**)$$

From (*) and (**)

$$\frac{a}{\sin A} = \frac{c}{\sin C} = b$$

$$\text{or, } \frac{a}{\sin A} = \frac{c}{\sin C} = \frac{b}{1}$$

$$\text{or, } \frac{a}{\sin A} = \frac{c}{\sin C} = \frac{b}{\sin B} \quad [\because \sin B = 1, \text{ from } ⓐ]$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

(Proved)

Note:-

* To find angle, we use

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

* To find sides, we use

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

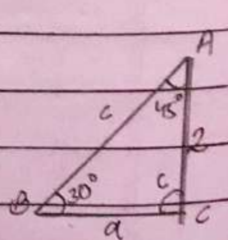
* $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

~~where R is radius of~~ circum circle

i.e. $a = 2R\sin A$, $b = 2R\sin B$, & $c = 2R\sin C$.

Ex. Find the remaining sides.

Sol; From figure,




$$45° + 30° + \measuredangle C = 180°$$

$$\Rightarrow \measuredangle C = 180° - 75°$$

$$\therefore \measuredangle C = 105°$$

By sine law,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

So, Putting values

$$\frac{a}{\sin 45^\circ} = \frac{2}{\sin 30^\circ} = \frac{c}{\sin 105^\circ}$$

$$\Rightarrow \frac{a}{1/\sqrt{2}} = \frac{2}{1/2} = \frac{c}{0.965}$$

Taking first two ratios,

$$\frac{a}{1/\sqrt{2}} = \frac{2}{1/2}$$

$$\Rightarrow a = \frac{1}{\sqrt{2}} \times 4$$

$$= 4/\sqrt{2}$$

$$= \frac{2 \times \sqrt{2} \times \cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$$

$$\therefore a = 2\sqrt{2}$$

Taking second and third ratios

$$\frac{2}{\sin 30^\circ} = \frac{c}{\sin 105^\circ}$$

$$\Rightarrow c = \frac{\sin 105^\circ \times 2}{\sin 30^\circ}$$

$$= \frac{0.965 \times 2}{1/2}$$

$$= 0.965 \times 4$$

$$= 3.36$$

$$\therefore a = 2\sqrt{2},\ c = 3.36$$

# Cosine law

For any triangle ABC,

(a) $a^2 = b^2 + c^2 - 2bc\cos A$

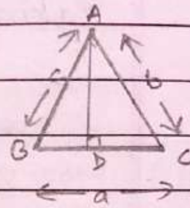
(b) $b^2 = c^2 + a^2 - 2ac\cos B$

(c) $c^2 = a^2 + b^2 - 2ab\cos C$

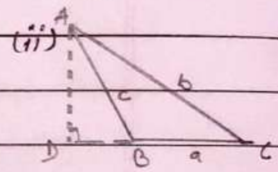
Proof:

Let ABC be any triangle. From A draw perpendicular AD to BC or produced BC

(i)




(ii)




From figure (i) :

In $\Delta ADB$, $\cos B = \dfrac{BD}{AB}$

$\Rightarrow BD = AB\cos B$

$\therefore BD = c\cos B$ ——— (i)

In $\Delta ADC$, $\cos C = \dfrac{CD}{AC}$

$\Rightarrow CD = AC\cos C$

$\therefore CD = b\cos C$ ——— (ii)

Now,

from $\Delta ADB$ (By using pythagoras theorem)

$(AB)^2 = (AD)^2 + (BD)^2$

$= (AD)^2 + (BC - CD)^2$ $[\because BD = BC - CD]$

$= (AD)^2 + (BC)^2 - 2(BC)(CD) + (CD)^2$

$= \underline{(AD)^2 + (CD)^2} + (BC)^2 - 2(BC)(CD)$

$= (AC)^2 + (BC)^2 - 2(BC)(CD)$ $\left[\begin{array}{l}\text{From } \Delta ADC, \\ (AD)^2 + (CD)^2 = (AC)^2\end{array}\right]$

$= b^2 + a^2 - 2a \cdot b \cos C$

$\therefore c^2 = a^2 + b^2 - 2ab\cos C$

Similarly, by drawing perpendiculars from B and C on opposite sides, we get

$a^2 = b^2 + c^2 - 2bc\cos A$ and

$b^2 = a^2 + c^2 - 2ac\cos B$

From figure (ii) :

In $\Delta ADC$, $\cos C = \frac{CD}{AC}$

$\Rightarrow CD = AC\cos C$

$\therefore CD = b\cos C$ ——— (iii)

In $\Delta ADB$, $\cos(\pi - B) = \frac{BD}{AB}$

$\Rightarrow BD = AB\cos(\pi - B)$

$\Rightarrow BD = c(-\cos B)$

$\therefore BD = -c\cos B$ ——— (iv)

Now,

from $\Delta ADB$, by using pythagoras theorem:

$(AB)^2 = (AD)^2 + (BD)^2$

$= (AD)^2 + (CD - BC)^2$ $[\because BD = CD - BC]$

$= (AD)^2 + (CD)^2 - 2(CD)(BC) + (BC)^2$

$= (AC)^2 - 2(CD)(BC) + (BC)^2$ $[\because (AD)^2 + (CD)^2 = (AC)^2]$

$= b^2 - 2b\cos C \cdot a + a^2$

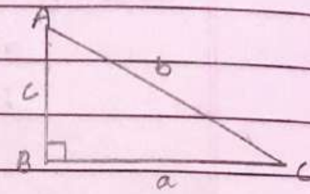
$\therefore c^2 = a^2 + b^2 - 2ab\cos C$

Similarly, by drawing perpendicular from B and C on opposite sides, we obtain

$$b^2 = c^2 + a^2 - 2ac\cos B \quad \text{and}$$
$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{(Proved)}$$

Note :

For right angled triangle




By using pythagoras theorem,

$$(AC)^2 = (AB)^2 + (BC)^2$$
$$\Rightarrow b^2 = c^2 + a^2$$
$$\Rightarrow b^2 = c^2 + a^2 - 2ac\cos B \quad \left[\angle B = 90^\circ, \cos B = 0, \; 2ac\cos B = 0\right]$$

Similarly, we can show,

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{and}$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

(Proved)

Note :- (cosine law)

(a) $a^2 = b^2 + c^2 - 2bc\cos A$

$\Rightarrow 2bc\cos A = b^2 + c^2 - a^2$

i.e. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

(b) $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

(c) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

# Projection law

In any triangle ABC,

(a) $a = b\cos C + c\cos B$

(b) $b = c\cos A + a\cos C$

(c) $c = a\cos B + b\cos A$

Proof:

(a) By sine law, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\Rightarrow \frac{a}{\sin A} = 2R$$

$$\Rightarrow a = 2R\sin A$$

$$= 2R\sin(\pi - (B+C)) \quad [\angle A + \angle B + \angle C = 180^\circ (\pi)]$$

$$= 2R\sin(B+C) \quad [\sin(180^\circ - \theta) \Rightarrow \sin\theta]$$

$$= 2R(\sin B\cos C + \cos B\sin C)$$

$$= 2R\sin B\cos C + 2R\cos B\sin C$$

$$= 2R\sin B\cos C + 2R\cancel{\cos B}\sin C\cos B$$

$$= b\cos C + c\cos B \quad [\because b = 2R\sin B \;\&\; c = 2R\sin C$$

$$\therefore a = b\cos C + c\cos B \;//$$

(b) By sine law, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\Rightarrow \frac{b}{\sin B} = 2R$$

$$\Rightarrow b = 2R\sin B$$

$$= 2R\sin(\pi - (A+C))$$

$$= 2R\sin(A+C)$$

$$= 2R(\sin A\cos C + \cos A\sin C)$$
$$= 2R\sin A\cos C + 2R\sin C\cos A$$
$$\therefore b = a\cos C + c\cos A$$

(c) By sine law, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\Rightarrow \frac{c}{\sin C} = 2R$$

$$\Rightarrow c = 2R\sin C$$
$$= 2R\sin(\pi - (A+B))$$
$$= 2R\sin(A+B)$$
$$= 2R(\sin A\cos B + \cos A\sin B)$$
$$= 2R\sin A\cos B + 2R\sin B\cos A$$
$$\therefore c = a\cos B + b\cos A$$

(Proved)

# Tangent law:

In any triangle,

(a) $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$

(b) $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

(c) $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$

Proof:

(a)

$$RHS = \frac{b-c}{b+c}\cot\frac{A}{2}$$

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$$= \frac{2R\sin B - 2R\sin C}{2R\sin B + 2R\sin C} \cot\frac{A}{2}$$

$$= \frac{2R(\sin B - \sin C)}{2R(\sin B + \sin C)} \cot\frac{A}{2}$$

$$= \frac{\sin B - \sin C}{\sin B + \sin C} \cot\frac{A}{2}$$

$$= \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)} \cot\frac{A}{2}$$

$$= \cot\left(\frac{B+C}{2}\right)\tan\left(\frac{B-C}{2}\right)\cot\frac{A}{2}$$

$$= \cot\left(\frac{\pi - A}{2}\right)\tan\left(\frac{B-C}{2}\right)\cot\frac{A}{2} \quad \left[\begin{array}{l} A+B+C=\pi, \\ \Rightarrow B+C=\pi-A \end{array}\right]$$

$$= \cot\left(\frac{\pi}{2} - \frac{A}{2}\right)\tan\left(\frac{B-C}{2}\right)\cot\frac{A}{2}$$

$$= \tan\frac{A}{2}\tan\frac{B-C}{2}\cot\frac{A}{2} \quad \left[\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta\right]$$

$$= 1 \times \tan\left(\frac{B-C}{2}\right) \quad [\because \tan\theta\cdot\cot\theta = 1]$$

$$= \tan\left(\frac{B-C}{2}\right)$$

$= \cancel{RH}$ LHS (proved)

b) $RHS = \frac{a-b}{a+b}\cot\frac{C}{2}$

$$= \frac{2R\sin A - 2R\sin B}{2R\sin A + 2R\sin B}\cot\frac{C}{2}$$

$$= \frac{\sin A - \sin B}{\sin A + \sin B}\cot\frac{C}{2}$$

$$= \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}\cot\frac{C}{2}$$

$$= \cot\left(\frac{A+B}{2}\right)\tan\left(\frac{A-B}{2}\right)\cot\frac{C}{2}$$

$$= \cot\left(\frac{\pi - C}{2}\right) \tan\left(\frac{A-B}{2}\right) \cot\frac{C}{2}$$

$$= \cancel{\cot} \tan\frac{C}{2} \tan\frac{A-B}{2} \cot\frac{C}{2}$$

$$= 1 \times \tan\frac{A-B}{2}$$

$$= \tan\frac{A-B}{2}$$

$$= \text{LHS} \quad \text{(proved)}$$

c)

$$\text{RHS} = \frac{c-a}{c+a} \cot\frac{B}{2}$$

$$= \frac{2R\sin C - 2R\sin A}{2R\sin C + 2R\sin A} \cdot \cot\frac{B}{2}$$

$$= \frac{\sin C - \sin A}{\sin C + \sin A} \cot\frac{B}{2}$$

$$= \frac{2\cos\frac{C+A}{2}\sin\frac{C-A}{2}}{2\sin\frac{C+A}{2}\cos\frac{C-A}{2}} \cot\frac{B}{2}$$

$$= \cot\frac{C+A}{2} \tan\frac{C-A}{2} \cot\frac{B}{2}$$

$$= \tan\frac{B}{2} \tan\frac{C-A}{2} \cot\frac{B}{2}$$

$$= 1 \times \tan\frac{C-A}{2}$$

$$= \tan\frac{C-A}{2}$$

$$= \text{LHS} \quad \text{(proved)}$$

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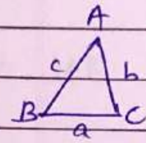
# Half angle formulae:-

Here,




$$2s = a+b+c$$

$$\Rightarrow s = \frac{a+b+c}{2} \text{, semi-perimeter.}$$

1(a) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

(b) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

(c) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

2(a) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$

$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

3(a) $\tan\frac{A}{B} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

(b) $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

(c) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Proving formulae:

1a) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

$\cos A = 1 - 2\sin^2\frac{A}{2}$
or
$= 2\cos^2\frac{A}{2} - 1$

Proof:
We know that,
$\cos A = 1 - 2\sin^2\frac{A}{2}$

or, $2\sin^2\frac{A}{2} = 1 - \cos A$

or, $\sin^2\frac{A}{2} = \frac{1}{2}(1 - \cos A)$

$= \frac{1}{2}\left(1 - \frac{b^2+c^2-a^2}{2bc}\right)$ $\left[\cos A = \frac{b^2+c^2-a^2}{2bc}\right]$

$= \frac{1}{2}\left(\frac{2bc - b^2 - c^2 + a^2}{2bc}\right)$

$= \frac{1}{4bc}\left(a^2 - (b^2+c^2-2bc)\right)$

$= \frac{1}{4bc}\left(a^2 - (b-c)^2\right)$

$= \frac{1}{4bc}(a-b+c)(a+b-c)$

$= \frac{1}{4bc}(a+b+c-2b)(a+b+c-2c)$

$= \frac{1}{4bc}(2s-2b)(2s-2c)$

$= \frac{\cancel{4}(s-b)(s-c)}{\cancel{4}bc}$

or, $\sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$

$\therefore \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (proved)

b) $\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}$

Proof:

We know that,

$\cos B = 1 - 2\sin^2\frac{B}{2}$

or, $2\sin^2\frac{B}{2} = 1 - \cos B$

or, $\sin^2\frac{B}{2} = \frac{1}{2}(1 - \cos B)$

$= \frac{1}{2}\left(1 - \frac{a^2+c^2-b^2}{2ac}\right)$

$= \frac{1}{2}\left(\frac{2ac-a^2-c^2+b^2}{2ac}\right)$

$= \frac{1}{4ac}\left(b^2 - \{a^2+c^2-2ac\}\right)$

$= \frac{1}{4ac}\{(b)^2 - (a-c)^2\}$

$= \frac{1}{4ac}(b-a+c)(b+a-c)$

$= \frac{1}{4ac}(a+b+c-2a)(a+b+c-2c)$

$= \frac{1}{4ac}(2s-2a)(2s-2c)$

$= \frac{\cancel{4}}{\cancel{4}ac}(s-a)(s-c)$

or, $\sin^2\frac{B}{2} = \frac{(s-a)(s-c)}{ac}$

$\therefore \sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-a)}{ac}}$

(Proved)

2a) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

Proof:

we know that

$\cos A = 2\cos^2\frac{A}{2} - 1$

$\Rightarrow 2\cos^2\frac{A}{2} = 1+\cos A$

$\Rightarrow \cos^2\frac{A}{2} = \frac{1}{2}(1+\cos A)$

$= \frac{1}{2}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$

$= \frac{1}{2}\left\{\frac{(b^2+2bc+c^2)-a^2}{2bc}\right\}$

$= \frac{1}{4bc}\left\{(b+c)^2-(a)^2\right\}$

$= \frac{1}{4bc}(b+c-a)(b+c+a)$

$= \frac{1}{4bc}(a+b+c-2a)(a+b+c)$

$= \frac{1}{4bc}(2s-2a)2s$

$= \frac{4}{4bc}\cdot s(s-a)$

or, $\cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$

$\therefore \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(Proved)

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3a) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Proof:
We know,

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

$$= \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$= RHS.$

(Proved),

# Note

In any triangle ABC:

(i) The area of $\Delta = \frac{1}{2} ab \sin C$
or
$= \frac{1}{2} bc \sin A$
or
$= \frac{1}{2} ac \sin B$

(ii) Area of $\Delta = \frac{abc}{4R}$ $\left[\because \Delta = \frac{1}{2} ab \sin C = \frac{1}{2} ab \frac{c}{2R} = \frac{abc}{4R}\right]$

(iii) ~~S~~ Area of $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

(iv) Area of $\Delta = \frac{1}{4}\sqrt{2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4}$

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Exercise 8.1

1. In any $\Delta ABC$, prove that

a) $2a \sin\frac{B}{2} \cdot \sin\frac{C}{2} = (b+c-a)\sin\frac{A}{2}$

Sol; Proof

We have,

$$LHS = 2a \sin\frac{B}{2} \sin\frac{C}{2}$$

$$= 2a \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= 2a \sqrt{\cancel{(s-a)\{(s-c)} = \frac{(s-a)^2(s-b)(s-c)}{a^2bc}}$$

$$= \frac{2a}{a}(s-a)\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$= (2s-2a)\sin\frac{A}{2}$$

$$= (a+b+c-2a)\sin\frac{A}{2}$$

$$= (b+c-a)\sin\frac{A}{2}$$

$= RHS$

(Proved)

b) $ac\cos B - bc\cos A = a^2 - b^2$

Sol; we have,

$$LHS = ac\cos B - bc\cos A$$

$$= \cancel{ac}\cdot\frac{a^2+c^2-b^2}{2\cancel{ac}} - \cancel{bc}\cdot\frac{b^2+c^2-a^2}{2\cancel{bc}}$$

$$= \frac{a^2+\cancel{c^2}-b^2-b^2-\cancel{c^2}+a^2}{2}$$

$$= \frac{\cancel{2}(a^2-b^2)}{\cancel{2}}$$

$$= a^2-b^2$$

$$= RHS$$

(Proved)

c) $a\cos^2\frac{B}{2} + b\cos^2\frac{A}{2} = s$

Sol; $LHS = a\cos^2\frac{B}{2} + b\cos^2\frac{A}{2}$

$$= \cancel{a}\,\frac{s(s-b)}{\cancel{a}c} + \cancel{b}\,\frac{s(s-a)}{\cancel{b}c}$$

$$= \frac{s(s-b+s-a)}{c}$$

$$= \frac{s(\cancel{a}+\cancel{b}+c-\cancel{a}-\cancel{b})}{c}$$

$$= s\,\frac{\cancel{c}}{\cancel{c}}$$

$$= s$$

$$= RHS$$

(Proved)

d) $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$

Sol; $LHS = (b+c)\cos A + (c+a)\cos B + (a+b)\cos C$

$$= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C$$

$$= b\cos A + a\cos B + c\cos A + a\cos C + c\cos B + b\cos C$$

$$= c+b+a$$

$$= a+b+c$$

$$= RHS$$

(Proved)

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e) $2\left(a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2}\right) = c + a - b$

Sol: $LHS = 2\left(a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2}\right)$

$= 2\left(a \frac{(s-a)(s-b)}{ab} + c \frac{(s-b)(s-c)}{bc}\right)$

$= 2\left\{\frac{(s-b)(s-a+s-c)}{b}\right\}$

$= 2\left\{\frac{(s-b)(a+b+\cancel{c}-\cancel{a}-\cancel{c})}{b}\right\}$

$= 2 \cdot \frac{\cancel{b}(s-b)}{\cancel{b}}$

$= 2s - 2b$

$= a+b+c-2b$

$= a-b+c$

$= c+a-b$

$= RHS$

(Proved)

2. Prove the followings:

a) $\frac{b-c}{a} \cos \frac{A}{2} = \sin \frac{B-C}{2}$

Sol: $LHS = \frac{b-c}{a} \cos \frac{A}{2}$

$= \frac{2R \sin B - 2R \sin C}{2R \sin A} \cos \frac{A}{2}$

$= \frac{\cancel{2R}(\sin B - \sin C)}{\cancel{2R} \sin A} \cos \frac{A}{2}$

$= \frac{2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}}{2 \sin \frac{A}{2} \cancel{\cos \frac{A}{2}}} \cancel{\cos \frac{A}{2}}$

$$= \frac{\cos\left(90-\frac{A}{2}\right)\sin\frac{B-C}{2}}{\sin\frac{A}{2}}$$

$$A+B+C=180$$
$$\frac{B+C}{2}=\frac{180}{2}-\frac{A}{2}$$
$$\Rightarrow \frac{B+C}{2}=90-\frac{A}{2}$$

$$= \frac{\cancel{\sin\frac{A}{2}}\sin\frac{B-C}{2}}{\cancel{\sin\frac{A}{2}}}$$

$$= \sin\frac{B-C}{2}$$

$= RHS$ (proved)

b) $\frac{c-b\cos A}{b-c\cos A} = \frac{\cos B}{\cos C}$

Sol: $LHS = \frac{c-b\cos A}{b-c\cos A}$

$$= \frac{a\cos B+\cancel{b\cos A}-\cancel{b\cos A}}{a\cos C+\cancel{c\cos A}-\cancel{c\cos A}}$$

$$= \frac{\cancel{a}\cos B}{\cancel{a}\cos C}$$

$$= \frac{\cos B}{\cos C}$$

$= RHS$ (proved)

c) $\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$

Sol: $LHS = \frac{a\sin(B-C)}{b^2-c^2}$

$$= \frac{2R\sin A\sin(B-C)}{b^2-c^2}$$

$$= R\,\frac{2\sin(B+C)\sin(B-C)}{b^2-c^2}$$

$$= R\,\frac{\cos(B+C-(B-C)) - \cos(B+C+(B-C))}{b^2-c^2}$$

$$2\sin A\sin B \Rightarrow \cos(A-B)-\cos(A+B)$$

$$= R\,\frac{\cos 2C-\cos 2B}{b^2-c^2}$$

$$= R\,\frac{1-2\sin^2 C-(1-2\sin^2 B)}{b^2-c^2} \qquad [\cos 2C = 1-2\sin^2 C]$$

$$= R\,\frac{2\sin^2 B-2\sin^2 C}{b^2-c^2}$$

$$= \frac{2R\,(\sin^2 B-\sin^2 C)}{(2R\sin B)^2-(2R\sin C)^2}$$

$$= \frac{\cancel{2}\cancel{R}\,\cancel{(\sin^2 B-\sin^2 C)}}{\cancel{4}2R^{\cancel{2}}\,\cancel{(\sin^2 B-\sin^2 C)}}$$

$$= \frac{1}{2R}$$

$$MHS = \frac{b\sin(C-A)}{c^2-a^2}$$

$$= \frac{2R\sin B\sin(C-A)}{c^2-a^2}$$

$$= \frac{2R\sin(C+A)\sin(C-A)}{c^2-a^2}$$

$$= R\,\frac{2\sin(C+A)\sin(C-A)}{c^2-a^2}$$

$$= R\,\frac{\cos(\cancel{C}+A-\cancel{C}+A)-\cos(C+A+C-\cancel{A})}{c^2-a^2} \qquad \boxed{2\sin A\sin B = \cos(A-B)-\cos(A+B)}$$

$$= R\,\frac{\cos 2A-\cos 2C}{c^2-a^2}$$

$$= R\,\frac{\cancel{1}-2\sin^2 A-\cancel{1}+2\sin^2 C}{c^2-a^2} \qquad [\cos 2A = 1-2\sin^2 A]$$

$$= \frac{2R\,\cancel{(\sin^2 C-\sin^2 A)}}{24R^2\,\cancel{(\sin^2 C-\sin^2 A)}}$$

$$= \frac{1}{2R}$$

$$RHS = \frac{c\sin(A-B)}{a^2-b^2}$$

$$= \frac{2R\sin C\sin A-B}{a^2-b^2}$$

$$= \frac{2R \sin(A+B) \sin(A-B)}{a^2 - b^2}$$

$$= R \frac{\cos(A+B-A+B) - \cos(A+B+A-B)}{a^2 - b^2}$$

$$= \frac{R \cos 2B - \cos 2A}{a^2 - b^2}$$

$$= R \frac{\cancel{1} - 2\sin^2 B - \cancel{1} + 2\sin^2 A}{(2R\sin A)^2 - (2R\sin B)^2}$$

$$= \frac{2R \, \cancel{(\sin^2 A - \sin^2 B)}}{4R^2 \, \cancel{(\sin^2 A - \sin^2 B)}}$$

$$= \frac{1}{2R}$$

Hence,

LHS = MHS = RHS

(proved)

3a) $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2} = \frac{1}{2}\, a+b+c$

$$LHS = b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}$$

$$= \cancel{b}\,\frac{s(s-c)}{a\cancel{b}} + \cancel{c}\,\frac{s(s-b)}{a\cancel{c}}$$

$$= \frac{s(s-c+s-b)}{a}$$

$$= \frac{s(a+\cancel{b}+\cancel{c}-\cancel{b}-\cancel{c})}{a}$$

$$= \frac{s\cancel{a}}{\cancel{a}}$$

$$= \frac{1}{2}\, a+b+c$$

= RHS (proved)

b. $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$

$LHS = a\sin(B-C) + b\sin(C-A) + c\sin(A-B)$

$= 2R\sin A\sin(B-C) + 2R\sin B\sin(C-A) + 2R\sin C\sin(A-B)$

$= 2R\sin(B+C)\sin(B-C) + 2R\sin(C+A)\sin(C-A) + 2R\sin(A+B)\sin(A-B)$

$= 2R(\sin^2 B - \sin^2 C) + 2R(\sin^2 C - \sin^2 A) + 2R(\sin^2 A - \sin^2 B)$

$[\sin(C+A)\sin(C-A) = \sin^2 C - \sin^2 A]$

$= 2R(\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B)$

$= 2R \times 0$

$= 0$

$= RHS$ (proved)

c. $\sin(A+B) : \sin(A-B) = c^2 : (a^2 - b^2)$

$\Rightarrow \dfrac{\sin(A+B)}{\sin(A-B)} = \dfrac{c^2}{a^2 - b^2}$

$LHS = \dfrac{\sin(A+B)}{\sin(A-B)}$

$= \dfrac{\sin(A+B)}{\sin(A-B)} \times \dfrac{\sin(A+B)}{\sin(A+B)}$

$= \dfrac{\sin^2(A+B)}{\sin(A-B)\sin(A+B)}$

$= \dfrac{\sin^2 C}{\sin^2 A - \sin^2 B}$

$= \dfrac{\left(\frac{c}{2R}\right)^2}{\left(\frac{a}{2R}\right)^2 - \left(\frac{b}{2R}\right)^2}$

$= \dfrac{\frac{c^2}{4R^2}}{\frac{a^2 - b^2}{4R^2}}$

$= \dfrac{c^2}{a^2 - b^2}$

$= RHS$ (proved)

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d. $1-\tan\frac{A}{2}\cdot\tan\frac{B}{2} = \frac{2c}{a+b+c}$

$$LHS = 1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$= 1-\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\times\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$= 1-\sqrt{\frac{(s-c)^2\cancel{(s-a)}\cancel{(s-b)}}{s^2\cancel{(s-a)}\cancel{(s-b)}}}$$

$$= 1-\frac{s-c}{s}$$

$$= \frac{\cancel{s}-\cancel{s}+c}{s}$$

$$= \frac{c}{\frac{a+b+c}{2}}$$

$$= \frac{2c}{a+b+c}$$

$$= RHS \text{ (Proved)},$$

e. $\frac{\sin(B-C)}{\sin(B+C)} = \frac{b^2-c^2}{a^2}$

$$LHS = \frac{\sin(B-C)}{\sin(B+C)}$$

$$= \frac{\sin(B-C)}{\sin(B+C)}\times\frac{\sin(B+C)}{\sin(B+C)}$$

$$= \frac{\sin(B-C)\sin(B+C)}{\sin^2(B+C)}$$

$$= \frac{\sin^2 B-\sin^2 C}{\sin^2 A}$$

$$= \frac{\frac{b^2-c^2}{4R^2}}{\frac{a^2}{4R^2}}$$

$$= \frac{b^2-c^2}{a^2} = RHS \text{ (Proved)},$$

4a. $\frac{a+b-c}{a+b+c} = \tan\frac{A}{2}\tan\frac{B}{2}$

$RHS = \tan\frac{A}{2}\tan\frac{B}{2}$

$= \sqrt{\frac{\cancel{(s-b)}(s-c)\cancel{(s-a)}(s-c)}{s\cancel{(s-a)}\cdot s\cancel{(s-b)}}}$

$= \sqrt{\frac{(s-c)^2}{s^2}}$

$= \frac{s-c}{s}$

$= \frac{\frac{a+b+c}{2}-c}{\frac{a+b+c}{2}}$

$= \frac{a+b+c-2c}{a+b+c}$

$= \frac{a+b-c}{a+b+c}$

$= LHS$ (Proved)

b. $\frac{b^2+c^2-a^2}{4\cot A} = \Delta$

$LHS = \frac{b^2+c^2-a^2}{4\cot A}$

$= \frac{b^2+c^2-a^2}{4\cdot\cancel{\frac{\sin A}{\cos A}}\frac{\cos A}{\sin A}}$

$= \frac{b^2+c^2-a^2}{4} \times \frac{\sin A}{\cos A}$

$= \frac{\cancel{(b^2+c^2-a^2)}}{4} \times \frac{a}{2R} \times \frac{2bc}{\cancel{(b^2+c^2-a^2)}}$

$= \frac{abc}{2R}$

$= \Delta = RHS$ (Proved)

c. $\dfrac{\cos A}{a} + \dfrac{a}{bc} = \dfrac{\cos B}{b} + \dfrac{b}{ca} = \dfrac{\cos C}{c} + \dfrac{c}{ab}$

$$LHS = \frac{\cos A}{a} + \frac{a}{bc}$$

$$= \frac{bc\cos A + a^2}{abc}$$

$$= \frac{\cancel{bc} \cdot \frac{b^2+c^2-a^2}{2\cancel{bc}} + a^2}{abc}$$

$$= \frac{b^2+c^2-a^2+2a^2}{2abc}$$

$$= \frac{a^2+b^2+c^2}{2abc}$$

$$MHS = \frac{\cos B}{b} + \frac{b}{ca}$$

$$= \frac{ca\cos B + b^2}{abc}$$

$$= \frac{\cancel{ca} \cdot \frac{a^2+c^2-b^2}{2\cancel{ac}} + b^2}{abc}$$

$$= \frac{a^2+c^2-b^2+2b^2}{2abc}$$

$$= \frac{a^2+b^2+c^2}{2abc}$$

$$RHS = \frac{\cos C}{c} + \frac{c}{ab}$$

$$= \frac{ab\cos C + c^2}{abc} = \frac{ab\left(\frac{a^2+b^2-c^2}{2ab}\right) + c^2}{abc}$$

$$= \frac{a^2+b^2-c^2+2c^2}{2abc}$$

$$= \frac{a^2+b^2+c^2}{2abc}$$

$\therefore LHS = MHS = RHS$ (Proved)

d) $\dfrac{a^2\sin(B-C)}{\sin A} + \dfrac{b^2\sin(C-A)}{\sin B} + \dfrac{c^2\sin(A-B)}{\sin C} = 0$

$$LHS = \frac{a^2\sin(B-C)}{\sin A} + \frac{b^2\sin(C-A)}{\sin B} + \frac{c^2\sin(A-B)}{\sin C}$$

$$= \frac{(2R\sin A)^2\sin(B-C)}{\sin A} + \frac{(2R\sin B)^2\sin(C-A)}{\sin B} + \frac{(2R\sin C)^2\sin(A-B)}{\sin C}$$

$$= \frac{4R^2\sin^{\cancel{2}}A\sin(B-C)}{\cancel{\sin A}} + \frac{4R^2\sin^{\cancel{2}}B\sin(C-A)}{\cancel{\sin B}} + \frac{4R^2\sin^{\cancel{2}}C\sin(A-B)}{\cancel{\sin C}}$$

$$= 4R^2\{\sin(B+C)\sin(B-C) + \sin(C+A)\sin(C-A) + \sin(A+B)\sin(A-B)\}$$

$$= 4R^2\{\sin^2B - \cancel{\sin^2C} + \cancel{\sin^2C} - \cancel{\sin^2A} + \cancel{\sin^2A} - \sin^2B\}$$

$$= 4R^2 \times 0$$

$$= 0$$

$= RHS$ (Proved)

5a. $\dfrac{1}{s-a} + \dfrac{1}{s-b} + \dfrac{1}{s-c} - \dfrac{1}{s} = \dfrac{4R}{\Delta}$

$$LHS = \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s}$$

$$= \frac{s(s-b)(s-c) + s(s-a)(s-c) + s(s-a)(s-b) - (s-a)(s-b)(s-c)}{s(s-a)(s-b)(s-c)}$$

$$= \frac{s\{s^2 - sc - sb + bc + s^2 - sc + ac - as + s^2 - sb - as + ab - (s-a)(s^2 - sc - sb + bc)}{s(s-a)(s-b)(s-c)}$$

$$= s\{3s^2 - 2sc - 2sb + bc + ac - 2as + ab - s^3 + s^2c$$

$$= \frac{1}{s-a} - \frac{1}{s} + \frac{1}{s-b} + \frac{1}{s-c}$$

$$= \frac{s - s + a}{s(s-a)} + \frac{s - c + s - b}{(s-b)(s-c)}$$

$$= \frac{a}{s(s-a)} + \frac{2s - b - c}{(s-b)(s-c)}$$

$$= \frac{a(s-b)(s-c) + (a + b + c - b - c)(s(s-a))}{s(s-a)(s-b)(s-c)}$$

$$= \frac{a\,\{(s-b)(s-c)+s(s-a)\}}{\left(\sqrt{s(s-a)(s-b)(s-c)}\right)^2}$$

$$= \cancel{a\{s^2}$$

$$= \frac{a\left\{\frac{(s-b)(s-c)}{bc}\times bc + \frac{s(s-a)}{bc}\times bc\right\}}{\Delta^2}$$

$$= \frac{a\,(\sin^2\frac{A}{2}\,bc + \cos^2\frac{A}{2}\,bc)}{\Delta^2}$$

$$= abc$$

$$= \frac{abc\left(\sin^2\frac{A}{2}+\cos^2\frac{A}{2}\right)}{\Delta^2}$$

$$= \frac{abc}{\Delta^2}$$

$$= \frac{abc}{\frac{(abc)^2}{(4R)^2}}$$

$$= \frac{(4R)^2}{abc}$$

$$= \frac{4R}{\frac{abc}{4R}}$$

$$= \frac{4R}{\Delta}$$

= RHS (Proved)

b. $\tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2} = \left(\frac{s-a}{s}\right)\left(\frac{s-b}{s}\right)\left(\frac{s-c}{s}\right)$

$$LHS = \tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2}$$

$$= \frac{\cancel{(s-b)}\cancel{(s-c)}}{s\cancel{(s-a)}}\;\frac{\cancel{(s-a)}(s-c)}{s\cancel{(s-b)}}\;\frac{(s-a)(s-b)}{s\cancel{(s-c)}}$$

$$= \left(\frac{s-a}{s}\right)\left(\frac{s-b}{s}\right)\left(\frac{s-c}{s}\right)$$

= RHS (proved)