Derivatives

→ $\frac{dy}{dx}$: the derivative of y with respect to x,

→ $\frac{dy}{dx}$: the rate of change of y with respect to x.

- If $y = f(x)$ then

$$\boxed{\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

where h is small increment

Definition

A function $y = f(x)$ is said to be differentiable with respect to x, if $\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$ exists
(i)

The value of limit (i) is called derivative of $f(x)$. It is denoted by $\frac{dy}{dx}$ or $\frac{df(x)}{dx}$ or $f'(x)$ or y'

$$\therefore \boxed{\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} \text{ ——— (ii)}$$

Note:- To find derivative from definition or from first principle, we calculate the derivative by using formula (ii).

* The derivatives of x^n is nx^{n-1}
i.e. $\boxed{\frac{dx^n}{dx} = nx^{n-1}}$

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(i) The derivative of x^n is nx^{n-1}
i.e. $\frac{dx^n}{dx} = nx^{n-1}$

Sol; Let $f(x) = x^n$, then
$f(x+h) = (x+h)^n$, where h is small increment in x

Now,
From definition

$$\frac{d\,f(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

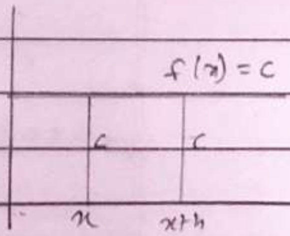
$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{(x+h) - x} \quad [\because x+h-x = h]$$

$$= nx^{n-1} \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

$\therefore \boxed{\frac{dx^n}{dx} = nx^{n-1}}$ Eg. $\frac{dx^7}{dx} = 7x^{7-1} = 7x^6$

(ii) The derivative of constant function is zero
i.e. if $f(x) = c$, then $\frac{df(x)}{dx} = \frac{dc}{dx} = 0$

Sol; We have,
$f(x) = c$
$f(x+h) = c$ [$\because$ constant function]



From definition,

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{dc}{dx} = \lim_{h \to 0} \frac{c-c}{h}$$

$\therefore \frac{dc}{dx} = 0$

Eg:- $\frac{da^n}{dx} = 0$, $\frac{d5}{dx} = 0$

Exercises:-

1) $f(x) = ax^2 + bx + c$

Sol: We have,

$f(x) = ax^2 + bx + c$,

then,

$f(x+h) = a(x+h)^2 + b(x+h) + c$

where h is small increment in x.

From definition,

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h}$$

$$= \lim_{h \to 0} \frac{a(x+h)^2 + b(x+h) + \cancel{c} - ax^2 - bx - \cancel{c}}{h}$$

$$= \lim_{h \to 0} \frac{a[(x+h)^2 - x^2] + b(x+h-\cancel{x})}{h}$$

$$= \lim_{h \to 0} \frac{a(x+h+x)(\cancel{x}+h-\cancel{x}) + bh}{h}$$

$$= \lim_{h \to 0} \frac{a(2x+h) \cdot h + bh}{h}$$

$$= \lim_{h \to 0} \frac{h[a(2x+h) + b]}{h}$$

$$= a(2x) + b = 2ax + b$$

$$\therefore \boxed{\frac{d(ax^2 + bx + c)}{dx} = 2ax + b}$$

2) $\sqrt{ax+b}$

Sol;

let $f(x) = \sqrt{ax+b}$

then $f(x+h) = \sqrt{a(x+h)+b}$

where h is a small increment in x

From definition, [Method 1]

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{a(x+h)+b} - \sqrt{ax+b}}{h}$$

$$= \lim_{h\to 0} \frac{(ax+ah+b)^{1/2} - (ax+b)^{1/2}}{h}$$

$$= \lim_{h\to 0} \frac{(ax+ah+b)^{1/2} - (ax+b)^{1/2}}{ah} \times a$$

$$= \lim_{h\to 0} \frac{(ax+ah+b)^{1/2} - (ax+b)^{1/2}}{(ax+ah+b)-(ax+b)} \times a$$

$$= \frac{1}{2}(ax+b)^{1/2-1} \times a$$

$$= \frac{1}{2(ax+b)^{+1/2}} \times a$$

$$= \frac{a}{2\sqrt{ax+b}}$$

[Method 2]

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{\sqrt{ax+ah+b} - \sqrt{ax+b}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{ax+ah+b} - \sqrt{ax+b}}{h} \times \frac{\sqrt{ax+ah+b} + \sqrt{ax+b}}{\sqrt{ax+ah+b} + \sqrt{ax+b}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{ax+ah+b})^2 - (\sqrt{ax+b})^2}{h(\sqrt{ax+ah+b} + \sqrt{ax+b})}$$

$$= \lim_{h\to 0} \frac{\cancel{ax}+ah+\cancel{b}-\cancel{ax}-\cancel{b}}{h(\sqrt{ax+ah+b} + \sqrt{ax+b})}$$

$$= \lim_{h\to 0} \frac{\cancel{h(a+} \, a\cancel{h}}{\cancel{h}(\sqrt{ax+ah+b}+\sqrt{ax+b})}$$

$$= \frac{a}{\sqrt{ax+0+b}+\sqrt{ax+b}}$$

$$= \frac{a}{2\sqrt{ax+b}}$$

Rules:-

1) Power Rule

$$\boxed{\frac{dx^n}{dx} = nx^{n-1}}$$

eg:- $\frac{dx^7}{dx} = 7x^6$

2) Sum Rule

$$\boxed{\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}}$$

Eg:- $y = x^7 + x^{5/2}$

Diff. both sides w.r. to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^7 + x^{5/2})$$

$$= \frac{dx^7}{dx} + \frac{dx^{5/2}}{dx}$$

$$= 7x^6 + \frac{5}{2}x^{5/2-1}$$

$$= 7x^6 + \frac{5}{2}x^{3/2}$$

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3) Coefficient rule:

$y = cu$, u is function of x,

$$\boxed{\frac{dy}{dx} = \frac{d\,cu}{dx} = \frac{c.du}{dx}}$$ [c → constant]

Eg: $y = 7x^{4/7}$

Diff. both sides w.r. to x,

$$\frac{dy}{dx} = \frac{d}{dx} 7x^{4/7}$$

$$= 7 \cdot \frac{dx^{4/7}}{dx}$$

$$= \cancel{7} \cdot \frac{4}{\cancel{7}} x^{4/7 - 1}$$

$$= 4x^{-3/7}$$

4) Product rule

If $y = uv$, where 'u' and 'v' are functions of x, then

$$\boxed{\frac{dy}{dx} = \frac{d(uv)}{dx} = u\frac{dv}{dx} + \frac{v\,du}{dx}}$$

Eg: $y = (x+1)^2 \cdot x^{3/2}$

Diff. both sides w.r. to x, we get

$$\frac{dy}{dx} = \frac{d}{dx} (x+1)^2 \cdot x^{3/2}$$

$$= (x+1)^2 \frac{d}{dx} x^{3/2} + x^{3/2} \cdot \frac{d}{dx} (x+1)^2$$

$$= (x+1)^2 \frac{3}{2} x^{3/2 - 1} + x^{3/2} \cdot 2(x+1)$$

$$= \frac{3}{2} x^{1/2} (x+1)^2 + 2x^{3/2}(x+1)$$

5) Quotient Rule

$y = \frac{u}{v}$, where u and v are functions of x

$$\boxed{\frac{dy}{dx} = \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

Eg:- $y = \frac{x^3+2x^2+x}{x+5}$

Diff. both sides w.r. to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3+2x^2+x}{x+5}\right)$$

$$= \frac{(x+5)\frac{d}{dx}(x^3+2x^2+x) - (x^3+2x^2+x)\frac{d}{dx}(x+5)}{(x+5)^2}$$

$$= \frac{(x+5)(3x^2+4x+1) - (x^3+2x^2+x)}{(x+5)^2}$$ //

Exercise 16.1

1. Find from first principle the derivitive of

a) $4x^2$

Sol; let $f(x) = 4x^2$

so $f(x+h) = 4(x+h)^2$, where h is a small increment in x

From definition,

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{4(x+h)^2 - 4x^2}{h}$$

$$= \lim_{h\to 0}\frac{4(x^2+2hx+h^2-x^2)}{h}$$

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$$= \lim_{h \to 0} \frac{4\cancel{h}(2x+h)}{\cancel{h}}$$

$$= 4(2x+0)$$

$$\therefore \frac{d\,4x^2}{dx} = 8x$$

*

$$\frac{1}{\sqrt{ax+b}}$$

let

$$f(x) = \frac{1}{\sqrt{ax+b}}$$

then,

$$f(x+h) = \frac{1}{\sqrt{a(x+h)+b}}$$

where h is small increment in x

We know that,

$$\frac{d\,f(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{a(x+h)+b}} - \frac{1}{\sqrt{ax+b}}}{h}$$

$$= \lim_{h \to 0} \frac{(ax+ah+b)^{-1/2} - (ax+b)^{-1/2}}{(ax+ah+b) - (ax+b)} \times a$$

$$= -\frac{1}{2}(ax+b)^{-1/2-1} \times a$$

$$= -\frac{a}{2}(ax+b)^{-3/2}$$

$$= \frac{-a}{2(ax+b)^{3/2}}$$

* $\frac{1}{\sqrt{3x+5}}$

Let

$$f(x) = \frac{1}{\sqrt{3x+5}}$$

Then

$$f(x+h) = \frac{1}{\sqrt{3(x+h)+5}}$$

where h is small increment in x

From definition,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{3(x+h)+5}} - \frac{1}{\sqrt{3x+5}}}{h}$$

$$= \lim_{h\to 0} \frac{(3x+3h+5)^{-1/2} - (3x+5)^{-1/2}}{(3x+3h+5)-(3x+5)} \times 3$$

$$= \frac{-1}{2}(3x+5)^{-1/2-1}$$

$$= \frac{-1}{2}(3x+5)^{-3/2} \times 3$$

$$= \frac{-3}{2(3x+5)^{3/2}}$$ //

V.V.I

4h) $\frac{ax+b}{\sqrt{x}}$

Sol;

$$\text{Let } f(x) = \frac{ax+b}{\sqrt{x}}$$

$$= \frac{ax}{\sqrt{x}} + \frac{b}{\sqrt{x}}$$

$$= a\sqrt{x} + \frac{b}{\sqrt{x}}$$

So,

$$f(x+h) = a\sqrt{x+h} + \frac{b}{\sqrt{x+h}}$$

where h is a small increment in x

From definition,

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a\sqrt{x+h} + \frac{b}{\sqrt{x+h}} - \left(a\sqrt{x} + \frac{b}{\sqrt{x}}\right)}{h}$$

$$= \lim_{h\to 0} \frac{a\sqrt{x+h} + \frac{b}{\sqrt{x+h}} - a\sqrt{x} - \frac{b}{\sqrt{x}}}{h}$$

$$= \lim_{h\to 0} \frac{a\left(\sqrt{x+h} - \sqrt{x}\right) + b\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right)}{h}$$

$$= \lim_{h\to 0} \frac{a\left\{(x+h)^{1/2} - (x)^{1/2}\right\}}{h} + \lim_{h\to 0} b\,\frac{\left\{(x+h)^{-1/2} - (x)^{-1/2}\right\}}{h}$$

$$= a\lim_{h\to 0} \frac{(x+h)^{1/2} - (x)^{1/2}}{(x+h) - (x)} + b\lim_{h\to 0} \frac{(x+h)^{-1/2} - (x)^{-1/2}}{(x+h) - (x)}$$

$$= a \cdot \frac{1}{2} x^{1/2 - 1} + b\left(-\frac{1}{2}\right) x^{-1/2 - 1}$$

$$= \frac{ax^{-1/2}}{2} - \frac{bx^{-3/2}}{2}$$

$$= \frac{a}{2\sqrt{x}} - \frac{b}{2x^{1} \cdot x^{1/2}}$$

$$= \frac{a}{2\sqrt{x}} - \frac{b}{2x\sqrt{x}}$$

$$= \frac{ax - b}{2x\sqrt{x}}$$

(11) $\frac{2x+5}{\sqrt{x}}$

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Let $f(x) = \frac{2x+5}{\sqrt{x}}$

$= \frac{2x}{\sqrt{x}} + \frac{5}{\sqrt{x}}$

$= 2\sqrt{x} + \frac{5}{\sqrt{x}}$

Then $f(x+h) = 2\sqrt{x+h} + \frac{5}{\sqrt{x+h}}$

where h is a small increment in x.

By definition,

$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{2\sqrt{x+h} + \frac{5}{\sqrt{x+h}} - 2\sqrt{x} - \frac{5}{\sqrt{x}}}{h}$

$= \lim_{h\to 0} \frac{2(\sqrt{x+h} - \sqrt{x}) + 5\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}\right)}{h}$

$= 2\lim_{h\to 0} \frac{(x+h)^{1/2} - (x)^{1/2}}{(x+h)-(x)} + 5\lim_{h\to 0} \frac{(x+h)^{-1/2} - (x)^{-1/2}}{(x+h)-(x)}$

$= 2\cdot\frac{1}{2}x^{1/2-1} + 5\left(-\frac{1}{2}\right)x^{-1/2-1}$

$= x^{-1/2} - \frac{5}{2}x^{-3/2}$

$= \frac{1}{\sqrt{x}} - \frac{5}{2x\sqrt{x}}$

$= \frac{2x-5}{2x\sqrt{x}}$ //

(f)

let $y = \left(\frac{x^2+1}{x^2-1}\right)^6$

Diff. both sides w.r. to x,
we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)^6$$

$$= \frac{d\left(\frac{x^2+1}{x^2-1}\right)^6}{d\left(\frac{x^2+1}{x^2-1}\right)} \times \frac{d\left(\frac{x^2+1}{x^2-1}\right)}{dx}$$

$$= 6\left(\frac{x^2+1}{x^2-1}\right)^5 \times \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= 6\left(\frac{x^2+1}{x^2-1}\right)^5 \times \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2}$$

$$= 6\left(\frac{x^2+1}{x^2-1}\right)^5 \times \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}$$

$$= \frac{6(x^2+1)^5}{(x^2-1)^5} \times \frac{-4x}{(x^2-1)^2}$$

$$= \frac{-24x\,(x^2+1)^5}{(x^2-1)^7}$$

h)

Sol. let $y = \frac{1}{\sqrt[5]{a^n - x^n}}$

$$= \frac{1}{(a^n-x^n)^{1/5}}$$

$$= (a^n-x^n)^{-1/5}$$

Diff. both sides w.r. to x

$$\frac{dy}{dx} = \frac{d}{dx}(a^n-x^n)^{-1/5}$$

$$= \frac{d(a^n-x^n)^{-1/5}}{d(a^n-x^n)} \times \frac{d(a^n-x^n)}{dx}$$

$$= \frac{-1}{5}(a^n-x^n)^{-1/5-1} \times (-nx^{n-1})$$

$$= \frac{-1}{5}(a^n-x^n)^{-6/5} \times (-nx^{n-1})$$

$$= \frac{1}{5}(a^n-x^n)^{-6/5} \cdot nx^{n-1}$$

$$= \frac{n}{5}(a^n-x^n)^{-6/5} \cdot x^{n-1}$$

8(n)

Let $y = \sqrt{\frac{1+x}{1-x}}$

$$= \left(\frac{1+x}{1-x}\right)^{1/2}$$

Diff. both sides w.r. to x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1+x}{1-x}\right)^{1/2}$$

$$= \frac{d\left(\frac{1+x}{1-x}\right)^{1/2}}{d\left(\frac{1+x}{1-x}\right)} \times \frac{d\left(\frac{1+x}{1-x}\right)}{dx}$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2} \times \frac{(1-x)\cdot\frac{d(1+x)}{dx} - (1+x)\frac{d(1-x)}{dx}}{(1-x)^2}$$

$$= \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{1/2} \times \frac{(1-x) - (1+x)(0-1)}{(1-x)^2}$$

$$= \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{1/2} \times \frac{(1-\cancel{x}+1+\cancel{x})}{(1-x)^2}$$

$$= \frac{1}{\cancel{2}} \frac{(1-x)^{1/2}}{(1+x)^{1/2}} \times \frac{\cancel{2}}{(1-x)^2}$$

$$= \frac{(1-x)^{1/2}}{(1+x)^{1/2}(1-x)^2}$$

Chain rule:

If $y = f(t)$ and $t = g(x)$

w.r. to t   w.r. to x

$\frac{dy}{dt} = \ldots$ $\frac{dt}{dx} = \ldots$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$$

6a) $y = 3t^2 + 4t - 1$, $t = x + 3$

Sol;

We have,

$y = 3t^2 + 4t - 1$ ——— (i)

and $t = x + 3$ ——— (ii)

Diff. both sides of (i) w.r. to t

$$\frac{dy}{dt} = \frac{d}{dt}(3t^2 + 4t - 1)$$

$$= 6t + 4$$

Diff. both sides of (ii) w.r. to x

$$\frac{dt}{dx} = \frac{d}{dx}(x + 3)$$

$$= 1$$

Now,

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= (6t + 4) \times 1$$

(from (ii))

$$= 6(x+3) + 4 = 6x + 18 + 4 = 6x + 22$$

e)

Sol; we have,

$$y = \sqrt{z} + \frac{1}{\sqrt{z}}$$

$$\therefore y = (z)^{1/2} + (z)^{-1/2} \text{ ——— (i)}$$

and $z = x + \frac{1}{x}$

$$\therefore z = x + x^{-1} \text{ ——— (ii)}$$

Diff. eq^n (i) w.r. to z,

$$\frac{dy}{dz} = \frac{d}{dz}\left(z^{1/2} + z^{-1/2}\right)$$

$$= \frac{1}{2} z^{-1/2} + \left(\frac{-1}{2}\right) z^{-3/2}$$

$$= \frac{z^{-1/2}}{2} - \frac{1}{2} z^{-3/2}$$

$$= \frac{z^{-1/2} - z^{-3/2}}{2}$$

Diff. eq (ii) w.r. to x

$$\frac{dz}{dx} = \frac{d}{dx}\left(x + x^{-1}\right)$$

$$= 1 - x^{-2}$$

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

$$= \frac{z^{-1/2} - z^{-3/2}}{2} \times 1 - x^{-2}$$

$$= \frac{\left(z^{-1/2} - z^{-3/2}\right)\left(1 - x^{-2}\right)}{2}$$