Sequence:- A set of numbers which are presented in a definite order (not necessarily according to definite rule) and separated by commas is called sequence. The numbers which form a sequence are called terms of sequence.

The terms of the sequence are usually denoted by a_1 , a_2 , a_3 ………… or by t_1 , t_2 , t_3 , ……..

The number occurring at the n^{th} place of a sequence i.e. t_n is called general term of the sequence .

A sequence is finite or infinite according as the number of terms in it is finite or infinite . A sequence is denoted by $\{t_n\}$ or $\{a_n\}$.

Progression:-

If the terms of a sequence follows certain pattern , then the sequence is called a progression.

Consider the following sequence :

(i) 1 , 3 , 5 , 7 , ….. 21

(ii) 3 , 9 , 27 , ………

(iii) 1 , 4 , 9 , 16 , ……….

(iv) $1 , \frac{1}{3} , \frac{1}{5} , \frac{1}{7} , \frac{1}{9}$ ………

We observe that in each of the example from (i) to (iv) there is a fixed rule to get the succeeding term . However , to define a sequence , we need not always have an explicit formula for the n^{th} term. For instance the infinite sequence 2 , 3 , 5 , 7 , 11 , 13 , 17 , ……. … of all positive prime numbers, we may not have an explicit formula for the n^{th} term.

Series:- The algebraic sum of the term of a sequence is called series and it is denoted by $\sum_{i=1}^{n} ti$ for finite series and $\sum_{i=1}^{\infty} ti$ or $\sum t\, i$ for an infinite series. A series is finite or infinite according as the terms in the corresponding sequence is finite or infinite.

(i) $1 + 3 + 5 + 7 + \ldots\ldots + 21$

(ii) $3 + 9 + 27 + \ldots\ldots\ldots$

(iii) $1 + 4 + 9 + 16 + \ldots\ldots\ldots\ldots$

(iv) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} \ldots\ldots\ldots$

These are the series corresponding to the sequence in above example.

Types of sequence:-

Sequences may be of various types . But,

- Arithmetic sequence
- Geometric sequence
- Harmonic sequence

are the most familiar type of sequences.

Arithmetic Progression (A.P)

A sequence is said to be A.P. if the difference of any term and its preceding term is constant throughout the whole sequence.

Common difference (d) = any term – its preceding term

$= 2^{nd}$ term – 1^{st} term

General term (n^{th} term) of an A.P.

1^{st} term $(t_1) = a = a + (1-1)d$

2^{nd} term $(t_2) = a + (2-1)\, d$

3^{rd} term $(t_3) = a + (3-1)\, d$

……………………………………………………..

n^{th} term $(t_n) = a + (n-1)d$

or, last term $(l) = a + (n-1)d$

Sum of the first n terms of an A.P.

Sum of the first n terms (S_n) = $\frac{n}{2}$ [2a + (n - 1)d]

$$\text{or,}\quad S_n = \frac{n}{2}\,[2a + (n-1)d]$$

$$= \frac{n}{2}\,[a + a + (n-1)d]$$

$$= \frac{n}{2}\,(a + l)$$

Where a = first term

n = number of terms

d = common difference

l = last term

S_n = sum to n terms of an A.P.

Arithmetic Mean (A.M)

If three numbers are in A.P then the middle term is called the arithmetic mean of the other two numbers.

If a, M , b are in A.P then M is the arithmetic mean of the numbers a and b.

So, M - a = b - M

$$\Rightarrow M = \frac{a+b}{2}$$

If a_1, a_2 , a_3, …….. , a_n are "n" arithmetic mean inserted between two numbers 'a' and 'b' then a , a_1, a_2 , a_3, …….. , a_n , b are in A.P and total number of terms is n + 2 and common difference 'd' is obtain by the formula $d = \frac{b-a}{n+1}$

Geometric Progression (G.P.)

A sequence or series is said to be G.P. if the ratio of any term(except the first) and its preceding term is always same.

n^{th} term of a G.P.

1^{st} term$(t_1) = a = ar^{1-1}$

2^{nd} term$(t_2) = ar = ar^{2-1}$

3^{rd} term$(t_3) = ar^2 = ar^{3-1}$

……………………….

n^{th} term$(t_n) = ar^{n-1}$ or, last term$(l) = ar^{n-1}$

Common ratio (r) = $\dfrac{a_{k+1}}{a_k}$, $k \geq 1$

$$= \frac{\textit{Second term}}{\textit{first term}}$$

The sum of the first n terms of a G.P.

(i) $S_n = \frac{a(r^n - 1)}{r-1}$ $[\because r > 1]$

or

$S_n = \frac{ar^n - a}{r-1} = \frac{ar^{n-1}.r - a}{r-1} = \frac{lr - a}{r-1}$ $[\because r > 1]$

(ii) $S_n = \frac{a(1-r^n)}{1-r}$ $[\because r < 1]$

or

$S_n = \frac{a - ar^n}{1-r} = \frac{a - ar^{n-1}.r}{1-r} = \frac{a - lr}{1-r}$ $[\because r < 1]$

Three numbers in G.P. may always be taken as

$$\frac{a}{r}, a, ar$$

Four numbers in G.P. may always be taken as

$$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$$

Five numbers in G.P may always be taken as

$$\frac{a}{r^3}, \frac{a}{r}, a, ar, ar^3$$

Geometric Means :-

If the numbers a , G.M. , b are in G.P.

Then, $\frac{G.M.}{a} = \frac{b}{G.M.}$

$GM^2 = ab$

$GM = \sqrt{ab}$

G.M. between a and b is $\sqrt{ab}$

If a_1, a_2 , a_3, …….. , a_n are "n" geometric mean inserted between two numbers 'a' and 'b' then a , a_1, a_2 , a_3, …….. , a_n , b are in G.P and total number of terms is n + 2 and common ratio 'r' is obtain by the formula $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

Harmonic Progression (H.P)

A sequence of (non zero) numbers is said to be a harmonic sequence or progression ,if the reciprocal of its terms form an arithmetic sequence .

Example :-

$\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\ldots\ldots\ldots\ldots\ldots$ is a harmonic sequence ,

since 3 , 5 , 7 , 9 , …….. are in A.P.

General term (n th term) of H.P

Let the A.P be a , a + d , a + 2d , a + 3d , ……. then its

n th term is = a + (n – 1)d .

Then , the corresponding H.P. be $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots\ldots\ldots$

and it's n th term is = $\frac{1}{a+(n-1)d}$

Note: There is no formula for obtaining the sum of a harmonic series.

Harmonic Means :-

If the numbers a , H.M. , b are in H.P.

Then, by definition of H.P $\frac{1}{a}$, $\frac{1}{H.M.}$, $\frac{1}{b}$ are in A.P

so, $\frac{1}{H.M.} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H.M.}$

or, $\frac{1}{H.M.} + \frac{1}{H.M.} = \frac{1}{b} + \frac{1}{a}$

or, $\frac{2}{H.M.} = \frac{a+b}{ab}$ $\Rightarrow$ $\text{H.M} = \frac{2ab}{a+b}$

Arithmetic, Geometric and Harmonic Mean of the sequence

If a and b are postive numbers, then

$$\text{Arithmetic Mean (AM)} = \frac{a+b}{2}$$

$$\text{Geometric Mean (GM)} = \sqrt{ab}$$

$$\text{Harmonic Mean (HM)} = \frac{2ab}{a+b}$$

Comparison between AP, GP and HP

The relations between AP, GP and HP is connected by their means. The arithmetic mean, geometric mean and harmonic mean of any two positive and unequal quantities are in descending order of magnitudes. They have vast differences as given below:

SN	Arithmetic Progression	Geometric Progression	Harmonic Progression
1.	A sequence of numbers in which each differ by a constant value.	A sequence of number in which the successor is multiplied by constant value.	A sequence of numbers in which the reciprocals differ by a constant value.
2.	AM of any two distinct numbers a and b is $\frac{a+b}{2}$	AM of any two distinct numbers a and b is $\sqrt{ab}$	AM of any two distinct numbers a and b is $\frac{2ab}{a+b}$
3.	AM is greater than GM and HM for any two distinct numbers.	GM is greater than HM and lesser than AM for any two distinct numbers.	HM is smaller than GM and AM for any two distinct numbers.
4.	AM is the quotient of the square of GM and HM.	GM is the geometric mean of AM and HM.	HM is the quotient of the square of GM and AM.
5.	The general term of AP is in linear form as a + (n – 1)d, where a = first term, n = no. of terms and d = common difference.	The general term of GP is in index form as ar^{n-1}, where a = first term, n = no. of terms and r = common ratio.	The general term of HP is in reciprocal of linear form as , $\frac{1}{a+(n-1)d}$ where a = first term, n = no. of terms and d = common difference.
6.	AP is the reciprocal of HP.	GP is distinct from AP and HP.	HP is the reciprocal of AP.

Theorem : (Relation between A.M. , G.M. and H.M.)

The A.M. , G.M. and H.M. between any two unequal positive numbers satisfy the following relations

(a) $(\boldsymbol{G.M.})^2 = (\boldsymbol{A.M.}) \times (\boldsymbol{H.M.})$ **(b)** $\boldsymbol{A.M.} > \boldsymbol{G.M.} > \boldsymbol{H.M.}$

Proof:

Let ‘a’ and ‘b’ be two unequal positive numbers. Then,

$A.M. = A = \frac{a+b}{2}$ ....(i)

$G.M. = G = \sqrt{ab}$ ....(ii)

$H.M. = H = \frac{2ab}{a+b}$ .... (iii)

(a) We have, $(A.M.) \times (H.M.) = \left(\frac{a+b}{2}\right) \times \left(\frac{2ab}{a+b}\right) = ab = \left(\sqrt{ab}\right)^2 = (G.M.)^2$

Therefore, $(A.M.) \times (H.M.) = (G.M.)^2$

(b) $A.M. - G.M.$

$= \frac{a+b}{2} - \sqrt{ab}$

$= \frac{a+b-2\sqrt{ab}}{2}$

$= \frac{1}{2}\left((\sqrt{a})^2 - 2\sqrt{a}.\sqrt{b} + \left(\sqrt{b}\right)^2\right) = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$

$[\because \textit{square of difference of any two unequal real number is always positive}]$

i.e. $A.M. - G.M. > 0$

i.e. $A.M. > G.M.$ ………………………… (iv)

Again, $(A.M.) \times (H.M.) = (G.M.)^2 = (G.M.) \times (G.M.)$

i.e. $(A.M.) \times (H.M.) = (G.M.) \times (G.M.)$

i.e. $\frac{A.M.}{G.M.} = \frac{G.M.}{H.M.}$ …………… (v)

From (iv) $A.M. > G.M.$ this imply that $G.M. > H.M.$ …………… (vi)

$[\because\ using\ (iv)\ and\ (v)]$

Combining (iv) and (vi) we have,

$A.M. > G.M. > H.M.$ Hence proved.

Note: If a, b, c are in $A.P.$ then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in $H.P.$

Similarly if, a, b, c are in $H.P.$ then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in $A.P.$

Example 1: Find two numbers whose arithmetic mean is 20 and geometric mean is 16.

Solution: Let a and b are two numbers. then

A.M. = 20

or, $\frac{a+b}{2} = 20$

or, a + b = 40 …(i)

Also, GM = $\sqrt{ab}$

or, 16 = $\sqrt{ab}$

or, ab = 256

or, a(40 – a) = 256

or, $40a - a^2 = 256$

or, $a^2 - 40a + 256 = 0$

or, $a^2 - 32a - 8a + 256 = 0$

or, $a(a - 32) - 8(a - 32) = 0$

or, $(a - 32)(a - 8) = 0$

$\therefore$ $a = 8, 32$

When $a = 8$, $b = 40 - 8 = 32$

When $a = 32$, $b = 40 - 32 = 8$

$\therefore$ Required numbers are 8, 32 or 32, 8.

Exercise: 5.1

1.(c) The A.M. between two numbers exceeds their G.M. by 1 and the G.M. exceeds the H.M. by 0.8 , Find the numbers.

Solution:

Let a and b be the given two numbers.

Then, $A.M. = \frac{a+b}{2}$, $G.M. = \sqrt{ab}$ and $H.M. = \frac{2ab}{a+b}$

By question,

$$A.M. = 1 + G.M.$$

or, $\frac{a+b}{2} = 1 + \sqrt{ab}$

so, $a + b = 2 + 2\sqrt{ab}$ ……………… (i)

Again, $G.M. - H.M. = 0.8$

or, $\sqrt{ab} - \frac{2ab}{a+b} = 0.8$

or, $\sqrt{ab} - \frac{2ab}{2+2\sqrt{ab}} = 0.8 \qquad [\because from\ (i)]$

or, $\frac{2\sqrt{ab}+2ab-2ab}{2+2\sqrt{ab}} = 0.8$

or, $2\sqrt{ab} + 2ab - 2ab = 1.6 + 1.6\sqrt{ab}$

or, $2\sqrt{ab} - 1.6\sqrt{ab} = 1.6$

or, $0.4\sqrt{ab} = 1.6$

or, $\sqrt{ab} = 4$

i.e. $ab = 16$ ……………… (ii) $\qquad$ {i.e. a = $\frac{16}{b}$ $\qquad$ .. .. .. (ii)}

Using (ii) in (i) we have,

$$a + b = 2 + 2\sqrt{16} = 2 + 2 \times 4 = 10$$

i.e. $a + b = 10$ …………….. (iii)

Solving (ii) and (iii) by elimination method we have,

$$a + \frac{16}{a} = 10$$

or, $a^2 + 16 = 10a$

or, $a^2 - 10a + 16 = 0$

or, $a^2 - 8a - 2a + 16 = 0$

or, $a(a - 8) - 2(a - 8) = 0$

or, $(a - 8)(a - 2) = 0$

Therefore, $a = 8 \ or \ a = 2$

When a = 8 from (ii) b = 2

When a = 2 from (ii) b = 8

Hence the required numbers are, 2, 8 or 8 , 2 .

Q.N 2. (b) The sum of three numbers in an arithmetic sequence is 15. If 1, 3 and 9 be added to them respectively, then they form a geometric sequence. Find the numbers.

Solution: Let a – d, a, a + d be three numbers in an AS, then

$$a - d + a + a + d = 15$$

or, $3a = 15$ $\quad\therefore a = 5.$

If 1, 3 and 9 be added to them respectively then the three numbers

$a - d + 1, a + 3, a + d + 9$ are in GS,

$$\text{so, } \frac{a+3}{a-d+1} = \frac{a+d+9}{a+3}$$

$$\text{or, } \frac{5+3}{5-d+1} = \frac{5+d+9}{5+3} \quad [\therefore a = 5]$$

$$\text{or, } \frac{8}{6-d} = \frac{14+d}{8}$$

$$\text{or, } 64 = 84 - 8d - d^2$$

$$\text{or, } d^2 + 8d - 20 = 0$$

$$\text{or, } d^2 + 10d - 2d - 20 = 0$$

$$\text{or, } d(d + 10)-2(d + 10) = 0$$

$$\text{or, } (d + 10)(d - 2) = 0$$

$$\therefore \quad d = -10 \quad \text{or, } \; 2$$

If $a = 5$ and $d = -10$ then the required three numbers are $5 + 10, 5, 5 - 10 = 15, 5, -5$.

If $a = 5$ and $d = 2$ then the required three numbers are $5 - 2, 5, 5 + 2 = 3, 5, 7$.

2. (c) The sum of three numbers in A.P. is 36. When the numbers are increased by 1, 4, 43 respectively, the resulting numbers are in G.P. Find the numbers.

Solution:

Let, $a - d, a, a + d$ are in A.P.

Then, $a - d + a + a + d = 36$

So, $a = 12$

By question, $a - d + 1, a + 4, a + d + 43$ are in G.P.

i.e. $12 - d + 1, 12 + 4, 12 + d + 43$ are in G.P.

i.e. $13 - d, 16, 55 + d$ are in G.P.

$$(16)^2 = (13 - d)(55 + d)$$

or, $256 = 715 + 13d - 55d - d^2$

or, $256 – 715 = 13d – 55d – d^2$

or, $d^2 + 42d – 459 = 0$

or, $d^2 + 51d – 9d - 459 = 0$

or, $d(d + 51) – 9 (d – 51) = 0$

or, $(d – 9) (d + 51) = 0$

Therefore, $d = 9$ or, -51

when $d = 9$, required numbers are

$a – d , a , a + d = 12 – 9 , 12 , 12 + 9 = 3 , 12 , 21$.

when $d = -51$, required numbers are

$a – d , a , a + d = 12 + 51 , 12 , 12 – 51 = 63 , 12 , -39$.

3 a) If H be the H.M between a and b , prove that (H – 2a) (H – 2b) = H^2

Solution :- Since, H be the H.M between a and b

So, $H.M. = H = \frac{2ab}{a+b}$

Now, L.H.S = (H – 2a) (H – 2b)

$$= \left(\frac{2ab}{a+b} - 2a\right)\left(\frac{2ab}{a+b} - 2b\right)$$

$$= \left(\frac{2ab - 2a^2 - 2ab}{a+b}\right)\left(\frac{2ab - 2b^2 - 2ab}{a+b}\right)$$

$$= \left(\frac{-2a^2}{a+b}\right)\left(\frac{-2b^2}{a+b}\right)$$

$$= \frac{4a^2b^2}{(a+b)^2}$$

$$= \left(\frac{2ab}{a+b}\right)^2 = H^2 = \text{R.H.S. proved.}$$

3. (b) If H be the H.M between a and b , prove that

$$\frac{H+a}{H-a}+\frac{H+b}{H-b}=2$$

Solution:- Given, H be the H.M between a and b

So, $H=\frac{2ab}{a+b}$

Now, L.H.S. $=\frac{H+a}{H-a}+\frac{H+b}{H-b}$

$$=\frac{\frac{2ab}{a+b}+a}{\frac{2ab}{a+b}-a}+\frac{\frac{2ab}{a+b}+b}{\frac{2ab}{a+b}-b}$$

$$=\frac{2ab+a^2+ab}{2ab-a^2-ab}+\frac{2ab+b^2+ab}{2ab-b^2-ab}$$

$$=\frac{3ab+a^2}{ab-a^2}+\frac{3ab+b^2}{ab-b^2}$$

$$=\frac{a(3b+a)}{a(b-a)}+\frac{b(3a+b)}{-b(b-a)}$$

$$= \frac{(3b + a)}{(b - a)} - \frac{(3a + b)}{(b - a)}$$

$$= \frac{3b + a - 3a - b}{(b - a)}$$

$$= \frac{2b - 2a}{(b - a)}$$

$= 2 =$ R.H.S. proved.

Q.N 3 (c) : If A be the arithmetic mean and H, the HM between two quantities a and b, show that : $\frac{a-A}{a-H} \times \frac{b-A}{b-H} = \frac{A}{H}$

Solution: If A is the AM , H be the HM between a and b, then

$$A.M. = A = \frac{a+b}{2} \quad \ldots\ldots\ldots\ldots(i)$$

$$H.M. = H = \frac{2ab}{a+b} \quad \ldots\ldots\ldots\ldots(ii)$$

$$\text{L.H.S} = \frac{a-A}{a+H} \times \frac{b-A}{b-H}$$

$$= \frac{a - \left(\frac{a+b}{2}\right)}{a - \left(\frac{2ab}{a+b}\right)} \times \frac{b - \left(\frac{a+b}{2}\right)}{b - \left(\frac{2ab}{a+b}\right)}$$

$$= \frac{\frac{a-b}{2}}{\frac{a^2+ab-2ab}{a+b}} \times \frac{\frac{b-a}{2}}{\frac{b^2+ab-2ab}{a+b}}$$

$$= \frac{(a-b)(a+b)}{2(a^2-ab)} \times \frac{(b-a)(a+b)}{2(b^2-ab)}$$

$$= \frac{(a-b)(a+b)}{2a(a-b)} \times \frac{(b-a)(a+b)}{2b(b-a)}$$

$$= \frac{(a+b)}{2a} \times \frac{(a+b)}{2b}$$

$$= \frac{(a+b)}{2} \times \frac{(a+b)}{2ab}$$

$$= \mathrm{A} \times \frac{1}{H} \qquad \text{[using (i) and (ii)]}$$

$$= \frac{A}{H}$$

4 (a) If $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$ and a, b, c are in G.P., prove that x, y, z are in A.P.

Solution:

Given, $a^{\frac{1}{x}} = b^{\frac{1}{y}} \Rightarrow a = b^{x/y}$

Also, $c^{\frac{1}{z}} = b^{\frac{1}{y}} \Rightarrow c = b^{z/y}$

since, a, b, c are in G.P.

So, $b^2 = ac = b^{x/y}\ b^{z/y} = b^{\frac{x}{y}+\frac{z}{y}}$

or, $b^2 = b^{\frac{x}{y}+\frac{z}{y}}$

or, $2 = \frac{x}{y} + \frac{z}{y}$

or, $2 = \frac{x+z}{y}$

i.e. $y = \frac{x+z}{2}$ This shows y is A.M between x and z

Hence x, y, z are in A.P proved.

4 (b) Show that b^2 is greater than, equal to or less than ac, according as a, b, c are in A.P., G.P. or H.P.

Solution:

(i) a, b, c are in A.P. then we have to show: $b^2 > ac$

(ii) a, b, c are in G.P. then we have to show: $b^2 = ac$

(iii) a, b, c are in H.P. then we have to show: $b^2 < ac$

For (i), Since, a, b, c are in A.P. so $b = \frac{a+c}{2}$

Now, $b^2 - ac = \left(\frac{a+c}{2}\right)^2 - ac$

$$= \frac{(a+c)^2}{4} - ac$$

$$= \frac{a^2+2ac+c^2-4ac}{4}$$

$$= \frac{a^2-2ac+c^2}{4}$$

$$= \left(\frac{a-c}{2}\right)^2 > 0$$

[Square of difference of any two unequal real number is always positive]

i.e. $b^2 - ac > 0$

Therefore, $b^2 > ac.$ proved.

For (ii),

if a, b, c are in G.P. then clearly, $b^2 = ac$

For (iii),

Since a, b, c are in H.P.

so, $b = \frac{2ac}{a+c}$

Now, $\frac{1}{b^2} - \frac{1}{ac} = \frac{(a+c)^2}{4a^2c^2} - \frac{1}{ac}$

$$= \frac{a^2+2ac+c^2-4ac}{4a^2c^2}$$

$$= \frac{a^2-2ac+c^2}{4a^2c^2}$$

$$= \left(\frac{a-c}{2ac}\right)^2 > 0$$

[Square of difference of any two unequal real number is always positive]

i.e. $\frac{1}{b^2} - \frac{1}{ac} > 0$

i.e. $\frac{1}{b^2} > \frac{1}{ac}$

Therefore, $b^2 < ac$.

5 (a) If a , b , c are in H.P , prove that 2a – b , b , 2c – b are in G.P

Solution:- Given a , b , c are in H.P

So, b is H.M between a and c

i.e. $b = \frac{2ac}{a+c}$

Now we prove, 2a – b , b , 2c – b are in G.P

i.e. b is the G.M between 2a – b and 2c – b

i.e. $b^2 = (2a – b)(2c – b)$

Now, here we prove, $b^2 = (2a – b)(2c – b)$

Here, R.H.S. = $(2a – b)(2c – b)$

$$= \left(2a - \frac{2ac}{a+c}\right)\left(2c - \frac{2ac}{a+c}\right)$$

$$= \left(\frac{2a^2 + 2ac - 2ac}{a+c}\right)\left(\frac{2c^2 + 2ac - 2ac}{a+c}\right)$$

$$= \frac{2a^2}{a+c} \times \frac{2c^2}{a+c} = \left(\frac{2ac}{(a+c)}\right)^2 = b^2 = \text{L.H.S. proved}$$

5 (b) If a, b, c are in H.P. prove that: $a(b+c), b(c+a), c(a+b)$ are in A.P.

Solution: Given,

a, b, c are in H.P.

i.e. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. [subtracting $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ from each term]

i.e. $\frac{1}{a}-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right), \frac{1}{b}-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right), \frac{1}{c}-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ are in A.P.

i.e. $-\left(\frac{1}{b}+\frac{1}{c}\right), -\left(\frac{1}{a}+\frac{1}{c}\right), -\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

i.e. $\left(\frac{1}{b}+\frac{1}{c}\right), \left(\frac{1}{a}+\frac{1}{c}\right), \left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

i.e. $\frac{b+c}{bc}, \frac{a+c}{ac}, \frac{a+b}{ab}$ are in A.P. [multiplying each term by abc]

i.e. $\frac{b+c}{bc} \times (abc), \frac{a+c}{ac} \times (abc), \frac{a+b}{ab} \times (abc)$ are in A.P.

i.e. $a(b+c), b(c+a), c(a+b)$ are in A.P.

Q.N 5 (c) If a , b , c are in H.P proved that

$\frac{a}{b+c}$, $\frac{b}{c+a}$, $\frac{c}{a+b}$ are in H.P

Solution:- Given, a , b , c are in H.P

So, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ $are\ in\ A.P$

by the property of A.P multiplying each term by (a + b + c)

i.e. $\frac{(a+b+c)}{a}, \frac{(a+b+c)}{b}, \frac{(a+b+c)}{c}$ $are\ in\ A.P$

i.e. $1+\frac{(b+c)}{a}, 1+\frac{(a+c)}{b}, 1+\frac{(a+b)}{c}$ $are\ in\ A.P$

i.e. $\frac{(b+c)}{a}, \frac{(a+c)}{b}, \frac{(a+b)}{c}$ $are\ in\ A.P$ (subtracting 1 from each term)

i.e. $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P proved.

6 (a) If $\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$ are in A. P , prove that a^2 , b^2 , c^2 are in A.P

Solution:- Given $\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$ are in A. P

so, $\frac{1}{c+a} = \frac{\frac{1}{b+c} + \frac{1}{a+b}}{2}$

or, $\frac{2}{c+a} = \frac{1}{b+c} + \frac{1}{a+b}$

or, $\frac{2}{c+a} = \frac{a+b+b+c}{(b+c)(a+b)}$

or, $\frac{2}{c+a} = \frac{a+2b+c}{ab+b^2+ac+bc}$

or, $2(ab + b^2 + ac + bc) = (c+a)(a+2b+c)$

or $2ab + 2b^2 + 2ac + 2bc = ac + 2bc + c^2 + a^2 + 2ab + ac$

or, $2b^2 = c^2 + a^2$

or, $b^2 = \frac{c^2+a^2}{2}$ This shows b^2 is A.M between a^2 and c^2

Hence a^2 , b^2 , c^2 are in A.P proved.

6 (b) If $\frac{1}{2}(x+y)$, y and $\frac{1}{2}(y+z)$ be in H.P., show that x, y, z are in G.P.

Solution:

Since, $\frac{1}{2}(x+y)$, y and $\frac{1}{2}(y+z)$ be in H.P.

So, $y = \frac{2\cdot\frac{1}{2}(x+y)\cdot\frac{1}{2}(y+z)}{\frac{1}{2}(x+y)+\frac{1}{2}(y+z)}$ [$H.M$ $betwwen$ a and b is $H.M = \frac{2ab}{a+b}$]

$= \frac{(x+y)(y+z)}{x+y+y+z}$

$= \frac{xy+xz+y^2+yz}{x+2y+z}$

or, $xy + 2y^2 + yz = xy + xz + y^2 + yz$

or, $y^2 = xz$

i.e. $y = \sqrt{xz}$

This shows that x, y, z are in G.P.

6 (c) If $x + y$, $2y$, $y + z$ are in H.P. , prove that x , y , z are in G.P.

Solution:- Given $x + y$, $2y$, $y + z$

So, $2y = \dfrac{2(x+y)(y+z)}{x+y+y+z}$

or, $2y = \dfrac{2(xy + xz + y^2 + yz)}{x + 2y + z}$

or, $y(x + 2y + z) = xy + xz + y^2 + yz$

or, $xy + 2y^2 + yz = xy + xz + y^2 + yz$

or, $2y^2 - y^2 = xz$

or, $y^2 = xz$

This shows y is the G.M. between x and z

Hence x , y , z are in G.P.

6 (d) If a , b , c are in G.P. , a , x , b are in A.P, and b , y , c are in A.P.

prove that $\frac{1}{x} + \frac{1}{y} = \frac{2}{b}$

Solution: Given a , b , c are in G.P.

So, $b^2 = ac$ …… … (i)

Again, a , x , b are in A.P.

So, $x = \frac{a+b}{2}$ ….. ….. (ii)

Similarly, b , y , c are in A.P.

So, $y = \frac{b+c}{2}$ ….. …. (iii)

Now, L.H.S. $= \frac{1}{x} + \frac{1}{y}$

$$= \frac{2}{(a+b)} + \frac{2}{(b+c)}$$

$$= \frac{2(b+c+a+b)}{(a+b)(b+c)}$$

$$= \frac{2(2b+c+a)}{(ab+ac+\ b^2+bc)}$$

$$= \frac{2(2b+c+a)}{(ab+b^2+\ b^2+bc)}$$ [using (i) $b^2 = ac$]

$$= \frac{2(2b+c+a)}{(ab+2b^2+bc)}$$

$$= \frac{2(2b+c+a)}{b(a+2b+c)}$$

$$= \frac{2}{b}$$ = R.H.S.

7 (a) If a, b, c be in A.P. , b, c, d in G.P. and c, d, e in H.P. , prove that a, c, e are in G.P.

Hint:

$b = \frac{a+c}{2}$ .... (i)

$c^2 = bd$ ....(ii)

$d = \frac{2ce}{c+e}$ .... (iii)

To show: a, c, e are in G.P.

Using (i) and (iii) in (ii) we have,

$c^2 = \left(\frac{a+c}{2}\right)\left(\frac{2ce}{c+e}\right)$

or, $c^3 + c^2e = ace + c^2e$

or, $c^2 = ae$ This shows a, c, e are in G.P.

7 (b) If p is the A.M between q and r , q is the G.M between r and p, then prove that r will be the H.M between p and q.

Solution:- Given p is the A.M between q and r

So, $p = \frac{q+r}{2}$ ……… (i)

Again, q is the G.M between r and p

So, $q^2 = rp$ ….. …. (ii)

Now we prove, r will be the H.M between p and q.

i.e. we prove $r = \frac{2pq}{(p+q)}$

$$\text{R.H S} = \frac{2pq}{(p+q)}$$

$$= \frac{2\left(\frac{q+r}{2}\right)q}{(p+q)} \qquad [\text{ using (i) }]$$

$$= \frac{q(q + r)}{(p + q)}$$

$$= \frac{q(\sqrt{pr} + r)}{(p + \sqrt{pr})}$$ [using (ii) $q^2 = rp$]

$$= \frac{q\sqrt{r}\ (\sqrt{p} + \sqrt{r})}{\sqrt{p}\ (\sqrt{p} + \sqrt{r})}$$

$$= \frac{q\sqrt{r}}{\sqrt{p}}$$

$$= \frac{\sqrt{pr}\sqrt{r}}{\sqrt{p}}$$

$$= \sqrt{r}\ .\ \sqrt{r}$$

$= r$ L.H.S

$\therefore r = \dfrac{2pq}{(p+q)}$ Hence r is H.M between p and q. Proved.

Q.N 7(c):- Replace p , q , r by x , y , z in this solution.

7 (d) If the p^{th}, q^{th}, r^{th} term of an A.P. is a, b, c respectively, prove that

$$a(q-r) + b(r-p) + c(p-q) = 0$$

Solution:

Let A = first term and d = common difference of an A.P.

By question, $t_p = A + (n-1)d = A + (p-1)d$

i.e. $a = A + (p-1)d$ ………………… (i)

$t_q = A + (q-1)d$

i.e. $b = A + (q-1)d$ ………………… (ii)

Similarly, $c = A + (r-1)d$ ………………… (iii)

Now, L.H.S $= a(q-r) + b(r-p) + c(p-q)$

$= \{A + (p-1)d\}(q-r) + \{A + (q-1)d\}(r-p) + \{A + (r-1)d\}(p-q)$

(Using (i), (ii) and (iii))

$= Aq - Ar + pqd - prd - dq + dr + Ar - Ap + qrd - qdp - dr + dp$

$+ Ap - Aq + rdp - rdq - dp + dq$

$= 0 =$ R.H.S. proved.

Q.N 8. If three unequal positive numbers a , b , c are in H.P. , prove that

(a) $a^2 + c^2 > 2b^2$

(b) $a^3 + c^3 > 2b^3$

(c) $a^n + c^n > 2b^n$

Solution:- Given , a , b , c are in H.P.

Here , we first prove if a , b , c are in H.P. then $b^2 < ac$ [i.e. $b^2 - ac < 0$]

Since, a , b , c are in H..P

So, $b = \frac{2ac}{a+c}$

Now, $b^2 - ac = \left(\frac{2ac}{a+c}\right)^2 - ac$

$$= \frac{4a^2c^2 - ac(a+c)^2}{(a+c)^2}$$

$$= \frac{ac(4ac \ -(a+c)^2)}{(a+c)^2}$$

$$= \frac{-ac((a+c)^2 - 4ac)}{(a+c)^2}$$

$= \frac{-ac(a-c)^2}{(a+c)^2}$ which is always negative

$\therefore$ b^2 - ac < 0 .... (i) [i.e. ac > b^2]

(a) Again we know, for any two positive number a and c

A.M. between a^2 and c^2 > G.M. between a^2 and c^2

i.e. $\frac{a^2+c^2}{2} > \sqrt{a^2c^2}$

i.e. $a^2 + c^2 > 2ac$

i.e. $a^2 + c^2 > 2ac > 2b^2$ [using (i)]

i.e. $a^2 + c^2 > 2b^2$ proved.

(b) Again we know, for any two positive number a and c

A.M. between a^3 and c^3 > G.M. between a^3 and c^3

i.e. $\frac{a^3 + c^3}{2} > \sqrt{a^3 c^3}$

i.e. $a^3 + c^3 > 2\,(ac)^{\frac{3}{2}}$

i.e. $a^3 + c^3 > 2(ac)^{\frac{3}{2}} > 2(b^2)^{\frac{3}{2}}$ [using (i) $ac > b^2$]

i.e. $a^3 + c^3 > 2b^3$ proved.

(c) Again we know, for any two positive number a and c

A.M. between a^n and c^n > G.M. between a^n and c^n

i.e. $\frac{a^n + c^n}{2} > \sqrt{a^n c^n}$

i.e. $a^n + c^n > 2\,(ac)^{\frac{n}{2}}$

i.e. $a^n + c^n > 2(ac)^{\frac{n}{2}} > 2(b^2)^{\frac{n}{2}}$ [using (i) $ac > b^2$]

i.e. $a^n + c^n > 2b^n$ proved.

Q.N 9. Find n such that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the arithmetic mean between a and b.

Solution:- Given $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the arithmetic mean between a and b.

So, $\frac{a+b}{2} = \frac{a^{n+1} + b^{n+1}}{a^n + b^n}$

or, $a^{n+1} + ab^n + ba^n + b^{n+1} = 2a^{n+1} + 2b^{n+1}$

or, $ab^n + ba^n = a^{n+1} + b^{n+1}$

or, $a^{n+1} - ba^n - ab^n + b^{n+1} = 0$

or, $a^n(a - b) - b^n(a - b) = 0$

or, $a^n(a - b) = b^n(a - b)$

or, $\left(\frac{a}{b}\right)^n = \frac{a-b}{a-b} = 1$

or, $\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0 \Rightarrow n = 0$ Ans.

Example 2: Insert 3 the harmonic means between 4 and 2.

Solution: Here, in the HP, First term = 4 and last term = 2, No. of harmonic means = 3, 3 harmonic means = ? Then,

In an AP, First term (a) = $\frac{1}{4}$ last term (b) = $\frac{1}{2}$, No. of harmonic means (n) = 3 and common difference = d

Now, we have

$$d = \frac{b-a}{n+1} = \frac{\frac{1}{2}-\frac{1}{4}}{3+1} = \frac{\frac{2-1}{4}}{4} = \frac{1}{16}$$

$\therefore$ First mean of the AP = a + d = $\frac{1}{4} + \frac{1}{16} = \frac{4+1}{16} = \frac{5}{16}$

Second mean of the AP = a + 2d = $\frac{1}{4} + 2 \times \frac{1}{16} = \frac{4+2}{16} = \frac{6}{16} = \frac{3}{8}$

Third mean of the AP = a + 3d = $\frac{1}{4} + 3 \times \frac{1}{16} = \frac{4+3}{16} = \frac{7}{16}$

Hence, the required 3 harmonic means of the HP are $\frac{16}{5}$, $\frac{8}{3}$ and $\frac{16}{7}$.

Example 3: If 4^{th}, 7^{th} and 10^{th} term of a GP are x, y and z respectively, show that $y^2 = zx$ and x, y, z are in GP.

Solution: Given, 4^{th} term $(t_4) = x$, 7^{th} term $(t_7) = y$ 10^{th} term $(t_{10}) = z$

or, $ar^3 = x$ .... (i) or, $ar^6 = y$........... (ii) or, $ar^9 = z$ .... (iii)

Now, dividing eq^n (ii) by eq^n (i) and eq^n (iii) by eq^n (ii), we get

$$\frac{ar^6}{ar^3} = \frac{y}{x} \text{ and } \frac{ar^9}{ar^6} = \frac{z}{y}$$

$$\text{or, } r^3 = \frac{y}{x} \text{ and } r^3 = \frac{z}{y}$$

$$\therefore \frac{y}{x} = \frac{z}{y}$$

or, $y^2 = zx$.

Hence, x, y, z are in GP.

Infinite Geometric Series

Sum of Infinite Geometric Series:

An infinite geometric series is the sum of an infinite geometric sequence.

This series have no last term. The general form of the infinite geometric series is $a + ar + ar^2 + ar^3 + \ldots$ to ∞,

where a is the first term and r is the common ratio.

For example,

The infinite geometric series will be $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \ldots\ldots$ to ∞.

Here, what is the sum of the is series ? How can we find it?

Consider the sum of an infinite geometric series as,

$S_\infty = a + ar + ar^2 + ar^3 + \ldots$ to ∞ .... (i)

Now, we know that the sum of the first n terms of a geometric series

$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^n$,

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{(a-ar^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r} \qquad \ldots\ldots\ (ii)$$

For r as the proper fraction, $|r| < 1$, i.e. $-1 < r < 1$ and $r \neq 0$.

Since $|r| < 1$, $r^n \to 0$ as $n \to \infty$

$$\Rightarrow \quad S_\infty = \frac{a}{1-r}$$

Hence, the sum of the infinite geometric series $a + ar + ar^2 + ar^3 + \ldots$ to ∞ exists, if $|r| < 1$ and $S_\infty = \frac{a}{1-r}$

Arithmetico – Geometric Series :-

A series is said to be an arithmetico – geometric series if it's each term is formed by multiplying the corresponding term of an A.P. and G.P.

i.e. A series in the form of

$a + (a + d) r + (a + 2d) r^2 + (a + 3d) r^3 + \ldots\ldots\ldots\ldots\ldots\ldots$

is called arithmetico – geometric series

For example, $1 + 3x + 5x^2 + 7x^3 + \ldots\ldots \ldots$

In which $1 + 3 + 5 + 7 + \ldots\ldots\ldots\ldots$ is in A.P. and

$1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots$ is in G.P.

Exercise: 5.2

1. Decide which infinite series have sums:

(a) $1 + 3 + 3^2 + 3^3 + \ldots\ldots\ldots\ldots$

Solution:

Here, common ratio $r = \frac{3}{1} = 3$ i.e. $|r| > 1$

So, given infinite series does not have the sum.

(d) 0.5 + 0.05 + 0.005 + …………

Solution:

Here, $r = \frac{0.05}{0.5} = \frac{0.005}{0.05} = 0.1 < 1$ i.e. $|r| < 1$

So, given infinite series have the sum.

2. Find the sum of the infinite series.

(a) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ \ldots\ldots\ldots$

Solution:- Here, first term a = $\frac{1}{2}$

Common ratio r = $\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

Here, |r| < 1 so the sum of infinity of given series exists.

Now, $S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$.

b) $2 + \sqrt{2} + 1 + \ldots\ldots$

Solution:- Here, first term a = 2 , Common ratio r = $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Here, |r| < 1 so the sum of infinity of given series exists.

Now, $S_\infty = \frac{a}{1-r} = \frac{2}{1-\frac{1}{\sqrt{2}}} = \frac{2\sqrt{2}}{\sqrt{2}-1} = \frac{2\sqrt{2}\,(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} = 2(2+\sqrt{2})$

(c) $3^{-1} + 3^{-2} + 3^{-3} + \ldots..$

Solution:- Here, $a = 3^{-1} = \frac{1}{3}$

Common ratio $r = \frac{3^{-2}}{3^{-1}} = \frac{1}{3}$

Here, $|r| < 1$ so the sum of infinity of given series exists.

Now, $S_\infty = \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$.

d) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots.. \ldots\ldots.$

Solution:- Here, $a = 1$, Common ratio $r = \frac{-\frac{1}{3}}{1} = -\frac{1}{3}$

Here, $|r| < 1$ so the sum of infinity of given series exists.

Now, $S_\infty = \frac{a}{1-r} = \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$

(e) $\frac{a}{x}+\frac{b}{x^2}+\frac{a}{x^3}+\frac{b}{x^4}+\ \dots\dots\dots$ **($|x| > 1$)**

Solution:

$$\frac{a}{x}+\frac{b}{x^2}+\frac{a}{x^3}+\frac{b}{x^4}+\ \dots\dots\dots=(\frac{a}{x}+\ \frac{a}{x^3}+\ \dots\dots.)+(\frac{b}{x^2}+\frac{b}{x^4}+\ \dots\dots.)$$

$$=\frac{a/x}{1-\frac{1}{x^2}}+\frac{b/x^2}{1-\frac{1}{x^2}}$$

$$=\frac{a/x}{\frac{x^2-1}{x^2}}+\frac{b/x^2}{\frac{x^2-1}{x^2}}$$

$$=\frac{ax}{x^2-1}+\frac{b}{x^2-1}$$

$$=\frac{ax+b}{x^2-1}$$

3. Sum to infinity of the following series:

(b) $1 + 3x + 5x^2 + 7x^3 + \ldots\ldots\ldots\ldots$ to ∞ $(|x| < 1)$

Solution: Which is arithmetico – geometric series with 1 , 3 , 5 , ……. is arithmetic series and $1, x, x^2, x^3, \ldots\ldots\ldots\ldots$ is geometric series

Here, common ratio of geometric series, $1, x, x^2, x^3, \ldots\ldots\ldots\ldots$ is $r = \frac{t_2}{t_1} = \frac{x}{1} = x$

Here, $S_\infty = 1 + 3x + 5x^2 + 7x^3 + \ldots\ldots\ldots\ldots$

$xS_\infty = x + 3x^2 + 5x^3 + 7x^4 + \ldots\ldots\ldots\ldots$

$[\because \textit{multiplying both sides by } x]$

Subtracting we have ,

$$S_\infty(1 - x) = 1 + 2x + 2x^2 + 2x^3 + \ldots\ldots\ldots\ldots$$

$$= 1 + 2(x + x^2 + x^3 + \ldots\ldots\ldots)$$

$$= 1 + 2\,\frac{x}{1-x} \qquad \left[\because S_\infty = \frac{a}{1-r} \text{ where, } r = x\right]$$

$$= \frac{1 - x + 2x}{1-x}$$

$$= \frac{1 + x}{1-x} \qquad \text{Hence, } S_\infty = \frac{1 + x}{(1-x)^2}$$

3. (c) $1 - 5x + 9x^2 - 13x^3 + \ldots\ldots\ldots\ldots$ **to** ∞ **($|x| < 1$)**

Solution: Which is arithmetico – geometric series with 1 , 5 , 9 , ……. is arithmetic series and 1, -x, x^2, -x^3, …………. is geometric series

Here, common ratio of geometric series, 1, -x, x^2, -x^3, …………. is r = $\frac{t_2}{t_1} = \frac{-x}{1} = -x$

$$\text{Here, } S_\infty = 1 - 5x + 9x^2 - 13x^3 + \ldots\ldots\ldots\ldots\ldots$$

$$-x\, S_\infty = \quad -x + 5x^2 - 9x^3 + \ldots\ldots\ldots\ldots\ldots.$$

$$[\because multiplying\ both\ sides\ by - x]$$

Subtracting we have,

$$(1 + x)S_\infty = 1 - 4x + 4x^2 - 4x^3 + \ldots\ldots\ldots\ldots..$$

$$= 1 - 4(x - x^2 + x^3 - \ldots\ldots\ldots to\ \infty)$$

$$= 1 - 4\frac{x}{1+x} \qquad \left[\because S_\infty = \frac{a}{1-r}\ where,\ r = -x\right]$$

$$= \frac{1+x-4x}{1+x}$$

$$= \frac{1-3x}{1+x}$$

$$\text{Therefore, } S_\infty = \frac{1-3x}{(1+x)^2}$$

3.(d) $1+\frac{3}{4}+\frac{7}{16}+\frac{15}{64}+\ \ldots\ldots\ldots\ldots.$

Solution:

$$S_\infty = 1+\frac{3}{4}+\frac{7}{16}+\frac{15}{64}+\ \ldots\ldots\ldots\ldots.$$

$$=1+\frac{2^2-1}{2^2}+\frac{2^3-1}{2^4}+\frac{2^4-1}{2^6}+\ \ldots\ldots\ldots$$

$$=1+\left(1-\frac{1}{2^2}\right)+\left(\frac{1}{2}-\frac{1}{2^4}\right)+\left(\frac{1}{2^2}-\frac{1}{2^6}\right)+\ \ldots\ldots\ldots\ldots$$

$$=1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\ \ldots\ldots..\right)-\left(\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\ \ldots\ldots\ldots.\right)$$

$$=1+\left(\frac{1}{1-\frac{1}{2}}\right)-\left(\frac{\frac{1}{2^2}}{1-\frac{1}{2^2}}\right)$$

$$=1+\frac{2}{1}-\frac{1}{3}$$

$$=1+2-\frac{1}{3}=\frac{8}{3}.$$

4. (b) The sum to an infinite number of terms of G.S. is 20 and the sum of their squares is 50 find the series.

Solution:- Let $a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots$ be the infinity series.

Given, $S_\infty = 20$

or, $\frac{a}{1-r} = 20$

or, $a = 20 - 20r \quad \ldots\ldots \text{(i)}$

Again, given $\frac{a^2}{1-r^2} = 50$

or, $a^2 = 50 - 50r^2$

or, $(20 - 20r)^2 = 50 - 50r^2$

or, $400 - 800r + 400r^2 = 50 - 50r^2$

or, $450r^2 - 800r + 350 = 0$

or, $9r^2 - 16r + 7 = 0$

or, $(r - 1)(9r - 7) = 0$

Hence either , $r - 1 = 0$

$\Rightarrow r = 1$ (Which is not possible)

or, $9r - 7 = 0$

$\Rightarrow r = \frac{7}{9}$

When $r = \frac{7}{9}$ from equation (i)

$a = 20 - 20r$

or, $a = 20 - 20 \times \frac{7}{9}$

or, $a = \frac{40}{9}$

Hence required series is

$a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots$

$= \frac{40}{9} + \frac{280}{81} + \ldots\ldots\ldots$

4. (c) The sum of the first two terms of an infinite G.S. is 5 and each term is equal to thrice the sum of all its succeeding terms . Find the sum of infinity of the series.

Solution:- Let $S_\infty = a + ar + ar^2 + ar^3 + \ldots$ to ∞ be the infinite geometric series.

By question, $a + ar = 5$

$$a = \frac{5}{1+r} \quad \ldots\ldots \ldots\ldots (i)$$

Again, $a = 3(ar + ar^2 + ar^3 + \ldots \ldots \text{ to } \infty)$

$$a = 3r(a + ar + ar^2 + ar^3 + \ldots \text{ to } \infty)$$

or, $a = 3r\, S_\infty$

or, $a = 3r \dfrac{a}{1-r}$

or, $1 - r = 3r$

or, $r = \dfrac{1}{4}$

When $r = \dfrac{1}{4}$, from equation (i) $a = 4$

Now, $S_\infty = \dfrac{a}{1-r} = \dfrac{4}{1-\frac{1}{4}} = \dfrac{16}{3}$

4.(d) Find the first term of a G.P. , whose second term is 2 and sum to infinity is 8.

Solution:- Let $a + ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots$ be the infinity series.

Given, $ar = 2 \Rightarrow r = \frac{2}{a}$

$$S_\infty = \frac{a}{1-r}$$

or, $8 = \frac{a}{1-r}$

or, $8 - 8r = a$

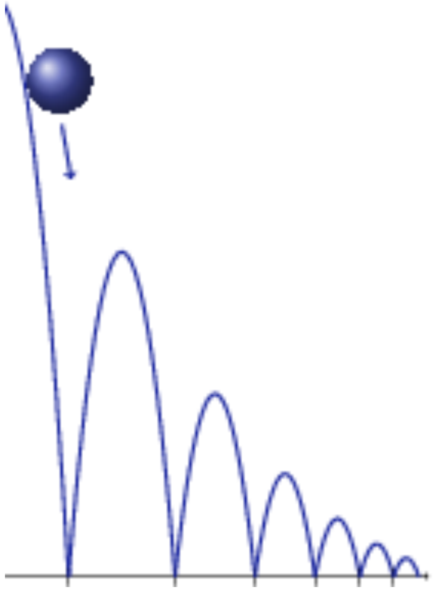
or, $8 - 8\times\frac{2}{a} = a$

or, $8a - 16 = a^2$

or, $a^2 - 8a + 16 = 0$

or, $(a-4)^2 = 0 \quad \Rightarrow \quad a = 4$ Hence, first term $a = 4$

5. A rubber ball is dropped from a roof of the height 24 feet. At each rebound, it rises to a height of $\frac{2}{3}$ of the height of the previous fall. If it continuous to fall and rebound in this way, find the total distance through which the ball will have moved before it finally comes to rest ?



Solution:- Taking downward motion of the ball,

$$h_1 = 24\text{ft},\ h_2 = 24 \times \frac{2}{3}\ \text{ft}\ ,\ h_3 = 24 \times \frac{2}{3} \times \frac{2}{3}\ \text{ft}\ ,\ \ldots\ldots\ldots\ldots \text{ to } \infty$$

$$\text{Their sum, } S_\infty = 24 + 24 \times \frac{2}{3} + 24 \times \frac{2}{3} \times \frac{2}{3}\ \ldots\ldots\ldots\ldots \text{ to } \infty$$

$$= \frac{24}{1 - \frac{2}{3}} = 24 \times 3 = 72 \text{ ft}$$

Taking upward motion of the ball,

$$H_1 = 24 \times \frac{2}{3}\ \text{ft}\ ,\ H_2 = 24 \times \frac{2}{3} \times \frac{2}{3}\ \text{ft}\ ,\ H_3 = 24 \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}\ \text{ft.}\quad \ldots\ldots\ldots\ldots \text{ to } \infty$$

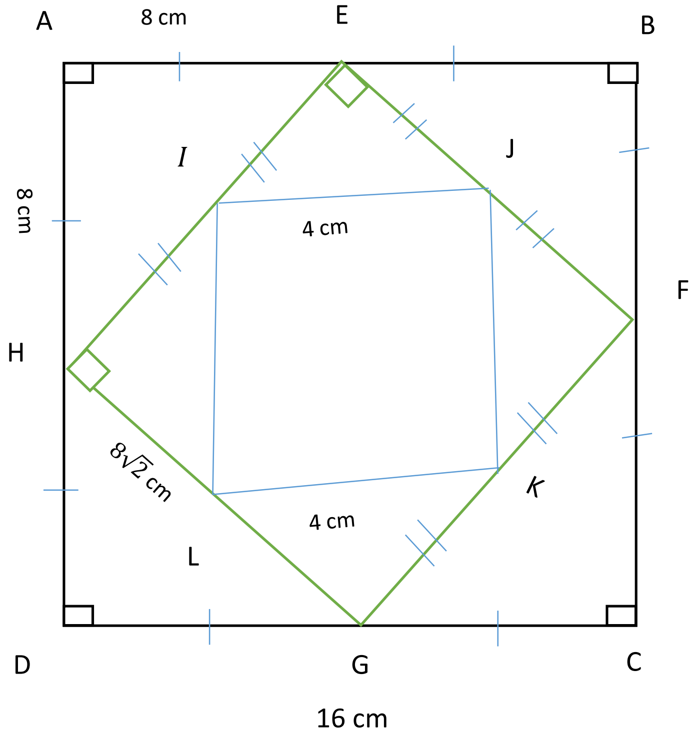
Their sum $S_\infty = 24 \times \frac{2}{3} + 24 \times \frac{2}{3} \times \frac{2}{3} + 24 \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$ .... to ∞

$$= \frac{24 \cdot \frac{2}{3}}{1 - \frac{2}{3}}$$

$$= 48 \text{ ft}$$

Hence, the total distance through which the ball will have moved before it finally comes to rest is $72 + 48 = 120$ ft.

6. The side of a square is each 16 cm. A second square is inscribed by joining the mid-points of the sides successively. A third square is drawn inside the second square in the same way and this process is continued indefinitely many times. Find the sum of the perimeters of all the squares.



Solution:- Given that, length of first square AB = 16 cm

According to the question,

Length of second square EH = $\sqrt{8^2+8^2} = 8\sqrt{2}$ cm

Length of third square IL = $\sqrt{(4\sqrt{2})^2+(4\sqrt{2})^2} = 8\ cm$

And so on.

Here, Perimeter of First Square $t_1 = 4\times 16 = 64$

Perimeter of Second Square $t_2 = 4\times 8\sqrt{2} = 32\sqrt{2}$

Perimeter of Third Square $t_3 = 4\times 8 = 32$

And so on.

Infinite sum of perimeter of all squares = $64 + 32\sqrt{2} + 32 + \ldots\ldots\ldots\ldots$

Here, $a = 64, r = \frac{t_2}{t_1} = \frac{32\sqrt{2}}{64} = \frac{1}{\sqrt{2}}$

Required infinite sum $= S_\infty = \frac{a}{1-r} = \frac{64}{1-\frac{1}{\sqrt{2}}} = \frac{64\sqrt{2}}{\sqrt{2}-1} = \frac{64\sqrt{2}}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{64(2+\sqrt{2})}{2-1} = 64(2+\sqrt{2})$.